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Operads and combinatorics

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In some problems it is necessary to consider series indexed, not by integers, but by some other combinatorial objects like trees for instance. In order to multiply and compose these series one needs to be able to add trees and to multiply them. Most of the time the structure which unravel these constructions is a new type of algebras, or, equivalently, an *operad*.

For instance, consider the algebras equipped with two binary operations $\prec$, called *left*, and $\succ$, called *right*, satisfying the following relations

$$\begin{cases} (x \prec y) \prec z &= x \prec (y * z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x * y) \succ z &= x \succ (y \succ z), \end{cases}$$

where $x * y := x \prec y + x \succ y$. They are called *dendriform algebras* (cf. [**L1**]). It can be shown that the free dendriform algebra on one generator is the vector space spanned by the planar binary rooted trees. The two products are described by means of grafting. From this result we can construct a product and a composition on the series $\sum_t a(t)x^t$, where the sum runs over the planar binary rooted trees (no constant term). For the product we simply use the fact that the product $*$ in a dendriform algebra is associative (check it !). For the composition we use the explicit description of the free dendriform algebra. In fact these series form a group for composition. This is the *renormalisation group* of Quantum Electro-Dynamics.

There are several examples of this type, few of them have been studied so far.

There are many more problems where operads can help in combinatorics. Let me just mention two of them. A free algebra of some sort is, in general, graded and one can form its generating series. In terms of operads, when the space of n-ary operations $\mathcal{P}(n)$ is finite dimensional one defines

$$f^{\mathcal{P}}(x) := \sum_{n\geq 1} (-1)^n \frac{\dim \mathcal{P}(n)}{n!} x^n \ .$$

An important theory in the operad framework is the Koszul duality. Well-known for associative algebras it has been generalized to operads by Ginzburg and Kapranov (cf. [**G-K**], [**F**], see [**L1**] appendix 2 for a short overview on operads and Koszul duality). One of the consequences is the following. To any quadratic operad $\mathcal{P}$ one can associate its dual $\mathcal{P}^!$ and a certain chain complex, called the Koszul complex. When the Koszul complex is acyclic, then the generating series of $\mathcal{P}$ and $\mathcal{P}^!$ are inverse to each other for composition:

$$f^{\mathcal{P}}(f^{\mathcal{P}^!}(x)) = x\ .$$

This nice theorem has many applications. One of them is, in some instances, to provide a combinatorial interpretation of some integer sequences (cf. [**L2**]).

Here is another application. The partition lattice gives rise to a chain complex whose homology is a representation of the symmetric group. It is not that easy to compute. However there is an operadic way of looking at it, which, by using Koszul duality, permits us to identify this homology group to the space of operations of a dual operad (cf. [**F**]). This interpretation can be generalized to many variations of the classical partition lattice provided that the operad involved is Koszul (cf. [**V1**]).

If, instead of looking only at operations, we want look at operations *and* co-operations, then the notion of operad has to be replaced by the notion of *prop.* At this point there is a need for a Koszul duality theory in the prop framework. This has recently been achieved in [**V2**].

References

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