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Strict Partitions and Discrete Dynamical Systems

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Abstract. We prove that the set of partitions with distinct parts of a given positive integer under dominance ordering can be considered as a configuration space of a discrete dynamical model with two transition rules and with initial configuration being the singleton partition. This allows us to characterize its lattice structure, fixed point, longest chains as well as their length, using Chip Firing Game theory. Finally, two extensions and their applications are discussed.

Résumé. Nous montrons que l'ensemble des partitions avec differents parts d'un entier donné n muni de l'ordre de dominance peut être considérećomm l'espace de configurations d'un système dynamique discret avec deux règles de transitions et avec la configuration initialle étant la partition (n). Cela nous permet de caractériser sa structure de treillis, son point fixe, les chaînes les plus longues ainsi que leurs longueur, en utilisant la theorie de Chip Firing Game. Enfin, deux extensions et leurs applications sont données.

1. Introduction

A partition of a positive integer n is a sequence of non-increasing positive integers $a = (a_1, \dots a_m)$ such that $a_1 + \dots + a_m = n$. The set of all such partitions of n is denoted by $\mathcal{P}(n)$. $\mathcal{P}(n)$ is equipped with a partial order called *dominance order* as follows : $a \geq b$ if its partial sums is greater than that of b, *i.e.* $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$. This order has been showed to have many applications to problems in combinatorics as well as group representation theory, among other fields. The structure of this poset was studied by Brylawski [**Bry73**] who showed in particular that it is a lattice. Since then, other properties such as maximal chains, fixed point have also been characterized in [**Bry73, GK86, GK93**]. In [**LP01**], Phan and Latapy constructed its infinite extension and obtained a construction algorithm.

In this paper, we study the structure of an interesting class $\mathcal{SP}(n)$ of partitions of n called strict partitions, or partitions with distinct parts, from the point of view of discrete dynamical systems. For any strict partition a of n, one can apply on a the following transition rules so that the resulting partition is also strict :

- Vertical transition (V-transition):
$$(a_1, \dots, a_i, a_{i+1}, \dots, a_n) \to (a_1, \dots, a_i - 1, a_{i+1} + 1, \dots, a_n),$$
if $a_i - a_{i+1} \geq 3$.
- Horizontal transition (H-transition):
$$(a_1, \dots, p+l+1, p+l-1, p+l-2, \dots, p+2, p+1, p-1, \dots, a_n) \to$$
$$(a_1, \dots, p+l, p+l-1, p+l-2, \dots, p+2, p+1, p, \dots, a_n).$$

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FIGURE 1. Vertical transition and horizontal transition

These rules then define a partial order on $\mathcal{SP}(n)$ by declaring that $b \leq_S a$ if b can be obtained from a via a sequence of transitions. In particular, we will show that all strict partitions can be obtained in this way if the initial configuration is the singleton partition (n). Moreover, the poset $\mathcal{SP}(n)$ which corresponds to the above order and initial configuration (n) turns out to be the same as the poset $\mathcal{SP}(n)$ with dominance order, so that the two orders can now be identified. We then show that $\mathcal{SP}(n)$ is also a lattice, but it is not a sublattice of $\mathcal{P}(n)$. Furthermore, unlike $\mathcal{P}(n)$, $\mathcal{SP}(n)$ is not self-dual. Using the fact that our dynamical model can be viewed as a "composition" of two Chip Firing Games in the sense of [**BL92**] (see also [**LP01**], [**GMP02**]), we are able to characterize explicitly the fixed point, longest chains as well as their length in $\mathcal{SP}(n)$. Finally, we present two generalizations : We obtain similar results for the set of k-strict partitions, *i.e.* partitions where two parts differ by at least $k > 0$. We also obtain an infinite extension of $\mathcal{SP}(n)$ and an algorithm to construct $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$ in linear time.

2. Lattice structure of $\mathcal{SP}(n)$

THEOREM 2.1. *The set $\mathcal{SP}(n)$ is exactly the set of all strict partitions reachables from (n) by applying two transitions rule V and H. .*

PROOF. Let $a = (a_1, \ldots, a_m)$ be a strict partition. It suffices to show that if a is different from (n) itself, then there exist another strict partition a' such that one can recover a by applying a transition on a'.

First of all, observe that if there is a subsequence $(a_i, a_{i+1}, \ldots, a_j)$ of consecutive numbers in a, where $i = 1$, or else $a_{i-1} - a_i \geq 2$, similarly $j = m$ or else $a_j - a_{j+1} \geq 2$. Then we can choose

$$a' = (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_m),$$

so that a' is again strict. Furthermore, one recovers a from a' by applying a H-transition.

On the other hand, if no such subsequence exists, then $a_1 - a_2 \geq 2$ and either $m = 2$ or $a_2 - a_3 \geq 2$. In this case, we can simply choose

$$a' = (a_1 + 1, a_2 - 1, a_3, \ldots, a_m).$$

It is easy to check that a' is a strict partition and that a V-transition applied on a' at the first position gives back a. The theorem is proved. □

PROPOSITION 2.2. *$\mathcal{SP}(n)$ is a subposet of $\mathcal{P}(n)$.*

PROOF. It is sufficient to show that if $a, b \in \mathcal{SP}(n)$ and $a > b$ then $a >_S b$, *i.e.* there exists a sequence of transitions from a to b. For this purpose, it suffices to prove that one can apply a transition on a to obtain a new strict partition a' such that one still has $a' \geq b$.

Since $a > b$, we have $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$ for all $1 \leq j \leq n$. Let j be the smallest index where $a_j > b_j$. Then let ℓ be the smallest index such that $\ell > j$ and $\sum_{i=1}^{l} a_i = \sum_{i=1}^{l} b_i$. Such a number ℓ exists because $\ell = n$ satisfies both conditions above. It is clear that $a_l < b_l$ because of the choice of ℓ.

We claim that we can apply a transition on a at some positions between j and ℓ, so that the newly constructed partition a' are identical with a outside this range . If this is possible, then we are done, because it is easy to verify, using the definition of j and ℓ,that $a' > b$ in $\mathcal{P}(n)$.

To construct a', observe that if there is an index $j \leq i \leq \ell$ such that $a_i - a_{i+1} \geq 3$, then a V-transition can be applied at position i and we are done.

Suppose now that $a_i - a_{i+1} \leq 2$ for all $j \leq i < \ell$. Since b is a strict partition and $b_i \geq 1$ for all i, we have $b_\ell - b_j \geq \ell - j$. But $a_\ell > b_\ell$ and a_j, b_j, hence $a_\ell - a_j \geq \ell - j + 2$. It follows that there exists at least two indices $j \leq r < s < \ell$ such that $a_r - a_{r+1} = a_s - a_{s+1} = 2$. Furthermore, by choosing a different pair of indices if necessary, we can even assume that $a_i - a_{i+1} = 1$ for all $r < i < s$. But in this case, the subsequence $(a_r, \ldots, a_s)$ is of exactly the form where one can apply a H-transition. The proof is finished. □

Because of the above result, we can now write $b \leq a$ instead of $b \leq_S a$ for any two strict partition a and b.

THEOREM 2.3. *$\mathcal{SP}(n)$ is a lattice. Moreover, the meet operation in $\mathcal{SP}(n)$ is the same as that in $\mathcal{P}(n)$, i.e. $a \wedge_S b = a \wedge b$ for any two strict partitions a and b.*

PROOF. Since $\mathcal{SP}(n)$ contains a maximal element, it is enough to prove that any pair of element in $\mathcal{SP}(n)$ has a greatest lower bound. Of course, their greatest lower bound $c = a \wedge b$ in $\mathcal{P}(n)$ does exist, but is it true that c is again a strict partition? We will show that this is the case for any pair of strict partitions a and b.

By definition, c is a partition defined by the formulae

$$\sum_{i=1}^{m} c_i = min(\sum_{i=1}^{m} a_i, \sum_{i=1}^{m} b_i)$$

for all $1 \leq m$. Suppose that $c_m > 0$. Without loss of generality, assume that $\sum_{i=1}^{m} c_i = \sum_{i=1}^{m} a_i$. Then $c_{m+1} \leq a_{m+1}$ while $a_m \leq c_m$. Thus $c_{m+1} < c_m$ because $a_{m+1} < a_m$. Hence c is also a strict partition. The proof above clearly also implies that the meet operation in $\mathcal{SP}(n)$ is the same as that in $\mathcal{P}(n)$. □

REMARK 2.4. $\mathcal{SP}(n)$ is **not** a sublattice of $\mathcal{P}(n)$. In fact, the joint operations in $\mathcal{SP}(n)$ and $\mathcal{P}(n)$ are different. For example, $(8,4,3,1) \vee (7,5,4) = (8,4,4)$ which is not a strict partition. Nevertheless, we still have $a \vee_S b \geq a \vee b$ for any a and b.

Since $\mathcal{SP}(n)$ is a lattice, it has an unique minimal element (or fixed point). We finish this section by giving an explicit formula for this minimal partition. Let p be the unique number such that

$$\frac{1}{2}p(p+1) \leq n < \frac{1}{2}(p+1)(p+2).$$

Then let $q = n - \frac{1}{2}p(p+1)$. One verifies easily that $q < p$. Now let Π be the following partition

$$\Pi = ((p+1), p, \ldots, (p-q+2), (p-q), (p-q-1), \ldots, 2, 1). \tag{2.1}$$

It is evident that Π is a strict partition on which no transition can be applied. Thus we have the following proposition:

PROPOSITION 2.5. Π *is the fixed point of the lattice $\mathcal{SP}(n)$.*

3. Longest chains

In this section, we characterize longest chains in $\mathcal{SP}(n)$ as well as their length. The longest chains in $\mathcal{P}(n)$ were characterized by Greene and Kleitman [**GK86**] where they introduced the notion of VH-chain (*i.e.* a chain of V-transitions followed by a chain of H-transitions) and proved that all VH-chains are longest chains. It turns out that the same is true for strict partitions. Our proof, however, is different. The proof in [**GK86**] makes use of a series of delicate lemmas which basically consider the differences of consecutive parts of partitions. We believe that our proof, which is based on the theory of Chip Firing Game on directed graph (CFG) [**BL92**], is simpler and probably can be adapted in other contexts.

3.1. V(H)-chain. Let us first introduce some definitions. A V(resp. H)-chain is a chain of V(resp. H)-transitions, and a VH-chain is a concatenation of a V-chain and a H-chain. If there is a V-chain from a strict partition a to another b, then we say that b is V-reachable from a. But a partition c is H-reachable from d means that there is an H-transitions from d back to c, or equivalently an inverse H-transition from c to d.

We will also need the two functions *V-weight* $w_V(a)$ and *H-weight* $w_H(a)$ on a strict partition a. From the Ferrers diagram for a, let

$$w_V(a) = \sum (i-1)a_i \tag{3.1}$$

and

$$w_H(a) = \sum (k-1)\tilde{a}_k, \tag{3.2}$$

where $\tilde{a}_k$ is the number of cells (i, j) on the segment $i + j = k + 1, i \geq 0, j \geq 0$. It is easy to see that a V-transition increases V-weight by 1, but decrease H-weight by at least 1. On the other hand, an H-transition decreases H-weight by 1, and increases V-weight by at least 1. This simple observation shows that V-chains (or H-chains) between two partitions are longest chains.

3.2. Chip Firing Game. We now give a brief overview of the theory of Chip Firing Game (CFG for short). In particular, we show that the dynamical model consisting of only the V-transition (resp. H-transition) are examples of CFG. For more details account of theory of Chip Firing Game, we refer to [**BLS91, BL92, LP01, GLM$^+$ar**].

A Chip Firing Game is a discrete dynamical system defined on a (directed) graph $G = (V, E)$, where each configuration consists of a partition of n chips on the vertices V, and obeys the following rule, called *firing rule*: a vertex containing at least at many chips as its outgoing degree (*i.e.* the number of outgoing edges) transfers one chip along each of its outgoing edges.

This rule defines a natural partial order on the space of configurations by declaring that a configuration b is smaller than a if b can be obtained from a by iterating the firing rule. A *fixed point* of a CFG is a configuration where no firing is possible. The following is the fundamental result in the theory of CFG,

THEOREM 3.1. [**BL92, LP01**] *The set of all configurations reachable from the initial one of a CFG with no closed component is a lattice.*

A closed component of a graph is a strongly connected component without outgoing edge.

One can also characterize the natural order defined above using the notion of shot vector. If $b < a$, then the *shot vector* $k(a, b)$ is the vector in $\mathbb{N}^{|V|}$ whose entry $k_v(a, b)$ is the number of firings at vertex v to obtain b from a. This vector depends only on a and b but not on a chosen sequence of firings. We then have:

LEMMA 3.2. [**LP01**] *Let c and d be two configurations reachable from the same initial configuration a in a CFG. Then $c \geq d$ if and only if $k_v(a, c) \leq k_v(a, d)$ for all vertices $v \in V$.*

Here are two important examples of CFG.

Example V: The dynamical model consisting to only the V-transition is a CFG. Indeed, consider the graph $G = (V, E)$ with $n + 1$ vertices defined pictorially as follows:

$$\overset{v_0}{\circ} \leftarrow \overset{v_1}{\circ} \leftrightarrows \overset{v_2}{\circ} \dots \leftrightarrows \overset{v_n}{\circ}$$

Thus each vertex of G, beside v_0 and v_n has outgoing degree 2. Now let a be a configuration *i.e.* a strict partition of n, we put $d_i = a_i - a_{i+1} - 1$ chips at vertex v_i for all $i \geq 1$ and no chip at v_0.

The necessary condition to apply a V-transition at position i on a is $a_i - a_{i+1} \geq 3$, or equivalently $d_i \geq 2$ which is the same as the condition to apply the CFG firing rule on v_i. It is easy to see that the space of reachable configurations of this CFG is exactly the set of partitions that are V-reachable from a. In particular, the unique fixed point of this CFG corresponds to the smallest partition which is V-reachable from a. In fact, in any interval $b \leq a$ in $\mathcal{SP}(n)$ there exists a unique smallest strict partition $\lfloor a \rfloor_b$ which is V-reachable from a.

Example H: The inverse H-transition defines a CFG on the same graph as in previous example in a similar way. For each initial configuration b, we put $\tilde{d}_i = \tilde{b}_i - \tilde{b}_{i+1}$ chips at vertex v_i for all $i \geq 1$ and no chip at v_0. Here $\tilde{b}_i$ is defined as in (3.1). One verifies again that the space of configurations of this game is the set of H-reachable from b and the fixed point corresponds to unique greatest partition which is H-reachable from b. Furthermore, there is a unique greatest strict partition $\lceil b \rceil^a$ which is H-reachable from b in any interval $b \leq a$.

The reader of [**GK86**] may find it interesting to compare the definition of H-transition there and ours, as in Example H. While an H-transition in the sense of [**GK86**] is dual to a V-transition in the usual sense (row vs column), our H-transition is defined in terms of the "diagonal" in the Ferrers diagram of the corresponding partition.

3.3. VH-chains are longest chains. First of all, it is not hard to show, as in [**GK86**, Lemma 3] that any longest chain must be a VH-chain. The point is that any sequence of two transitions (H,V) (H follows by V) equals sequence of the form either (V,H) or (V,V,H), proof by direct inspection. Thus for any chain of transitions between two partitions, there is a VH-chain of at least the same length.

It remains to show that any VH-chain is a longest chain. We begin with the following key lemma which explains the relevance of dominance order.

Lemma 3.3. *Let c and d be two partitions which are V-reachable from a. If $d \leq c$, then d is V-reachable from c.*

Proof. We compute the shot vector $k(a, c)$ and $k(a, d)$ in the corresponding CFG. It is easy to see that $k_i(a, c) = k_{i-1}(a, c) + a_i - c_i$ for all $i \geq 1$, which implies that $k_i(a, c) = \sum_{j=1}^{i} a_j - \sum_{j=1}^{i} c_j$. Similarly, $k_i(a, d) = \sum_{j=1}^{i} a_j - \sum_{j=1}^{i} d_j$. On the other hand, $\sum_{j=1}^{i} c_j \geq \sum_{j=1}^{i} d_j$ because $c \geq d$. It follows that $k_i(a, c) \leq k_i(a, d)$ and so d is V-reachable from c by Lemma 3.2. □

Lemma 3.4. *If $a \geq b$, then $\lfloor a \rfloor_b$ is H-reachable from b and $\lceil b \rceil^a$ is V-reachable from a.*

Proof. There is a VH-chain from $\lfloor a \rfloor_b$ to b. Since $\lfloor a \rfloor_b$ is the smallest strict partition which is V-reachable from a in interval $a \leq b$, there can not be no V-transition in this chain and $\lfloor a \rfloor_b$ is H-reachable from b. Similar argument applied for $\lceil b \rceil^a$. □

As an immediate corollary, we see that there is a VH-chain $a \to \lceil b \rceil^a \to b$ from a to b of length $w_V(a, \lceil b \rceil^a) + w_H(\lceil b \rceil^a, b)$.

We can now state the main result of this section:

Theorem 3.5. *All VH-chains from a to b in $\mathcal{SP}(n)$ have the same length and this length is maximal.*

Proof. Suppose that $a \xrightarrow{V} c \xrightarrow{H} b$ is a VH-chain from a to b with length $w_V(c) - w_V(a) + w_H(b) - w_H(c)$. We will show that it has the same length as that of the VH-chain $a \xrightarrow{V} \lceil b \rceil^a \xrightarrow{H} b$. In particular, its length only depends on a and b and is maximal.

It is clear from the definition of $\lfloor a \rfloor_b$ and $\lceil b \rceil^a$ that $\lfloor a \rfloor_b \leq c \leq \lceil b \rceil^a$. Since both $\lceil b \rceil^a$ and c are V-reachable from a and $\lceil b \rceil^a \geq c$, then there is a V-chain from $\lceil b \rceil^a$ to c by Lemma 2. On the other hand, there is also an H-chain from $\lceil b \rceil^a$ to c because $\lceil b \rceil^a$ is the minimum element of the lattice of all H-reachable strict partitions from b which contains c. The two chains are both of maximal length, hence $w_V(c) - w_V(\lceil b \rceil^a) = w_H(\lceil b \rceil^a) - w_H(c)$. The required result immediately follows from the equalities :

$$\begin{aligned} w_V(c) - w_V(a) &= w_V(\lceil b \rceil^a) - w_V(a) + w_V(c) - w_V(\lceil b \rceil^a) \\ w_H(c) - w_H(b) &= w_H(\lceil b \rceil^a) - w_H(b) - (w_H(\lceil b \rceil^a) - w_H(c)). \end{aligned}$$

□

3.4. The length of a longest chain. Once we know all VH-chains are longest chains, it is sufficient to calculate the length of a well-chosen VH-chains from (n) to Π. The VH-chain that we will use is $(n) \xrightarrow{V} \lfloor (n) \rfloor_\Pi \xrightarrow{H} \Pi$. For the point $P = \lfloor (n) \rfloor_\Pi$, which is the fixed point of the CFG in Example V with initial configuration (n), together with the length of the V-chain from $(n) \to \lfloor (n) \rfloor_\Pi$ was already computed in [**GMP02**]. Our model corresponds to the model named $L(n,3)$ in that article. To describe P and $w_V((n),P)$, first write n in the form $n = k(k+1) + \ell(k+1) + h$, where $0 \le \ell \le 1, 0 \le h \le k$. The integers k, ℓ, h are all uniquely determined from n. We have

Proposition 3.6.

$$P = (\ell + 2k, \ell + 2(k-1), \dots, \ell + 2h, \ell + 2(h-1) + 1, \dots, \ell + 2 + 1, \ell + 1), \tag{3.3}$$

and

$$w_V(P) = \frac{(k-1)k(k+1)}{3} + \ell \frac{k(k+1)}{2} + h \frac{2k-h+1}{2}. \tag{3.4}$$

We can now state the following result:

Proposition 3.7. *Let p, q the unique integers such that $n = \frac{1}{2}p(p+1) + q, 0 \le q \le p$ and let k, ℓ, h the unique integers such that $n = k(k+1) + \ell(k+1) + h, 0 \le \ell \le 1, 0 \le h \le k$. We have the following formula for the length L of longest chains in $\mathcal{SP}(n)$:*

$$L = \frac{k(k+1)(8k-5)}{6} + 2\ell k(k+1) + (2k+l)h - \frac{(p-1)p(p+1)}{3} - qp.$$

Proof. Since $L = w_V(P) + w_H(P) - w_H(\Pi)$, we have from (2.1) and (3.2):

$$w_H(\Pi) = \sum_{i=1}^{p} (i-1)i + qp = \frac{(p-1)p(p+1)}{3} + qp.$$

and from (3.3) and (3.2):

$$\begin{aligned} w_H(P) &= \sum_{i=1}^{k} (i-1)i + \sum_{i=k}^{1} (2k-i)i + l \sum_{i=k}^{2k} i + \sum_{i=0}^{h-1} (k+\ell+i) \\ &= \frac{1}{2}k(k+1)(2k-1) + \frac{3}{2}\ell k(k+1) + \frac{1}{2}(2k+2\ell+h-1)h. \end{aligned}$$

□

4. Infinite extension of $\mathcal{SP}(n)$

It is natural to ask whether one can construct the lattice $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$. More generally, what is the precise relationship between the lattices $\mathcal{SP}(n)$ for various n. Our solution to these questions is to assemble them together into a lattice $\mathcal{SP}(\infty)$ called lattice of strict partitions of infinity. Indeed, this lattice is constructed in a similar way as $\mathcal{SP}(n)$ by pretending that n can be as large as needed. More precisely, it is the lattice obtained from the dynamical system with two transitions rules as those for $\mathcal{SP}(n)$, and the initial configuration is infinity. Equivalently, one can also define $\mathcal{SP}(\infty)$ in terms of dominance order : A strict partition of infinity is just a sequence of finitely many strictly decreasing positive integers, except the first entry : $(\infty, a_2, a_3, \dots a_k)$. The partial order is defined by declaring that $a \ge_\infty b$ if $\sum_{i \ge j} a_i \le \sum_{i \ge j} b_i$ for all $j \ge 2$. By convention, we put $a_n = 0$ for $n > k$.

Many results presented in this section are obtained initially in the case normal partitions in [**LP99**]. However, the proofs are not completely similar since we must be careful that our operations are within the

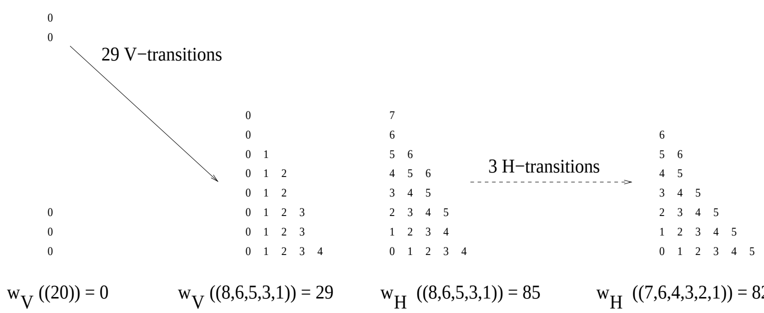

FIGURE 2. A longest chain in $\mathcal{SP}(23)$: $P = (8,6,5,3,1)$ and $\Pi = (7,6,4,3,2,1)$. A longest chain in $\mathcal{SP}(23)$ is a chain containing a V-chain from (23) to P and an H-chain from P to Π, and its length is $w_H(8,6,5,3,1)+w_V(8,6,5,3,1)-w_V(7,6,4,3,2,1) = 29-0+85-82 = 32$.

set of strict partition. In fact, even though $\mathcal{SP}(n)$ can be embedded in a $\mathcal{P}(n)$, the structure of the infinite lattices or infinity trees are different.

4.1. Notations and definitions. If $a = (a_1, a_2, \ldots, a_k)$ is a strict partition, then the partition obtained from a by adding one grain on its i-th column is denoted by $a^{\downarrow_i}$. Notice that $a^{\downarrow_i}$ is not necessarily a strict partition. If S is a set of strict partitions, then $S^{\downarrow_i}$ denotes the set $\{a^{\downarrow_i} | a \in S\}$. We denote $a \xrightarrow{i} b$ if b is obtained from a by applying a transition at position i and by $Succ(a)$ the set of configurations directly reachable from a.

Write $d_i(a) = a_i - a_{i+1}$ with the convention that $a_{k+1} = 0$. We say that a has a *cliff* at position i if $d_i(a) \geq 3$. If there exists an $\ell > i$ such that $d_j(a) = 1$ for all $i \leq j < \ell$ and $d_\ell(a) = 2$, then we say that a has a *slippery plateau* at i with *length* $(\ell - i)$. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = 1$ for all $i \leq j < \ell$ and it has a cliff at ℓ. The integer $\ell - i$ is called the *length* of the non-slippery plateau at i. The partition a has a *(non)-slippery step* at i if there is a strict partition b such that $b^{\downarrow_i} = a$ and b has a (non)-slippery plateau at i. See Figure 3 for some illustrations. The set of elements of $\mathcal{SP}(n)$ that begin

FIGURE 3. From left to right: a cliff, a slippery step, a non-slippery step, a slippery plateau and a non-slippery plateau.

with a cliff, a slippery step, a non-slippery step, a slippery plateau of length l and a non-slippery plateau of length l are denoted by C, SS, nSS, SP_l, nSP_l respectively.

4.2. Constructing $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$. Let $a = (a_1, a_2, \ldots, a_k)$ be a strict partition. It is clear that $a^{\downarrow_1}$ is again a strict partition. This define an embedding $\pi : \mathcal{SP}(n) \to \mathcal{SP}(n)^{\downarrow_1} \subset \mathcal{SP}(n+1)$ which can be proved, by using infimum formula of $\mathcal{SP}(n)$ and $\mathcal{SP}(n+1)$, as a lattice map.

PROPOSITION 4.1. *$\mathcal{SP}(n)^{\downarrow_1}$ is a sublattice of $\mathcal{SP}(n+1)$.*

Our next result characterizes the remaining elements of $\mathcal{SP}(n+1)$ that are not in $\mathcal{SP}(n)^{\downarrow_1}$.

THEOREM 4.2. *For all $n \geq 1$, we have*

$$\mathcal{SP}(n+1) = \mathcal{SP}(n)^{\downarrow_1} \sqcup SS^{\downarrow_2} \sqcup nSS^{\downarrow_2} \ \sqcup_l SP_l^{\downarrow_{l+1}}.$$

PROOF. It is easy to check that each element in one of the sets $\mathcal{SP}(n)^{\downarrow_1}$, $SS^{\downarrow_2}$, $nSS^{\downarrow_2}$ and $SP_l^{\downarrow_{l+1}}$ is an element of $\mathcal{SP}(n+1)$, and that these sets are disjoint.

Now let us consider an element b of $\mathcal{SP}(n+1)$. If b begins with a cliff or a step then b is in $\mathcal{SP}(n)^{\downarrow_1}$. If b begins with a slippery plateau of length 2 then b is in $SS^{\downarrow_2}$, if b begins with a non-slippery of length 2 then b is in $nSS^{\downarrow_2}$. And if b begins with a plateau of length $l+1, l \geq 2$, then b is in $SP_l^{\downarrow_{l+1}}$. □

Finally, we describe an algorithm to compute the successors of any given element of $\mathcal{SP}(n+1)$, thus giving a complete construction of $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$.

PROPOSITION 4.3. *Let x be an element of $\mathcal{SP}(n+1)$.*

(1) *Suppose $x = a^{\downarrow_1} \in \mathcal{SP}(n)^{\downarrow_1}$.*
  - *If a is in C or nSP then $Succ(a^{\downarrow_1}) = Succ(a)^{\downarrow_1}$,*
  - *If a is in SP_l then $Succ(a^{\downarrow_1}) = Succ(a)^{\downarrow_1} \cup \{a^{\downarrow_{l+1}}\}$,*
  - *If a is in SS then let b be such that $a \xrightarrow{1} b$. We have $Succ(a^{\downarrow_1}) = (Succ(a) \setminus \{b\})^{\downarrow_1} \cup \{a^{\downarrow_2}\}$.*

(2) *If $x = a^{\downarrow_2} \in SS^{\downarrow_2}$ where $a \in SS$: Let b be such that $a \xrightarrow{1} b$, then $Succ(a^{\downarrow_2}) = (Succ(a) \setminus \{b\})^{\downarrow_2} \cup \{b^{\downarrow_1}\}$.*

(3) *If $x = a^{\downarrow_2} \in nSS^{\downarrow_2}$ with $a \in nSS$, then $Succ(a^{\downarrow_2}) = Succ(a)^{\downarrow_2}$.*

(4) *Finally, if $x = a^{\downarrow_{l+1}} \in SP_l^{\downarrow_{l+1}}$ for some $a \in SP_l$, then*
  - *If a has a cliff at $l+1$ or a non-slippery step at l, then $Succ(a^{\downarrow_{l+1}}) = Succ(a)^{\downarrow_{l+1}}$,*
  - *If a has a slippery step at l, let b such that $a \xrightarrow{l} b$ in $\mathcal{SP}(n)$, then $Succ(a^{\downarrow_{l+1}}) = (Succ(a) \setminus \{b\})^{\downarrow_{l+1}} \cup \ \{b^{\downarrow_l}\}$.*

PROOF. We will give the proof for the two most difficult cases (1) and (4). Consider $x = a^{\downarrow_1}$ where $a \in C$: notice first that the transitions possible from a on columns other than the first one are still possible from $a^{\downarrow_1}$, and on the other hand the addition of one grain on a cliff does not allow any new transition from the first column, since such a transition was already possible.

In the last case: $x = a^{\downarrow_{l+1}}$ where $a \in SP_l^{\downarrow_{l+1}}$ and a has a slippery step of length l' at l. Then, $a \xrightarrow{l} b$ in $\mathcal{SP}(n)$. The possible transitions from $a^{\downarrow_{l+1}}$ are the same as the possible ones from a, except the transition on the column l. All the elements directly reachable from a except b have a slippery plateau at 1, therefore the elements of $(Succ(a) \setminus \{b\})^{\downarrow_{l+1}} \in Succ(a^{\downarrow_{l+1}})$ The only one missing transition is: $a^{\downarrow_{l+1}} \xrightarrow{l+1} a^{\downarrow_{l+l'+1}}$. But we can verify that $a^{\downarrow_{l+l'+1}} = b^{\downarrow_l}$. □

Proposition 4.3 makes it possible to write an algorithm to construct the lattice $\mathcal{SP}(n+1)$ in linear time (with respect to its size).

4.3. The infinite lattice $\mathcal{SP}(\infty)$. Imagine that (∞) is the initial configuration where the first column contains infinitely many grains and all the other columns contain no grain. Then the transitions V and H defined in the first section can be performed on (∞) just as if it is finite, and we call $\mathcal{SP}(\infty)$ as the set of all the configurations reachable from (∞). A typical element a of $\mathcal{SP}(\infty)$ has the form $(\infty, a_2, a_3, \ldots, a_k)$. As in the previous section, we find that the dominance ordering on $\mathcal{SP}(\infty)$ (when the first component is ignored) is equivalent to the order induced by the dynamical model. The first partitions in $\mathcal{SP}(\infty)$ are given in Figure 4, along with their covering relations (the first component, equal to ∞, is not represented on this diagram).

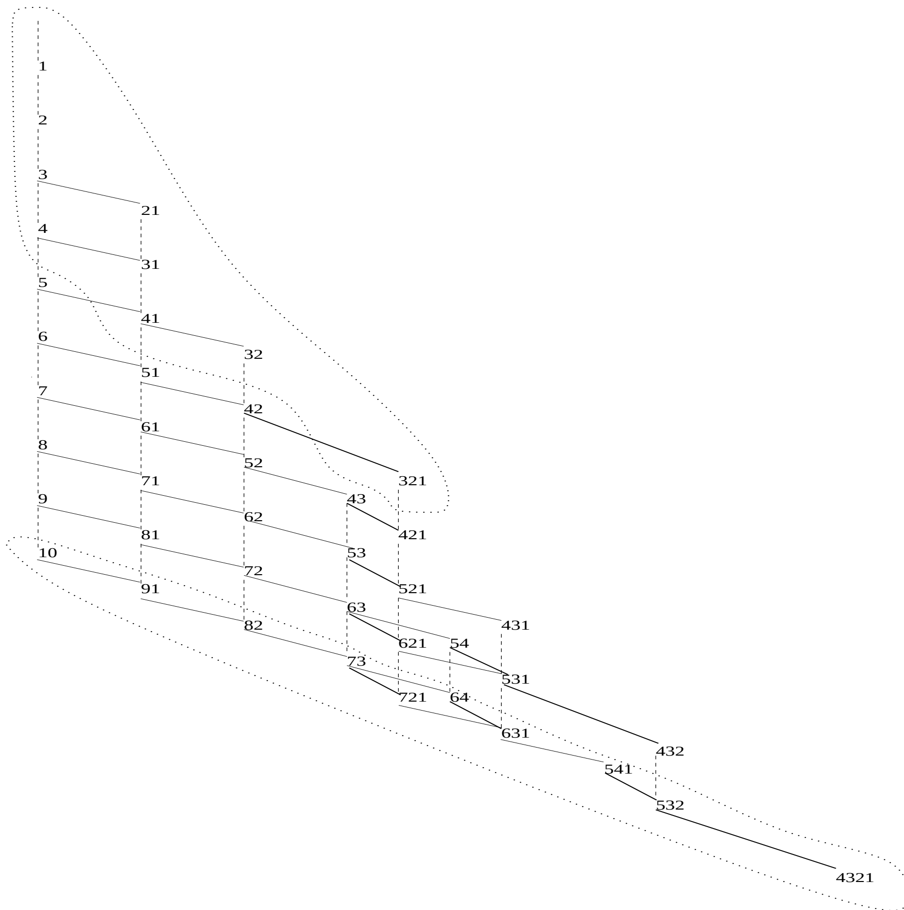

FIGURE 4. The first elements and transitions of $\mathcal{SP}(\infty)$. As shown on this figure for $n = 10$, we will see two ways to find parts of $\mathcal{SP}(\infty)$ isomorphic to $\mathcal{SP}(n)$ for any n.

We start by showing that $\mathcal{SP}(\infty)$ is a lattice. We also obtain a formula for the minimum in $\mathcal{SP}(\infty)$. Furthermore, for any n, there are two different ways to find sublattices of $\mathcal{SP}(\infty)$ isomorphic to $\mathcal{SP}(n)$. We will also give a way to compute some other special sublattices of $\mathcal{SP}(\infty)$, using its self-similarity.

THEOREM 4.4. *The set $\mathcal{SP}(\infty)$ is a lattice. Moreover, for any two elements $a = (\infty, a_2, \ldots, a_k)$ and $b = (\infty, b_2, \ldots, b_\ell)$ of $\mathcal{SP}(\infty)$, then $\inf_{\mathcal{SP}(\infty)}(a, b) = c$ in $\mathcal{SP}(\infty)$, where c is defined by:*

$$c_i = max(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) - \sum_{j > i} c_j \quad \textit{for all } i \textit{ such that } 2 \leq i \leq max(k, l).$$

PROOF. One just needs to check that c is an element of $\mathcal{SP}(\infty)$, *i.e.* $c_1 = \infty$ and $c_i > c_{i+1}$ for all $i > 1$. □

Now for any $n > 1$, there are two canonical embeddings of $\mathcal{SP}(n)$ in $\mathcal{SP}(\infty)$, defined by

$$\begin{array}{cccc} \pi : & \mathcal{SP}(n) & \longrightarrow & \mathcal{SP}(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \mapsto & \pi(a) = (\infty, a_2, \ldots, a_k) \end{array}$$

$$\begin{array}{cccc} \chi : & \mathcal{SP}(n) & \longrightarrow & \mathcal{SP}(\infty) \\ & a = (a_1, a_2, \dots, a_k) & \mapsto & \chi(a) = (\infty, a_1, a_2, \dots, a_k) \end{array}$$

THEOREM 4.5. *Both π and χ are embedding of lattices.*

PROOF. The first embedding π comes from our result in the Proposition 1. The second is clear by noting that for all $a, b \in \mathcal{SP}(n)$, $a \xrightarrow{l} b$ if and only if $\chi(a) \xrightarrow{l+1} \chi(b)$ in $\mathcal{SP}(\infty)$. □

COROLLARY 4.6. *Let*

$$\mathcal{SP}(\leq n) = \bigsqcup_{0 \leq i \leq n} \mathcal{SP}(i),$$

then $\mathcal{SP}(\leq n)$ is a sublattice of $\mathcal{SP}(\infty)$ (by the embedding χ).

So by using the embedding χ, one can consider $\mathcal{SP}(\infty)$ as the union disjoint of $\mathcal{SP}(n)$ for all n, $\mathcal{SP}(\infty) = \bigsqcup_{n \geq 0} \mathcal{SP}(n)$.

4.4. Self-reference property: the infinite binary tree $T_B(\infty)$. Observe that each element a of $\mathcal{SP}(n+1)$ can be obtained from an element b of $\mathcal{SP}(n)$ by addition of one grain at some position i; that is $a = b^{\downarrow_i}$. We will represent this relation by a tree where $a \in \mathcal{SP}(n+1)$ is a child of $b \in \mathcal{SP}(n)$ if and only if $a = b^{\downarrow_i}$ for some $i \geq 0$, and we label the edge $b \longrightarrow a$ by i. We denote this tree by $\mathcal{ST}(\infty)$. The root of $\mathcal{ST}(\infty)$ is the empty partition. We will describe two ways to compute all strict partitions of a given positive integer n in $\mathcal{ST}(\infty)$. As an application, we derive an efficient and simple algorithm to compute them. Moreover, this tree has a special property which we called 'self-reference' from which we can deduce a recursive formula for the cardinality of $\mathcal{SP}(n)$ and some special classes of strict partitions.

First of all, it is easy to see from the construction of $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$ that the each node $a \in \bigsqcup_{n \geq 0} \mathcal{SP}(n)$ has at least one child, which is $a^{\downarrow_1}$. Furthermore, if a begins with a slippery plateau of length l, then it has another child which is the element $a^{\downarrow_{l+1}}$. It follows that $\mathcal{ST}(\infty)$ is a binary tree. We will call *left child* the first of two children, and *right child* the other (if it exists). We call the level n of the tree the set of elements of depth n. The first levels of $\mathcal{ST}(\infty)$ are shown in Figure 5.

By using the embedding χ and π in Theorem 4.5, we have:

PROPOSITION 4.7. *The level n of $\mathcal{ST}(\infty)$ is exactly the set of the elements of $\mathcal{SP}(n)$. Moreover the set $\mathcal{SP}(n)$ is in a bijection with a subtree of $\mathcal{ST}(\infty)$ having the same root.*

We will now give a recursive description of $ST(\infty)$. This will allow us to obtain a new recursive formula to calculate the cardinality of $\mathcal{SP}(n)$, as well as for some special classes of strict partitions. We first define a certain kind of subtrees of $ST(\infty)$. Afterward, we show how the whole structure of $\mathcal{ST}(\infty)$ can be described in terms of such subtrees.

We call X_k *subtree* any left subtree of an element beginning with a slippery plateau of length k. Moreover, we define X_0 as a simple node.
The next proposition shows that all the X_k subtrees are isomorphic (see Figure 6).

PROPOSITION 4.8. *A X_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes (the rightmost chain) whose edges are labeled $1, 2, \dots, k$ and whose i-th node having an out going edge labeled with 1 to a X_i subtree for all i between 1 and k.*

This recursive structure and the above propositions allows us to give a compact representation of the tree $\mathcal{ST}(\infty)$ by a chain (see Figure 7).

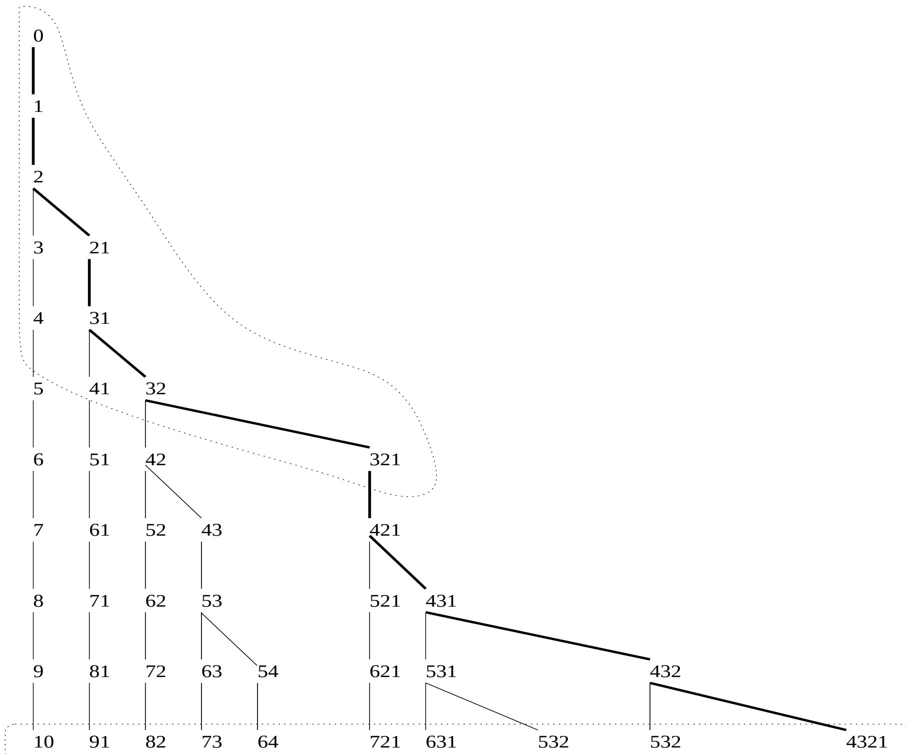

FIGURE 5. The first levels of the tree $\mathcal{ST}(\infty)$ (to clarify the picture, the labels are omitted). As shown on this figure for $n = 10$, we will see two ways to find the elements of $\mathcal{SP}(n)$ in $\mathcal{ST}(\infty)$ for any n.

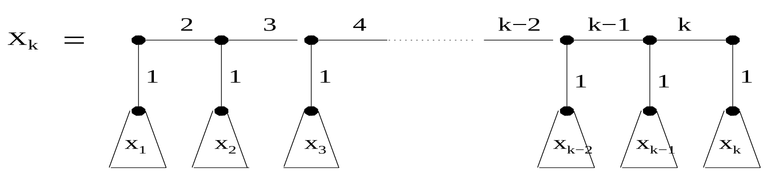

FIGURE 6. Self-referencing structure of X_k subtrees.

THEOREM 4.9. *The tree $\mathcal{ST}(\infty)$ can be represented by the infinite chain* $(), 1, 2, 21, 31, 32, 321, \ldots, (n-1, n-2, \ldots, 1), (n, n-2, \ldots, 2, 1), \ldots,$ $(n, n-1, \ldots, 3, 2), (n, n-1, \ldots, 3, 2, 1), \ldots$ *with corresponding edges* $1, 1, 2, 1, 2, 3, \ldots, 1, 2, \ldots, n, \ldots$*; each node before an edge k having an out going edge labeled with 1 to the root of a X_{k-1} subtree.*

Moreover, we can prove a stronger property of each subtree in this chain:

THEOREM 4.10. *The subtree (of the form $(k, k-1, \ldots 2, 1) \xrightarrow{1} X_k$) of $\mathcal{ST}(\infty)$ contains exactly the partitions of length k.*

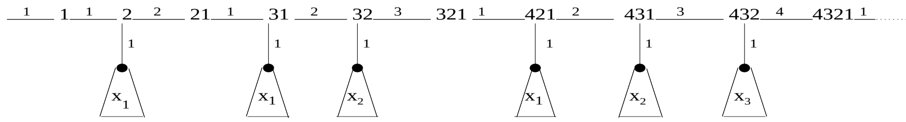

FIGURE 7. Representation of $\mathcal{ST}(\infty)$ as a chain.

Different to the case of infinite tree of partitions, the distance of this root to the root of $\mathcal{ST}(\infty)$ is equal to $\frac{k(k-1)}{2}$. We can now state our last result:

THEOREM 4.11. *Let $c(l,k)$ denote the number of paths in a X_k tree originating from the root and having length l. We have:*

$$c(l,k) = \begin{cases} 1 & \text{if } l = 0 \text{ or } k = 1 \\ \sum_{i=1}^{inf(l,k)} c(l-i,i) & \text{otherwise} \end{cases}$$

Moreover, $|\mathcal{SP}(n)| = \sum_{0 \leq k \leq \sqrt{(2n)}+1} c(n - \frac{k(k-1}{2}, k)$ and the number of partitions of n with length exactly k is $c(n - \frac{k(k-1)}{2}, k)$.

□

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