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q, t-Kostka Polynomials and the Affine Symmetric Group

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Abstract. The k-Young lattice Y^k is a partial order on partitions with no part larger than k that originated [**LLM**] from the study of k-Schur functions $s_\lambda^{(k)}$, symmetric functions that form a natural basis of the space spanned by homogeneous functions indexed by k-bounded partitions. The chains in the k-Young lattice are induced by a Pieri-type rule experimentally satisfied by the k-Schur functions. Here, using a natural bijection between k-bounded partitions and $k+1$-cores, we can identify chains in the k-Young lattice with certain tableaux on $k+1$ cores. This identification reveals that the k-Young lattice is isomorphic to the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup. From this, the conjectured k-Pieri rule implies that the k-Kostka matrix connecting the homogeneous basis $\{h_\lambda\}_{\lambda\in Y^k}$ to $\{s_\lambda^{(k)}\}_{\lambda\in Y^k}$ may now be obtained by counting appropriate classes of tableaux on $k+1$-cores. This suggests that the conjecturally positive k-Schur expansion coefficients for Macdonald polynomials (reducing to q,t-Kostka polynomials for large k) could be described by a q,t-statistic on these tableaux, or equivalently on reduced words for affine permutations.

Résumé. Un ordre partiel Y^k sur les partitions dont les parties ne dépassent pas un certain entier positif k tire son origine de l'étude de fonctions de Schur généralisées [**LLM**], fonctions symétriques formant une base de l'espace engendré par les fonctions homogènes indicées par des partitions k-bornées. Les chaînes dans le treillis Y^k sont induites par une règle du type Pieri que satisfont experimentalement les fonctions de k-Schur. En utilisant une bijection naturelle entre les partitions k-bornées et les $k+1$-cores, nous obtenons une correspondance entre les chaînes dans le treillis Y^k et certains remplissages de $k+1$-cores. Cette correspondance révèle que le treillis Y^k est isomorphique à l'ordre faible du groupe symétrique affine $\tilde{S}_{k+1}$ modulo un sous-groupe parabolique maximal. La règle de Pieri experimentale implique ainsi que la matrice de k-Kostka connectant les bases $\{h_\lambda\}_{\lambda\in Y^k}$ et $\{s_\lambda^{(k)}\}_{\lambda\in Y^k}$ peut être obtenue en énumérant certaines classes de tableaux sur les $k+1$-cores, et suggère entre autres que les coefficients de développements, que nous conjecturons positifs, des polynômes de Macdonald en termes de fonctions de k-Schur (se reduisant aux polynômes de q,t-Kostka lorsque k est grand) pourraient être décrits par une q,t-statistique sur ces tableaux, ou de façon équivalente, par une q,t-statistique sur les décompositions réduites de certaines permutations affines.

1. Introduction

1.1. The k-Young lattice. Recall that λ is a successor of a partition μ in the Young lattice when λ is obtained by adding an addable corner to μ where partitions are identified by their Ferrers diagrams. This

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relation, which we denote "$\mu \to \lambda$", occurs naturally in the classical Pieri rule

$$h_1[X]\, s_\mu[X] \,=\, \sum_{\lambda:\, \mu\to\lambda} s_\lambda[X]\,, \tag{1.1}$$

and the partial order of the Young lattice may be defined as the transitive closure of $\mu \to \lambda$. It was experimentally observed that the k-Schur functions [**LLM, LM1**] satisfy the rule

$$h_1[X]\, s_\mu^{(k)}[X] \,=\, \sum_{\lambda:\, \mu\to_k\lambda} s_\lambda^{(k)}[X]\,, \tag{1.2}$$

where "$\mu \to_k \lambda$" is a certain subrelation of "$\mu \to \lambda$". This given, the partial order of the k-Young lattice Y^k is defined as the transitive closure of $\mu \to_k \lambda$.

The precise definition of the relation $\mu \to_k \lambda$ stems from another "Schur" property of k-Schur functions. Computational evidence suggests that the usual ω-involution for symmetric functions acts on k-Schur functions according to the formula

$$\omega\, s_\mu^{(k)}[X] \,=\, s_{\mu^{\omega_k}}^{(k)}[X]\,, \tag{1.3}$$

where the map $\mu \mapsto \mu^{\omega_k}$ is an involution on k-bounded partitions called "*k-conjugation*" generalizing partition conjugation $\mu \mapsto \mu'$. Then viewing the covering relations on the Young lattice as

$$\mu \to \lambda \iff |\lambda| = |\mu| + 1 \quad \& \quad \mu \subseteq \lambda \quad \& \quad \mu' \subseteq \lambda'\,, \tag{1.4}$$

we accordingly, in our previous work [**LLM**], defined $\mu \to_k \lambda$ in terms of the involution $\mu \mapsto \mu^{\omega_k}$ by

$$\mu \to_k \lambda \iff |\lambda| = |\mu| + 1 \quad \& \quad \mu \subseteq \lambda \quad \& \quad \mu^{\omega_k} \subseteq \lambda^{\omega_k}\,. \tag{1.5}$$

Thus only certain addable corners may be added to a partition μ to obtain its successors in the k-Young lattice. We shall call such corners the "*k-addable corners*" of μ.

The precise determination of k-addable corners relies on a bijection between k-bounded partitions and the set of $k+1$-cores (partitions with no $k+1$-hooks). For any $k+1$-core γ, we define

$$\mathfrak{p}(\gamma) = (\lambda_1, \ldots, \lambda_\ell)$$

where λ_i is the number of cells with k-bounded hook length in row i of γ. It turns out that $\mathfrak{p}(\gamma)$ is a k-bounded partition and that the correspondence $\gamma \mapsto \mathfrak{p}(\gamma)$ bijectively maps $k+1$-cores onto k-bounded partitions. With $\lambda \mapsto \mathfrak{c}(\lambda)$ denoting the inverse of $\mathfrak{p}$, we define the k-conjugation of a k-bounded partition λ to be

$$\lambda^{\omega_k} \,=\, \mathfrak{p}\big(\mathfrak{c}(\lambda)'\big)\,. \tag{1.6}$$

That is, if γ is the $k+1$-core corresponding to λ, then λ^{ω_k} is the partition whose row lengths equal the number of k-bounded hooks in corresponding rows of γ'. This reveals that k-conjugation, which originally emerged from the action of the ω involution on k-Schur functions, is none other than the $\mathfrak{p}$-image of ordinary conjugation of $k+1$-cores.

The $\mathfrak{p}$-bijection then leads us to a characterization for k-addable corners that determine successors in the k-Young lattice. By labeling every square (i,j) in the i^{th} row and j^{th} column by its "*$k+1$-residue*", $j-i \mod k+1$, we find

(Theorem 4.1) *Let c be any addable corner of a k-bounded partition λ and c' (of $k+1$-residue i) be the addable corner of $\mathfrak{c}(\lambda)$ in the same row as c. c is k-addable if and only if c' is the highest addable corner of $\mathfrak{c}(\lambda)$ with $k+1$-residue i.*

This characterization of k-addability leads us to a notion of standard k-tableaux which we prove are in bijection with saturated chains in the k-Young lattice.

(**Definition 4.6)** *Let γ be a $k+1$-core and m be the number of k-bounded hooks of γ. A standard k-tableau of shape γ is a filling of the cells of γ with the letters $1, 2, \ldots, m$ which is strictly increasing in rows and columns and such that the cells filled with the same letter have the same $k+1$-residue.*

(Theorem 4.9) *The saturated chains in the k-Young lattice joining the empty partition $\emptyset$ to a given k-bounded partition λ are in bijection with the standard k-tableaux of shape $\mathfrak{c}(\lambda)$.*

We then consider the affine symmetric group $\tilde{S}_{k+1}$ modulo a maximal parabolic subgroup denoted by S_{k+1}. Bruhat order on the minimal coset representatives of $\tilde{S}_{k+1}/S_{k+1}$ can be defined in terms of simple containment of $k+1$-core diagrams (this connection is stated precisely by Lascoux in [**L**] and is equivalent to results in [**MM, BB**]). From this, stronger relations among $k+1$-core diagrams can be used to describe the weak order on such coset representatives. We are thus able to prove that our new characterization of the k-Young lattice chains implies an isomorphism between the k-Young lattice and the weak order on these coset representatives. Consequently, a bijection between the set of k-tableaux of a given shape $\mathfrak{c}(\lambda)$ and the set of reduced decompositions for a certain affine permutation $\sigma_\lambda \in \tilde{S}_{k+1}/S_{k+1}$ can be achieved by mapping:

$$\mathfrak{w} : T \mapsto s_{i_\ell} \cdots s_{i_2} s_{i_1} \,, \tag{1.7}$$

where i_a is the $k+1$-residue of letter a in the standard k-tableau T. A by-product of this result is a simple bijection between k-bounded partitions and affine permutations in $\tilde{S}_{k+1}/S_{k+1}$:

$$\phi : \lambda \mapsto \sigma_\lambda \,, \tag{1.8}$$

where σ_λ corresponds to the reduced decomposition obtained by reading the $k+1$-residues of λ from right to left and from top to bottom. It is shown in [**LMW**] that this bijection, although algorithmically distinct, is equivalent to the one presented by Björner and Brenti [**BB**] using a notion of inversions on affine permutations. It follows from our results that Eq. (1.2) reduces simply to

$$h_1[X]\, s^{(k)}_{\phi^{-1}(\sigma)}[X] = \sum_{\sigma \lessdot_w \tau} s^{(k)}_{\phi^{-1}(\tau)}[X] \,, \tag{1.9}$$

where the sum is over all permutations that cover σ in the weak order on $\tilde{S}_{k+1}/S_{k+1}$.

As will be detailed in § 1.2, Theorem 4.9 also plays a role in the theory of Macdonald polynomials and the study of k-Schur functions, thus motivating a semi-standard extension of Definition 4.6:

(Definition 6.1) *Let m be the number of k-bounded hooks in a $k+1$-core γ and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of m. A semi-standard k-tableau of shape γ and evaluation α is a column strict filling of γ with the letters $1, 2, \ldots, r$ such that the collection of cells filled with letter i is labeled with exactly α_i distinct $k+1$-residues.*

As with the ordinary semi-standard tableaux, we show that there are no semi-standard k-tableau under conditions relating to dominance order on the shape and evaluation. An analogue of Theorem 4.9 can then be used to show that this coincides with unitriangularity of coefficients in the k-Schur expansion of homogeneous symmetric functions and suggests that the k-tableaux should have statistics to combinatorially describe the k-Schur function expansion of the Hall-Littlewood polynomials. The analogue of Theorem 4.9 relies on the following definition: with the pair of k-bounded partitions λ, μ defined to be "*r-admissible* " if and only if λ/μ and $\lambda^{\omega_k}/\mu^{\omega_k}$ are respectively horizontal and vertical r-strips, we say a sequence of partitions

$$\emptyset = \lambda^{(0)} \longrightarrow \lambda^{(1)} \longrightarrow \lambda^{(2)} \longrightarrow \cdots \longrightarrow \lambda^{(\ell)}$$

is α-admissible when $\lambda^{(i)}, \lambda^{(i-1)}$ is a α_i-admissible pair for $i = 1, \ldots, \ell$. It turns out that all α-admissible sequences are in fact chains in the k-Young lattice and that Theorem 4.9 extends to:

(Theorem 6.6) *Let m be the number of k-bounded hooks in a $k+1$-core γ and let α be a composition of m. The collection of α-admissible chains joining $\emptyset$ to $\mathfrak{p}(\gamma)$ is in bijection with the semi-standard k-tableaux of shape γ and evaluation α.*

As mentioned, the root of our work lies in the study of symmetric functions. We conclude our introduction with a summary of these ideas.

1.2. Macdonald expansion coefficients. The k-Young lattice emerged from the experimental Pieri rule (1.2) satisfied by k-Schur functions. In turn, k-Schur functions have arisen from a close study of Macdonald polynomials. To appreciate the role of our findings in the theory of Macdonald polynomials we shall briefly review this connection. To begin, we consider a modification of the Macdonald integral form [**M**] $J_\lambda[X;q,t]$ obtained by plethystic substitution:

$$H_\mu[X;q,t\,] \;=\; J_\mu\big[\tfrac{X}{1-q};q,t\,\big] \;=\; \sum_{\lambda\vdash n} K_{\lambda\mu}(q,t) s_\lambda[X]\,, \tag{1.10}$$

where $K_{\lambda\mu}(q,t)\in\mathbb{N}[q,t]$ are known as the q,t-Kostka polynomials. Formula (1.10), when $q=t=1$, reduces to

$$h_1^n \;=\; \sum_{\lambda\vdash n} f_\lambda\, s_\lambda[X]\,, \tag{1.11}$$

where f_λ is the number of standard tableaux of shape λ. This given, one of the outstanding problems in algebraic combinatorics is to associate a pair of statistics $a_\mu(T), b_\mu(T)$ on standard tableaux to the partition μ so that

$$K_{\lambda\mu}(q,t) \;=\; \sum_{T\in ST(\lambda)} q^{a_\mu(T)} t^{b_\mu(T)}\,, \tag{1.12}$$

where "$ST(\lambda)$" denotes the collection of standard tableaux of shape λ.

In previous work [**LLM, LM1**], we proposed a new approach to the study of the q,t-Kostka polynomials. This approach is based on the discovery of a certain family of symmetric functions $\{s_\lambda^{(k)}[X;t]\}_{\lambda\in Y^k}$ for each integer $k\geq 1$, which we have shown [**LM1**] to be a basis for the space $\Lambda_t^{(k)}$ spanned by the Macdonald polynomials $H_\mu[X;q,t\,]$ indexed by k-bounded partitions. This revealed a mechanism underlying the structure of the coefficients $K_{\lambda\mu}(q,t)$. To be precise, for $\mu,\nu\in Y^k$, consider

$$H_\mu[X;q,t\,] \;=\; \sum_{\nu\in Y^k} K_{\nu\mu}^{(k)}(q,t)\, s_\nu^{(k)}[X;t\,]\ , \ \text{and}\ \ s_\nu^{(k)}[X;t\,] \;=\; \sum_\lambda \pi_{\lambda\nu}(t)\, s_\lambda[X]\,. \tag{1.13}$$

We then we have the factorization

$$K_{\lambda\mu}(q,t) \;=\; \sum_{\nu\in Y^k} \pi_{\lambda\nu}(t)\, K_{\nu\mu}^{(k)}(q,t)\,. \tag{1.14}$$

It was experimentally observed (proven for $k=2$ in [**LM0, LM1**]) that $K_{\nu\mu}^{(k)}(q,t)\in\mathbb{N}[q,t]$ and $\pi_{\lambda\nu}(t)\in\mathbb{N}[t]$. This suggests that the problem of finding statistics for $K_{\lambda\mu}(q,t)$ may be decomposed into two analogous problems for $K_{\nu\mu}^{(k)}(q,t)$ and $\pi_{\lambda\nu}(t)$. We also have experimental evidence to support that $K_{\lambda\mu}(q,t)-K_{\nu\mu}^{(k)}(q,t)\in\mathbb{N}[q,t]$ which implies that $s_\lambda^{(k)}[X;t\,]$-expansions are formally simpler.

These developments prompted a close study of the polynomials $s_\lambda^{(k)}[X;1\,]=s_\lambda^{(k)}[X]$. In addition to (1.2), it was also conjectured that these polynomials satisfy the more general rule

$$h_r[X]\, s_\mu^{(k)}[X] \;=\; \sum_{\substack{\lambda/\mu=\text{horizontal } r\text{-strip}\\ \lambda^{\omega_k}/\mu^{\omega_k}=\text{vertical } r\text{-strip}}} s_\lambda^{(k)}[X]\,. \tag{1.15}$$

Iteration of (1.2) starting from $s_\emptyset[X]=1$ yields

$$h_1^n[X] \;=\; \sum_{\lambda\in Y^k} K_{\lambda,1^n}^{(k)}\, s_\lambda^{(k)}[X]\,, \tag{1.16}$$

while iterating (1.15) for suitable choices of r gives the k-Schur function expansion of an h-basis element indexed by any k-bounded partition μ . That is,

$$h_\mu[X] = \sum_{\lambda\in Y^k} K^{(k)}_{\lambda\mu}\, s^{(k)}_\lambda[X] \tag{1.17}$$

Since $s^{(k)}_\lambda[X] = s_\lambda[X]$ when all the hooks of λ are k-bounded, we see that (1.16) reduces to (1.11) for a sufficiently large k. Similarly, the coefficient $K^{(k)}_{\lambda\mu}$ in (1.17) reduces to the classical Kostka number $K_{\lambda\mu}$ when k is large. Our definition of the k-Young lattice and its admissible chains, combined with the experimental Pieri rules (1.2) and (1.15), yield the following corollary of Theorems 4.9 and 6.6:

On the validity of (1.15)*,* $K^{(k)}_{\lambda,1^n}$ *equals the number of standard k-tableaux of shape* $\mathfrak{c}(\lambda)$*, or equivalently the number of reduced expressions for* σ_λ*, and the coefficient* $K^{(k)}_{\lambda\mu}$ *equals the number of semi-standard k-tableaux of shape* $\mathfrak{c}(\lambda)$ *and evaluation* μ*.*

Since (1.13) reduces to (1.16) when $q = t = 1$, this suggests that the positivity of $K^{(k)}_{\lambda\mu}(q,t)$ may be accounted for by q,t-counting standard k-tableaux of shape $\mathfrak{c}(\lambda)$, or reduced words of σ_λ, according to a suitable statistic depending on μ. More precisely, for $\mathcal{T}^k(\lambda)$ the set of k-tableaux of shape $\mathfrak{c}(\lambda)$ and $Red(\sigma)$ the reduced words for σ,

$$H_\mu[X;q,t] = \sum_{\lambda:\lambda_1\le k}\left(\sum_{T\in\mathcal{T}^k(\lambda)} q^{a_\mu(T)}\, t^{b_\mu(T)}\right) s^{(k)}_\lambda[X;t] \tag{1.18}$$

$$= \sum_{\sigma\in\tilde S_{k+1}/S_{k+1}}\left(\sum_{w\in Red(\sigma)} q^{a_{\sigma_\mu}(w)}\, t^{b_{\sigma_\mu}(w)}\right) s^{(k)}_{\phi^{-1}(\sigma)}[X;t]\,. \tag{1.19}$$

We should also mention that the relation in (1.17) was proven to be unitriangular [**LM1**] with respect to the dominance partial order "$\trianglerighteq$" as well as the t-analog of this relation, given by the Hall-Littlewood polynomials corresponding to the specialization $q = 0$ of the Macdonald polynomials:

$$H_\mu[X;0,t] = \sum_{\substack{\lambda\in Y^k\\ \lambda\trianglerighteq\mu}} K^{(k)}_{\lambda\mu}(t)\, s^{(k)}_\lambda[X;t]\,. \tag{1.20}$$

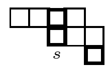
The conjecture that $K^{(k)}_{\lambda\mu}(q,t)\in\mathbb{N}[q,t]$ implies $K^{(k)}_{\lambda\mu}(t)$ would also have positive integer coefficients. Our work here then suggests that this positivity may be accounted for by defining the coefficients in terms of a k-charge statistic on semi-standard k-tableaux.

2. Definitions

A partition $\lambda = (\lambda_1,\ldots,\lambda_m)$ is a non-increasing sequence of positive integers with *"degree"* $|\lambda| = \lambda_1+\cdots+\lambda_m$ and *"length"* $\ell(\lambda) = m$. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top, and a *"conjugate"* diagram λ' obtained by reflecting λ about the diagonal. λ is "*k-bounded*" if $\lambda_1 \le k$. Any lattice square (i,j) in the ith row and jth column of a Ferrers diagram is called a cell. We say that $\lambda\subseteq\mu$ when $\lambda_i\le\mu_i$ for all i. The *"dominance order"* $\trianglerighteq$ on partitions is defined by $\lambda\trianglerighteq\mu$ when $\lambda_1+\cdots+\lambda_i \ge \mu_1+\cdots+\mu_i$ for all i, and $|\lambda| = |\mu|$. A "*removable*" corner of partition γ is a cell $(i,j)\in\gamma$ with $(i,j+1),(i+1,j)\notin\gamma$ and an "*addable*" corner is a square $(i,j)\notin\gamma$ with $(i,j-1),(i-1,j)\in\gamma$.

More generally, for $\rho\subseteq\gamma$, the skew shape γ/ρ is identified with its diagram $\{(i,j):\rho_i<j\le\gamma_i\}$. The degree of a skew shape is the number of cells in its diagram. Lattice squares that do not lie in γ/ρ will be simply called "*squares*". We shall say that any $c\in\rho$ lies "*below* " γ/ρ. The "*hook* " of any lattice square

$s \in \gamma$ is defined as the collection of cells of γ/ρ that lie inside the L with s as its corner. This is intended to apply to all $s \in \gamma$ including those below γ/ρ. In the example below the hook of $s = (1, 3)$ is depicted by the framed cells

(2.1)  .

We let $h_s(\gamma/\rho)$ denote the number of cells in the hook of s and say that the hook of a cell or a square is k-bounded if its length is not larger than k. We are particularly interested in *"$k+1$-cores*, partition that have no $k+1$-hooks (e.g.[**JK**]). The "$k+1$-*residue*" of square (i, j) is $j - i \mod k + 1$.

A *"composition"* α of an integer m is a vector of positive integers that sum to m. A "*tableau*" T of shape λ is a filling of T with integers that is weakly increasing in rows and strictly increasing in columns. The "*evaluation*" of T is given by a composition α where α_i is the multiplicity of i in T.

The affine symmetric group $\tilde{S}_{k+1}$ is generated by the $k+1$ elements $\hat{s}_0, \ldots, \hat{s}_k$ satisfying the affine Coxeter relations:

$$\hat{s}_i^2 = id, \qquad \hat{s}_i\hat{s}_j = \hat{s}_j\hat{s}_i \quad (i - j \neq \pm 1 \mod k+1), \quad \text{and} \quad \hat{s}_i\hat{s}_{i+1}\hat{s}_i = \hat{s}_{i+1}\hat{s}_i\hat{s}_{i+1}\,. \tag{2.2}$$

Note that $\hat{s}_i$ is understood as $\hat{s}_{i \bmod k+1}$ if $i \geq k+1$. Elements of $\tilde{S}_{k+1}$ are called affine permutations. A word $w = i_1 i_2 \cdots i_m$ in the alphabet $\{0, 1, \ldots, k\}$ corresponds to the permutation $\sigma \in \tilde{S}_{k+1}$ if $\sigma = \hat{s}_{i_1} \ldots \hat{s}_{i_m}$. The *"length"* of σ, denoted $\ell(\sigma)$, is the length of the shortest word corresponding to σ. Any word for σ with $\ell(\sigma)$ letters is said to be *"reduced"*. We denote by $Red(\sigma)$ the set of all reduced words of σ.

The weak order on $\tilde{S}_{k+1}$ is defined through the following covering relations:

$$\sigma \lessdot_w \tau \quad \Longleftrightarrow \quad \tau = \hat{s}_i\, \sigma \text{ for some } i \in \{0, \ldots, k\}\,, \text{ and } \ell(\tau) > \ell(\sigma)\,. \tag{2.3}$$

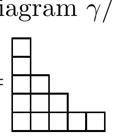
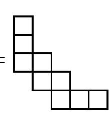
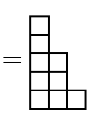
The subgroup of $\tilde{S}_{k+1}$ generated by the subset $\{\hat{s}_1, \ldots, \hat{s}_k\}$ is a maximal parabolic subgroup denoted by S_{k+1}. We consider the set of minimal coset representatives of $\tilde{S}_{k+1}/S_{k+1}$.

3. The k-Young lattice

Let $\mathcal{C}_{k+1}$ and $\mathcal{P}_k$ respectively denote the collections of $k+1$ cores and k-bounded partitions. We start with the map

$$\mathfrak{p} : \gamma \to (\lambda_1, \ldots, \lambda_\ell)$$

where λ_i is the number of cells with a k-bounded hook in row i of γ. If $\rho(\gamma)$ is the partition consisting only of the cells in γ whose hook lengths exceed k, then $\mathfrak{p}(\gamma) = \lambda$ is equivalently defined by letting λ_i denote the length of row i in the skew diagram $\gamma/\rho(\gamma)$. For example, with $k = 4$:

(3.1) $\gamma =$  $\gamma/\rho(\gamma) =$  $\mathfrak{p}(\gamma) =$ 

We prove that $\mathfrak{p}$ is a bijection by showing that each diagram $\gamma/\rho(\gamma)$ can be uniquely associated to a skew diagram constructed from some k-bounded partition λ.

Definition 3.1. For any $\lambda \in \mathcal{P}_k$, the "$k$-skew diagram of λ" is the diagram $\lambda/^k$ where
(i) row i has length λ_i for $i = 1, \ldots, \ell(\lambda)$
(ii) no cell of $\lambda/^k$ has hook-length exceeding k
(iii) all squares below $\lambda/^k$ have hook-length exceeding k.

The inverse of $\mathfrak{p}$ can thus be defined on any k-bounded partition λ, with $\lambda/^k = \gamma/\rho$, by $\mathfrak{c}(\lambda) = \gamma$. Note, there is an algorithm for constructing $\lambda/^k$ by attaching the row of length λ_ℓ to the bottom of $(\lambda_1, \ldots, \lambda_{\ell-1})/^k$, in the leftmost position so that no hooks-lengths exceeding k are created. In (3.1), we construct $\lambda/^k = \gamma/\rho$ from $\lambda = \mathfrak{p}(\gamma)$ according to this method and thus can easily find $\mathfrak{c}(\lambda) = \gamma$.

Theorem 3.2. *$\mathfrak{p}$ is a bijection from $\mathcal{C}_{k+1}$ onto $\mathcal{P}_k$ with inverse $\mathfrak{c}$.*

The notion of k-skew diagram gives rise to an involution on $\mathcal{P}_k$:

DEFINITION 3.3. For any $\lambda \in \mathcal{P}_k$, the "*$k$-conjugate*" of λ denoted λ^{ω_k} is the k-bounded partition given by the columns of $\lambda/^k$. Equivalently, $\lambda^{\omega_k} = \mathfrak{p}(\mathfrak{c}(\lambda)')$.

Given the k-conjugate, a partial order "$\preceq$" on the collection of k-bounded partitions arises:

DEFINITION 3.4. The "*k-Young lattice*" $\preceq$ on partitions in $\mathcal{P}_k$ is defined by the covering relation

$$\lambda \to_k \mu \quad \text{when} \quad \lambda \subseteq \mu \quad \text{and} \quad \lambda^{\omega_k} \subseteq \mu^{\omega_k} \tag{3.2}$$

for $\mu, \lambda \in \mathcal{P}_k$ where $|\mu| - |\lambda| = 1$. See Figure 1.

The k-Young lattice generalizes the Young lattice since $\lambda \preceq \mu$ reduces to $\lambda \leq \mu$ when μ is such that $h_{(1,1)}(\mu) \leq k$. It is also important to note that although the definition of $\preceq$ implies:

$$\lambda \preceq \mu \implies \lambda \subseteq \mu \quad \text{and} \quad \lambda^{\omega_k} \subseteq \mu^{\omega_k} ,$$

the converse of this statement does not hold. For example, with $k = 3$, $\lambda = (2,2)$ and $\mu = (3,2,1,1,1,1)$, we have $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$, but $\lambda \not\preceq \mu$. This can be verified by constructing all chains using Theorem 4.1, or follows immediately from [**LM2**, Th. 19].

While this poset on k-bounded partitions originally arose in connection to a rule for multiplying k-Schur functions, we show that it is isomorphic to the weak order on the quotient of the affine symmetric group by a maximal parabolic subgroup. Consequently, this poset is a lattice [**W**] ([**Ul**] gives a proof by identifying the k-Young lattice as a cone in a permutahedron-tiling of $\mathbb{R}^k$).

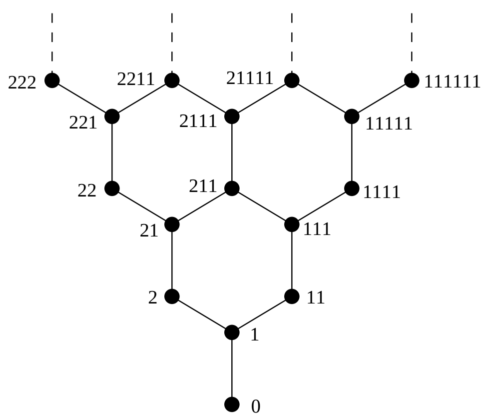


FIGURE 1. Hasse diagram of the k-Young lattice in the case $k = 2$.

4. k-Young lattice, $k+1$-cores, and k-tableaux

Since the set of μ such that $\mu \supset \lambda$ and $|\mu| = |\lambda| + 1$ consists of all partitions obtained by adding a corner to λ, a subset of these partitions will be the elements that cover λ with respect to $\preceq$.

THEOREM 4.1. *The order $\preceq$ can be characterized by the covering relation*

$$\lambda \to_k \mu \iff \lambda = \mu - e_r \tag{4.1}$$

where r is any row of $\mathfrak{c}(\mu)$ with a removable corner whose $k+1$-residue i does not occur in a higher removable corner.

EXAMPLE 4.2. With $k = 4$ and $\lambda = (4,2,1,1)$,

$$\mathfrak{c}(4,2,1,1) = \begin{array}{llllllll} 1 \\ \boxed{2} \\ \boxed{3} & 4 \\ \boxed{4} & \boxed{0} & 1 \\ \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{0} & 1 \end{array} \quad , \tag{4.2}$$

and thus the partitions that are covered by λ are $(4,1,1,1)$, and $(4,2,1)$, while those that cover it are $(4,2,1,1,1)$ and $(4,2,2,1)$.

We find the covering relations can be equivalently determined using operators on $\mathcal{C}_{k+1}$:

DEFINITION 4.3. The *"operator s_i"* acts on a $k+1$-core by:

(a) removing all removable corners with $k+1$-residue i if there is at least one removable corner of $k+1$-residue i
(b) adding all addable corners with $k+1$-residue i if there is at least one addable corner with $k+1$-residue i
(c) leaving it invariant when there are no addable or removable corners of $k+1$-residue i.

Recall that operators adding corners of a given residue to partitions arose in [**DJKMO**] and [**MM**] (see also [**L**]), and coincide with restricting and inducing operators introduced in [**Ro**].

COROLLARY 4.4. *Given k-bounded partitions λ and μ,*

$$\lambda \to_k \mu \iff \mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu) \ \text{and} \ s_i\big(\mathfrak{c}(\lambda)\big) = \mathfrak{c}(\mu) \ \text{for some } i \in \{0,\dots,k\}\,. \tag{4.3}$$

From this, we are able to provide a core-characterization of the saturated chains from the empty partition (hereafter $\emptyset = \lambda^{(0)}$) to any k-bounded partition $\lambda \vdash n$:

$$\mathcal{D}^k(\lambda) = \Big\{ (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \lambda) \ : \ \lambda^{(j)} \to_k \lambda^{(j+1)} \Big\}\,. \tag{4.4}$$

COROLLARY 4.5. *The saturated chains to the vertex $\lambda \vdash n$ in the k-lattice are given by*

$$\mathcal{D}^k(\lambda) = \Big\{ (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \lambda) \ : \ \mathfrak{c}(\lambda^{(j)}) \subset \mathfrak{c}(\lambda^{(j+1)}) \ \text{and}\ \mathfrak{c}(\lambda^{(j+1)}) = s_i\left(\mathfrak{c}(\lambda^{(j)})\right) \ \text{for some } i \Big\}$$

Motivated by the proposed role of k-lattice chains in the study of certain Macdonald polynomial expansion coefficients, we pursue a tableaux interpretation for these chains. We provide a bijection between the set of chains $\mathcal{D}^k(\lambda)$ and a new family of tableaux defined on cores.

DEFINITION 4.6. A k-tableau T of shape $\gamma \in \mathcal{C}_{k+1}$ with n k-bounded hooks is a filling of γ with integers $\{1,\dots,n\}$ such that
(i) rows and columns are strictly increasing
(ii) repeated letters have the same $k+1$-residue

The set of all k-tableaux of shape $\mathfrak{c}(\lambda)$ is denoted by $\mathcal{T}^k(\lambda)$.

EXAMPLE 4.7. $\mathcal{T}^3(3,2,1,1)$, or the set of 3-tableaux of shape $(6,3,1,1)$, is

$$\begin{array}{llllll} \boxed{7} \\ \boxed{5} \\ \boxed{4} & \boxed{6} & \boxed{7} \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} & \boxed{7} \end{array} \qquad \begin{array}{llllll} \boxed{7} \\ \boxed{6} \\ \boxed{4} & \boxed{5} & \boxed{7} \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{7} \end{array} \qquad \begin{array}{llllll} \boxed{7} \\ \boxed{4} \\ \boxed{3} & \boxed{6} & \boxed{7} \\ \boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} \end{array} \qquad \begin{array}{llllll} \boxed{7} \\ \boxed{4} \\ \boxed{2} & \boxed{6} & \boxed{7} \\ \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} \end{array} \tag{4.5}$$

The bijection between chains of $\mathcal{D}^k(\lambda)$ in the k-lattice and k-tableaux $\mathcal{T}^k(\lambda)$ is given by the following maps:

DEFINITION 4.8. For any path $P = (\lambda^{(0)}, \dots, \lambda^{(n)}) \in \mathcal{D}^k(\lambda)$, let $\Gamma(P)$ be the tableau constructed by putting letter j in positions $\mathfrak{c}(\lambda^{(j)})/\mathfrak{c}(\lambda^{(j-1)})$ for $j = 1,\dots,n$.
Given $T \in \mathcal{T}^k(\lambda)$, let $\bar\Gamma(T) = (\lambda^{(0)}, \dots, \lambda^{(n)})$ where $\mathfrak{c}(\lambda^{(j)})$ is the shape of the tableau obtained by deleting letters $j+1,\dots,n$ from T.

To compute the action of Γ on a path, we view the action of $\mathfrak{c}$ as a composition of maps on a partition – first skew the diagram and then add the squares below the skew to obtain a core.

Theorem 4.9. *Γ is a bijection between $\mathcal{D}^k(\lambda)$ and $\mathcal{T}^k(\lambda)$ with $\Gamma^{-1} = \bar{\Gamma}$.*

5. The k-Young lattice and the weak order on $\tilde{S}_{k+1}/S_{k+1}$

The $k+1$-core characterization of the k-Young lattice covering relations given in Corollary 4.4 leads to the identification of the k-Young lattice as the weak order on $\tilde{S}_{k+1}/S_{k+1}$. A by-product of this result is a simple bijection between reduced words and k-tableaux and one between k-bounded partitions and affine permutations in $\tilde{S}_{k+1}/S_{k+1}$.

Definition 5.1. For $\sigma \in \tilde{S}_{k+1}$, let $\mathfrak{s}$ send σ to a $k+1$-core by

$$\mathfrak{s} : \sigma = s_{i_1} \cdots s_{i_\ell} \cdot \emptyset \,, \tag{5.1}$$

where $i_1 \cdots i_\ell$ is any reduced word for σ and $\emptyset$ is the empty $k+1$-core.

Following from the characterization of Bruhat order in terms of cores (see [**L**]), we are able to obtain from our $k+1$-core characterization of the chains in the k-lattice that this lattice is isomorphic to the weak order on $\tilde{S}_{k+1}/S_{k+1}$:

Proposition 5.2. *Let $\sigma, \tau \in \tilde{S}_{k+1}/S_{k+1}$, and let $\lambda = \mathfrak{p}(\mathfrak{s}(\sigma))$ and $\mu = \mathfrak{p}(\mathfrak{s}(\tau))$. Then*

$$\sigma \lessdot_w \tau \qquad \Longleftrightarrow \qquad \lambda \to_k \mu \,. \tag{5.2}$$

We have seen in Theorem 4.9 that the saturated chains to shape λ in the k-lattice are in bijection with k-tableaux of shape $\mathfrak{p}(\gamma)$. On the other hand, the reduced words for $\sigma \in \tilde{S}_{k+1}/S_{k+1}$ encode the chains to σ. Proposition 5.2 thus implies there is a bijection between k-tableaux of shape γ and the reduced words for $\mathfrak{s}^{-1}(\gamma)$.

Definition 5.3. For a k-tableau T with m letters where i_a is the $k+1$-residue of the letter a, define

$$\mathfrak{w} : T \mapsto i_m \cdots i_1 \,.$$

For $w = i_m \cdots i_1 \in Red(\sigma)$, $\mathfrak{w}^{-1}(w)$ is the tableau with letter $\ell = 1, \ldots, m$ occupying the cells of $s_{i_\ell} \cdots s_{i_1} \cdot \emptyset / s_{i_{\ell-1}} \cdots s_{i_1} \cdot \emptyset$.

Example 5.4. With $k = 3$:

$$T = \begin{array}{|c|cccccc} \hline 7 \\ \cline{1-1} 6 \\ \cline{1-3} 4 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} \\ \hline \end{array} \quad \overset{\mathfrak{w}}{\leftrightarrow} \quad 1\,2\,0\,3\,2\,1\,0 \quad \text{since the 4-residues are} \quad \begin{array}{|c|cccccc} \hline 1 \\ \cline{1-1} 2 \\ \cline{1-3} 3 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline 0 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Proposition 5.5. *The map $\mathfrak{w} : \mathcal{T}^k(\lambda) \longrightarrow Red(\sigma)$ is a bijection, where $\sigma \in \tilde{S}_{k+1}/S_{k+1}$ is defined uniquely by $\mathfrak{c}(\lambda) = \mathfrak{s}(\sigma)$.*

We now make use of canonical chains in the k-Young lattice to obtain a simple bijection between k-bounded partitions and permutations in $\tilde{S}_{k+1}/S_{k+1}$.

Definition 5.6. For any partition λ, let "w_λ" be the word obtained by reading the $k+1$-residues in each row of λ, from right to left, starting with the highest removable corner and ending in the first cell of the first row. Further, let "σ_λ" be the affine permutation corresponding to w_λ.

Example 5.7. For $\lambda = (3, 2, 2, 1)$ and $k = 3$, $w_\lambda = 1\,3\,2\,0\,3\,2\,1\,0$ and $\sigma_\lambda = \hat{s}_1 \hat{s}_3 \hat{s}_2 \hat{s}_0 \hat{s}_3 \hat{s}_2 \hat{s}_1 \hat{s}_0$ since:

$$\lambda = \begin{array}{|c|cc} \hline 1 \\ \cline{1-2} 2 & \multicolumn{1}{|c|}{3} \\ \cline{1-2} 3 & \multicolumn{1}{|c|}{0} \\ \hline 0 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \tag{5.3}$$

Proposition 5.8. *The map $\phi : \mathcal{P}_k \to \tilde{S}_{k+1}/S_{k+1}$ where $\phi(\lambda) = \sigma_\lambda$ is a bijection whose inverse is $\phi^{-1} = \mathfrak{p} \circ \mathfrak{s}$.*

6. Generalized k-tableaux and the k-Young lattice

We now introduce a set of tableaux that serve as a semi-standard version of k-tableaux.

Definition 6.1. Let γ be a $k+1$-core, m be the number of k-bounded hooks of γ, and $\alpha=(\alpha_1,\ldots,\alpha_r)$ be a composition of m. A semi-standard k-tableau of shape γ and evaluation α is a filling of γ with integers $1,2,\ldots,r$ such that
(i) rows are weakly increasing and columns are strictly increasing
(ii) the collection of cells filled with letter i are labeled with exactly α_i distinct $k+1$-residues.

We denote the set of all semi-standard k-tableaux of shape $\mathfrak{c}(\lambda)$ and evaluation α by $\mathcal{T}^k_\alpha(\lambda)$. When $\alpha=(1^m)$, we call the k-tableaux "*standard*". In this case, $\mathcal{T}^k_{(1^m)}(\lambda)=\mathcal{T}^k(\lambda)$.

Example 6.2. For $k=3$, $\mathcal{T}^3_{(1,3,1,2,1,1)}(3,3,2,1)$ of shape $\mathfrak{c}\left((3,3,2,1)\right)=(8,5,2,1)$ is the set:

$$
\begin{array}{llllllll}
5 \\
4 & 6 \\
2 & 3 & 4 & 4 & 6 \\
1 & 2 & 2 & 2 & 3 & 4 & 4 & 6
\end{array}
\qquad
\begin{array}{llllllll}
6 \\
4 & 5 \\
2 & 3 & 4 & 4 & 5 \\
1 & 2 & 2 & 2 & 3 & 4 & 4 & 5
\end{array}
\qquad
\begin{array}{llllllll}
4 \\
3 & 6 \\
2 & 4 & 4 & 5 & 6 \\
1 & 2 & 2 & 2 & 4 & 4 & 5 & 6
\end{array}
\tag{6.1}
$$

It is known that there are no semi-standard tableaux of shape λ and evaluation μ when $\lambda \not\trianglerighteq \mu$ in dominance order. We have found that this is also true for the k-tableaux.

Theorem 6.3. *There are no semi-standard k-tableaux in $\mathcal{T}^k_\mu(\lambda)$ when $\lambda \not\trianglerighteq \mu$. Further, there is exactly one when $\lambda=\mu$.*

A rule for expanding the product of a k-Schur function with the homogeneous function h_ℓ (for $\ell \le k$) in terms of k-Schur functions was conjectured in [**LM1**]. We introduce certain sequences of partitions based on this generalized Pieri rule and show their connection to the semi-standard k-tableaux. The connection with symmetric functions is then discussed in the next section.

A pair of k-bounded partitions λ,μ is "*r-admissible*" if and only if λ/μ and $\lambda^{\omega_k}/\mu^{\omega_k}$ are respectively horizontal and vertical r-strips. For composition α, a sequence of partitions $\left(\lambda^{(0)},\lambda^{(1)},\cdots,\lambda^{(r)}\right)$ is "α-admissible" if $\lambda^{(j)},\lambda^{(j-1)}$ is a α_j-admissible pair for all j. It turns out that any α-admissible sequence must be a chain in the k-Young lattice. We are interested in the set of chains:

Definition 6.4. For any composition α, let

$$\mathcal{D}^k_\alpha(\lambda)=\left\{(\emptyset=\lambda^{(0)},\ldots,\lambda^{(r)}=\lambda)\ \text{ that are }\ \alpha\text{-admissible}\right\}.$$

The following maps provide a bijection between the chains in $\mathcal{D}^k_\alpha(\lambda)$ and the tableaux in $\mathcal{T}^k_\alpha(\lambda)$.

Definition 6.5. For any $P=(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(m)})\in\mathcal{D}^k_\alpha(\lambda)$, let $\Gamma(P)$ be the tableau of shape $\mathfrak{c}(\lambda)$ where letter j fills cells in positions $\mathfrak{c}(\lambda^{(j)})/\mathfrak{c}(\lambda^{(j-1)})$, for $j=1,\ldots,m$.

For a k-tableau $T\in\mathcal{T}^k_\alpha(\lambda)$ with $\alpha=(\alpha_1,\ldots,\alpha_m)$, let $\bar\Gamma(T)=(\lambda^{(0)},\ldots,\lambda^{(m)})$, where $\mathfrak{c}(\lambda^{(i)})$ is the shape of the tableau obtained by deleting the letters $i+1,\ldots,m$ from T.

Theorem 6.6. *Γ is a bijection between $\mathcal{T}^k_\alpha(\lambda)$ and $\mathcal{D}^k_\alpha(\lambda)$, with $\Gamma^{-1}=\bar\Gamma$.*

7. Symmetric functions and k-tableaux

Refer to [**M**] for details on Macdonald polynomials. Here, we are interested in the study of the q,t-Kostka polynomials $K_{\mu\lambda}(q,t)\in\mathbb{N}[q,t]$. These polynomials arise as expansion coefficients for the Macdonald polynomials $J_\lambda[X;q,t]$ in terms of a basis dual to the monomial basis with respect to the Hall-Littlewood scalar product. As in the introduction, we use the modification of $J_\lambda[X;q,t]$ whose expansion coefficients in terms of Schur functions are the q,t-Kostka coefficients:

$$H_\lambda[X;q,t]=\sum_\mu K_{\mu\lambda}(q,t)\,s_\mu[X]\,. \tag{7.1}$$

The q,t-Kostka coefficients also have a representation theoretic interpretation [**GH, H**], from which they were shown to lie in $\mathbb{N}[q,t]$. Since $J_\lambda[X;q,t]$ reduces to the Hall-Littlewood polynomial $Q_\lambda[X;t]$ when $q=0$, we obtain a modification of the Hall-Littlewood polynomials by taking:

$$H_\lambda[X;t] = H_\lambda[X;0,t] = \sum_{\mu \trianglerighteq \lambda} K_{\mu\lambda}(t)\, s_\mu[X]\,, \tag{7.2}$$

with the coefficients $K_{\mu\lambda}(t) \in \mathbb{N}[t]$ known as Kostka-Foulkes polynomials. We can then obtain the homogeneous symmetric functions by letting $t=1$:

$$h_\lambda[X] = H_\lambda[X;1] = \sum_{\mu \trianglerighteq \lambda} K_{\mu\lambda}\, s_\mu[X]\,, \tag{7.3}$$

where $K_{\mu\lambda} \in \mathbb{N}$ are the Kostka numbers.

Recent work in the theory of symmetric functions has led to a new approach in the study of the q,t-Kostka polynomials. The underlying mechanism for this approach relies on a family of polynomials that appear to have a remarkable kinship with the classical Schur functions [**LLM, LM1**]. More precisely, consider the filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$, given by linear spans of Hall-Littlewood polynomials indexed by k-bounded partitions. That is,

$$\Lambda_t^{(k)} = \mathcal{L}\{H_\lambda[X;t]\}_{\lambda;\lambda_1 \le k}\,, \qquad k = 1,2,3,\ldots\,.$$

A family of symmetric functions called the k-Schur functions, $s_\lambda^{(k)}[X;t]$, was introduced in [**LM1**] (these functions are conjectured to be precisely the polynomials defined using tableaux in [**LLM**]). It was shown that the k-Schur functions form a basis for $\Lambda_t^{(k)}$ and that, for λ a k-bounded partition,

$$H_\lambda[X;q,t\,] \;=\; \sum_{\mu;\mu_1\le k} K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}[X;t\,]\,, \qquad K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{Z}[q,t]\,, \tag{7.4}$$

and

$$H_\lambda[X;t\,] \;=\; s_\lambda^{(k)}[X;t\,] \;+\; \sum_{\substack{\mu;\mu_1\le k\\ \mu >_D \lambda}} K_{\mu\lambda}^{(k)}(0,t)\, s_\mu^{(k)}[X;t\,]\,, \qquad K_{\mu\lambda}^{(k)}(0,t) \in \mathbb{Z}[t]\,. \tag{7.5}$$

The study of the k-Schur functions is motivated in part by the conjecture that the expansion coefficients actually lie in $\mathbb{N}[q,t]$. That is,

$$K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{N}[q,t]\,. \tag{7.6}$$

Since it was shown that $s_\lambda^{(k)}[X;t] = s_\lambda[X]$ for k larger than the hook-length of λ, this conjecture generalizes Eq. (7.1). Also, there is evidence to support that $K_{\mu\lambda}(q,t) - K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{N}[q,t]$, suggesting that the k-Schur expansion coefficients are simpler than the q,t-Kostka polynomials.

The preceding developments on the k-lattice can be applied to the study of the generalized q,t-Kostka coefficients as follows: the k-Schur functions appear to obey a generalization of the Pieri rule on Schur functions. It was conjectured in [**LLM, LM1**] that for the complete symmetric function $h_\ell[X]$,

$$h_\ell[X]\, s_\lambda^{(k)}[X;1] = \sum_{\mu \in E_{\lambda,\ell}^{(k)}} s_\mu^{(k)}[X;1]\,, \tag{7.7}$$

where

$$E_{\lambda,\ell}^{(k)} = \big\{\mu \,|\, \mu/\lambda \text{ is a horizontal } \ell\text{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a vertical } \ell\text{-strip}\big\}\,. \tag{7.8}$$

Iteration, from $s_\emptyset^{(k)}[X;1]=1$, then yields that the expansion of $h_{\lambda_1}[X]h_{\lambda_2}[X]\cdots$ satisfies

$$h_\lambda[X]=\sum_\mu K^{(k)}_{\mu\lambda}\, s^{(k)}_\mu[X;1]\,, \tag{7.9}$$

where $K^{(k)}_{\mu\lambda}$ is a nonnegative integer reducing to the usual Kostka number $K_{\mu\lambda}$ when k is large since $s^{(k)}_\lambda[X;t]=s_\lambda[X]$ in this case. The definition of $E^{(k)}_{\lambda,\ell}$ in the k-Pieri expansion thus reveals the motivation behind the set of chains given in Definition 6.4. This connection implies that

$$K^{(k)}_{\mu\lambda} \;=\; \text{the number of chains of the } k\text{-lattice in } \mathcal{D}^k_\lambda(\mu)\,.$$

Equivalently, using the bijection between paths in $\mathcal{D}^k_\lambda(\mu)$ and $\mathcal{T}^k_\lambda(\mu)$ given in Theorem 6.6, we have

$$K^{(k)}_{\mu\lambda} \;=\; \text{ the number of } k\text{-tableaux of shape } \mathfrak{c}(\mu) \text{ and evaluation } \lambda\,.$$

Although this combinatorial interpretation relies on the conjectured Pieri rule (7.7), it was proven in [**LM1**] that the k-Schur functions are unitriangularly related to the homogeneous symmetric functions. That is, $K^{(k)}_{\lambda\mu}=0$ when $\mu\not\geq\lambda$ and $K^{(k)}_{\lambda\lambda}=1$. Therefore, Theorem 6.3 implies that the number of k-tableaux does correspond to $K^{(k)}_{\lambda\mu}$ in these cases.

More generally, note that letting $q=0$ in Eq. (7.6) gives that the coefficients in Hall-Littlewood expansion Eq. (7.5) satisfy $K^{(k)}_{\mu\lambda}(0,t)\in\mathbb{N}[t]$. However, since $H_\lambda[X;1]=h_\lambda[X]$, we have that $K^{(k)}_{\mu\lambda}(0,1)=K^{(k)}_{\mu\lambda}$ from Eq. (7.9). Therefore, since it appears that $K^{(k)}_{\mu\lambda}$ counts the number of semi-standard k-tableaux in $\mathcal{T}^k_\lambda(\mu)$, it is suggested that there exists a t-statistic on such k-tableaux giving a combinatorial interpretation for the generalized Kostka-Foulkes $K^{(k)}_{\mu\lambda}(0,t)$.

Alternatively, $H_\lambda[X;1,1]=h_{1^n}[X]$ for $\lambda\vdash n$ implies that $K^{(k)}_{\mu\lambda}(1,1)=K^{(k)}_{\mu\,1^n}$ by Eq. (7.9). This lends support to the idea that a q,t-statistic on the standard k-tableaux that would account for the apparently positive coefficients $K^{(k)}_{\mu\lambda}(q,t)$ in Eq. (7.6). That is,

$$K^{(k)}_{\mu\lambda}(1,1) \;=\; \text{ the number of standard } k\text{-tableaux of shape } \mathfrak{c}(\mu)\,.$$

Equivalently, our bijection between affine permutations and standard k-tableaux suggests there may be a q,t-statistic on reduced words that would account for the positivity:

$$K^{(k)}_{\mu\lambda}(1,1) \;=\; \text{ the number of reduced words of } \sigma_\mu\in\tilde{S}_{k+1}/S_{k+1}\;\;.$$

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