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Bruhat Order on the Involutions of Classical Weyl Groups

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Abstract. It is known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is EL-shellable, hence Cohen-Macaulay, and Eulerian. In this work we consider the subposet of W induced by the set of involutions of W, denoted by $Invol(W)$. Our main result is that, if W is a classical Weyl group, then the poset $Invol(W)$ is graded, with rank function given by the average between the length and the absolute length, and that it is EL-shellable, hence Cohen-Macaulay, and Eulerian. In particular we obtain, as new results, a combinatorial description of the covering relation in the Bruhat order of the hyperoctahedral group and the even-signed permutation group, and a combinatorial description of the absolute length of the involutions in classical Weyl groups.

Résumé. Il est bien connu qu'un groupe de Coxeter W, munis de l'ordre de Bruhat, est un poset gradué, avec fonction rang donnée par la longueur, et qu'il est EL-shellable, donc de Cohen-Macaulay, et Eulerien. Dans cet article on considère le sous-poset induit par l'ensemble des involutions de W, noté $Invol(W)$. Nous montrons que, si W est un groupe de Weyl classique, alors le poset $Invol(W)$ est gradué, avec fonction rang égale à la moyenne entre la longueur et la longueur absolue, et qu'il est EL-shellable, donc de Cohen-Macaulay, et Eulerien. Nous obtenons en particulier deux résultats nouveaux: une description combinatoire de la relation de couverture dans l'ordre de Bruhat de B_n et D_n, et une description combinatoire de la longueur absolue des involutions dans les groupes de Weyl classiques.

1. Introduction

It is known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is EL-shellable, hence Cohen-Macaulay, and Eulerian. The aim of this work is to investigate whether a particular subposet of W, namely that induced by the set of involutions of W, which we denote by $Invol(W)$, is endowed with similar properties.

The problem arises from a geometric question. It is known that the symmetric group, partially ordered by the Bruhat order, encodes the cell decomposition of Schubert varieties. Richardson and Springer ([**RS1**], [**RS2**]) introduced a vast generalization of this partial order, in relation to the cell decomposition of certain symmetric varieties. In a particular case they obtained the poset $Invol(S_n)$.

In this work the problem is completely solved for an important class of Coxeter groups, namely that of classical Weyl groups. Our main result is that, if W is a classical Weyl group, then the poset $Invol(W)$ is graded, with rank function given by the average between the length and the absolute length, and that it is EL-shellable, hence Cohen-Macaulay, and Eulerian.

The proofs (see [**Inc1**], [**Inc2**], [**Inc3**] for details) are combinatorial and use the descriptions of classical Weyl groups in terms of permutation groups: the symmetric group for type $\mathbf{A}_n$, the hyperoctahedral group for type $\mathbf{B}_n$ and the even-signed permutation group for type $\mathbf{D}_n$.

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In particular we obtain, as new results, a combinatorial description of the covering relation in the Bruhat order of the hyperoctahedral group and the even-signed permutation group, and a combinatorial description of the absolute length of the involutions in classical Weyl groups.

Finally it is conjectured that the result proved for classical Weyl groups actually holds for every Coxeter group.

2. Notation and preliminaries

We let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}$ be the set of integers. For $n, m \in \mathbf{Z}$, with $n \leq m$, we let $[n, m] = \{n, n+1, \ldots, m\}$. For $n \in \mathbf{N}$, we let $[n] = [1, n]$ and $[\pm n] = [-n, n] \backslash \{0\}$.

2.1. Posets.

We follow [**Sta1**, Chapter 3] for poset notation and terminology. In particular we denote by $\lhd$ the *covering relation*: $x \lhd y$ means that $x < y$ and there is no z such that $x < z < y$. A poset is *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively. If $x, y \in P$, with $x \leq y$, we let $[x, y] = \{z \in P : x \leq z \leq y\}$, and we call it an *interval* of P. If $x, y \in P$, with $x < y$, a *chain* from x to y of *length* k is a $(k+1)$-tuple $(x_0, x_1, ..., x_k)$ such that $x = x_0 < x_1 < ... < x_k = y$. A chain $x_0 < x_1 < ... < x_k$ is said to be *saturated* if all the relations in it are covering relations ($x_0 \lhd x_1 \lhd ... \lhd x_k$).

A poset is said to be *graded* of *rank* n if it is finite, bounded and if all maximal chains of P have the same length n. If P is a graded poset of rank n, then there is a unique *rank function* $\rho : P \to [0, n]$ such that $\rho(\hat{0}) = 0$, $\rho(\hat{1}) = n$ and $\rho(y) = \rho(x) + 1$ whenever y covers x in P. Conversely, if P is finite and bounded, and if such a function exists, then P is graded of rank n.

Let P be a graded poset and let Q be a totally ordered set. An *EL-labelling* of P is a function $\lambda : \{(x, y) \in P^2 : x \lhd y\} \to Q$ such that for every $x, y \in P$, with $x < y$, two properties hold:

1. there is exactly one saturated chain from x to y with non decreasing labels:
$$x = x_0 \underset{\lambda_1}{\lhd} x_1 \underset{\lambda_2}{\lhd} \ldots \underset{\lambda_k}{\lhd} x_k = y,$$
with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$;
2. this chain has the lexicographically minimal labelling: if
$$x = y_0 \underset{\mu_1}{\lhd} y_1 \underset{\mu_2}{\lhd} \ldots \underset{\mu_k}{\lhd} y_k = y$$
is a saturated chain from x to y different from the previous one, then
$$(\lambda_1, \lambda_2, \ldots, \lambda_k) < (\mu_1, \mu_2, \ldots, \mu_k).$$

A graded poset P is said to be *EL-shellable* if it has an EL-labelling.

Connections between EL-shellable posets and shellable complexes, Cohen-Macaulay complexes and Cohen-Macaulay rings can be found, for example, in [**Bac**], [**BGS**], [**Bjö**], [**Gar**], [**Hoc**], [**Rei**] and [**Sta2**]. Here we only recall the following important result, due to Björner.

Theorem 2.1. Let P be a graded poset. If P is EL-shellable then P is shellable and hence Cohen-Macaulay.

A graded poset P with rank function ρ is said to be *Eulerian* if
$$|\{z \in [x, y] : \rho(z) \text{ is even}\}| = \{z \in [x, y] : \rho(z)| \text{ is odd}\}|,$$
for every $x, y \in P$ such that $x < y$.

In an EL-shellable poset there is a necessary and sufficient condition for the poset to be Eulerian. We state it in the following form (see [**Bjö**, Theorem 2.7] and [**Sta3**, Theorem 1.2] for proofs of more general results).

Theorem 2.2. Let P be a graded EL-shellable poset and let λ be an EL-labelling of P. Then P is Eulerian if and only if for every $x, y \in P$, with $x < y$, there is exactly one saturated chain from x to y with decreasing labels.

2.2. Coxeter groups. About Coxeter groups we recall some basic definitions. Let W be a Coxeter group, with set of generators S. The *length* of an element $w \in W$, denoted by $l(w)$, is the minimal k such that w can be written as a product of k generators.
A *reflection* in a Coxeter group W is a conjugate of some element in S. The set of all reflections is usually denoted by T:

$$T = \{wsw^{-1} : s \in S,\ w \in W\}.$$

The *absolute length* of an element $w \in W$, denoted by $al(w)$, is the minimal k such that w can be written as a product of k reflections.

2.3. Bruhat order. Let W be a Coxeter group with set of generators S. Let $u, v \in W$. Then $u \to v$ if and only if $v = ut$, with $t \in T$, and $l(u) < l(v)$. The *Bruhat order* of W is the partial order relation so defined: given $u, v \in W$, then $u \leq v$ if and only if there is a chain

$$u = u_0 \to u_1 \to u_2 \to \ldots \to u_k = v.$$

The Bruhat order of Coxeter groups has been studied extensively (see, e.g., [**BW**], [**Deo**], [**Ede**], [**Ful**], [**Pro**], [**Rea**], [**Ver**]). In particular it is known that it gives to W the structure of a graded poset, whose rank function is the length. It has been also proved that this poset is EL-shellable, hence Cohen-Macaulay (see [**Ede**], [**Pro**], [**BW**]), and Eulerian (see [**Ver**]).[1] The aim of this work is to investigate whether the induced subposet $Invol(W)$ is endowed with similar properties. The problem is solved for classical Weyl groups, to which next subsection is dedicated.

2.4. Classical Weyl groups. The finite irreducible Coxeter groups have been completely classified (see, e.g., [**BB**], [**Hum**]). Among them we find the classical Weyl groups, which have nice combinatorial descriptions in terms of permutation groups: the symmetric group S_n is a representative for type $\mathbf{A}_{n-1}$, the hyperoctahedral group B_n for type $\mathbf{B}_n$ and the even-signed permutation group D_n for type $\mathbf{D}_n$.

2.4.1. *The symmetric group.* We denote by S_n the *symmetric group*, defined by

$$S_n = \{\sigma : [n] \to [n] : \sigma \text{ is a bijection}\}$$

and we call its elements *permutations*. To denote a permutation $\sigma \in S_n$ we often use the *one-line notation*: we write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$. We also write σ in *disjoint cycle form*, omitting to write the 1-cycles of σ: for example, if $\sigma = 364152$, then we also write $\sigma = (1, 3, 4)(2, 6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$. Given $\sigma \in S_n$, the *diagram* of σ is a square of $n \times n$ cells, with the cell (i, j) (that is, the cell in column i and row j, with the convention that the first column is the leftmost one and the first row is the lowest one) filled with a dot if and only if $\sigma(i) = j$. For example, in Figure 1 the diagram of $\sigma = 35124 \in S_5$ is represented.

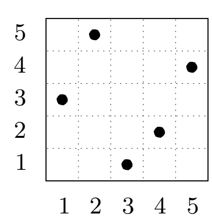


FIGURE 1. Diagram of $\sigma = 35124 \in S_5$.

The *diagonal* of the diagram is the set of cells $\{(i, i) : i \in [n]\}$.
As a set of generators for S_n, we take $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i + 1)$ for every $i \in [n - 1]$. It is known that the symmetric group S_n, with this set of generators, is a Coxeter group of type $\mathbf{A}_{n-1}$ (see, e.g., [**BB**]).
The length of a permutation $\sigma \in S_n$ is given by

$$l(\sigma) = inv(\sigma),$$

where

$$inv(\sigma) = |\{(i,j) \in [n]^2 : i < j,\ \sigma(i) > \sigma(j)\}|$$

is the number of *inversions* of σ.
In the symmetric group the reflections are the transpositions:

$$T = \{(i,j) \in [n]^2 : i < j\}.$$

In order to give a characterization of the covering relation in the Bruhat order of the symmetric group, we introduce the following definition.
DEFINITION 2.3. Let $\sigma \in S_n$. A *rise* of σ is a pair $(i,j) \in [n]^2$ such that

1. $i < j$,
2. $\sigma(i) < \sigma(j)$.

A rise (i,j) is said to be *free* if there is no $k \in [n]$ such that

1. $i < k < j$,
2. $\sigma(i) < \sigma(k) < \sigma(j)$.

For example, the rises of $\sigma = 35124 \in S_5$ are $(1,2)$, $(1,5)$, $(3,4)$, $(3,5)$ and $(4,5)$. They are all free except $(3,5)$. The following is a well-known result.
PROPOSITION 2.4. Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then $\sigma \lhd \tau$ in S_n if and only if

$$\tau = \sigma(i,j),$$

where (i,j) is a free rise of σ.

2.4.2. *The hyperoctahedral group.* We denote by $S_{\pm n}$ the symmetric group on the set $[\pm n]$:

$$S_{\pm n} = \{\sigma : [\pm n] \to [\pm n] : \sigma \text{ is a bijection}\}$$

(which is clearly isomorphic to S_{2n}), and by B_n the *hyperoctahedral group*, defined by

$$B_n = \{\sigma \in S_{\pm n} : \sigma(-i) = -\sigma(i) \text{ for every } i \in [n]\}$$

and we call its elements *signed permutations*. To denote a signed permutation $\sigma \in B_n$ we use the *window notation*: we write $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$ (the images of the negative entries are then uniquely determined). We also denote σ by the sequence $|\sigma_1|\,|\sigma_2| \ldots |\sigma_n|$, with the negative entries underlined. For example, $\underline{3}\,\underline{2}\,1$ denotes the signed permutation $[-3,-2,1]$. We also write σ in disjoint cycle form. Signed permutations are particular permutations of the set $[\pm n]$, so they inherit the notion of diagram. Note that the diagram of a signed permutation is symmetric with respect to the center. In Figure 2, the diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$ is represented.

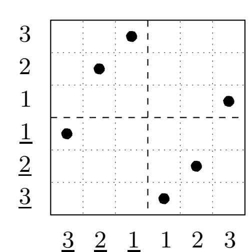


FIGURE 2. Diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$.

The (*main*) *diagonal* of the diagram is the set of cells $\{(i,i) : i \in [\pm n]\}$, and the *antidiagonal* is the set of cells $\{(i,-i) : i \in [\pm n]\}$.
As a set of generators for B_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1,-1)$ and $s_i = (i,i+1)(-i,-i-1)$ for every $i \in [n-1]$. It is known that the hyperoctahedral group B_n, with this set of generators, is a Coxeter group of type $\mathbf{B}_n$ (see, e.g., [**BB**]).

There are various known formulas for computing the length in B_n (see, e.g., [**BB**]). In [**Inc2**] we introduced a new one: the length of $\sigma \in B_n$ is given by

$$l_B(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2}, \tag{2.1}$$

where

$$inv(\sigma) = |\{(i,j) \in [\pm n]^2 : i < j,\ \sigma(i) > \sigma(j)\}|$$

(the length of σ in the symmetric group $S_{\pm n}$), and

$$neg(\sigma) = |\{i \in [n] : \sigma(i) < 0\}|.$$

For example, for $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$, we have $inv(\sigma) = 8$, $neg(\sigma) = 2$, so $l_B(\sigma) = 5$.
Finally, it is known (see, e.g., [**BB**]) that the set of reflections of B_n is

$$T = \{(i,-i) : i \in [n]\} \cup \{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\}.$$

2.4.3. *The even-signed permutation group.* We denote by D_n the *even-signed permutation group*, defined by

$$D_n = \{\sigma \in B_n : neg(\sigma) \text{ is even}\}.$$

Notation and terminology are ineherited from the hyperoctahedral group. For example the signed permutation $\sigma = \underline{3}\,\underline{2}\,1$, whose diagram is represented in Figure 2, is also in D_3.
As a set of generators for D_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1,-2)(-1,2)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$. It is known that the even-signed permutation group D_n, with this set of generators, is a Coxeter group of type $\mathbf{D}_n$ (see, e.g., [**BB**]).
About the length function in D_n, it is known (see, e.g., [**BB**]) that

$$l_D(\sigma) = l_B(\sigma) - neg(\sigma).$$

Thus, by (2.1), the length of $\sigma \in D_n$ is given by

$$l_D(\sigma) = \frac{inv(\sigma) - neg(\sigma)}{2}.$$

For example, for $\sigma = \underline{3}\,\underline{2}\,1 \in D_3$, we have $l_D(\sigma) = 3$.
Finally, it is known (see, e.g., [**BB**]) that the set of reflections of D_n is

$$T = \{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\}.$$

3. The main problem

It is known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is also EL-shellable, hence Cohen-Macaulay, and Eulerian.[1] The aim of this work is to investigate whether a particular subposet of W, namely that induced by the set of involutions of W, is endowed with similar properties.

3.1. Motivation. The problem arises from a geometric question. It is known that the symmetric group, partially ordered by the Bruhat order, encodes the cell decomposition of Schubert varieties (see [**Ful**]). In 1990 Richardson and Springer (see [**RS1**] and [**RS2**]) considered a vast generalization of this partial order, in relation to the cell decomposition of certain symmetric varieties. In a particular case they obtained the subposet of S_n induced by the involutions.

3.2. An example. In Figure 3 the example of the poset S_4 with the induced subposet $Invol(S_4)$ is illustrated. Even in this simple case it is not obvious why the poset $Invol(S_4)$ is graded and who the rank function is. Note that, for example, the involutions 2143 and 4231 have distance 3 in the Hasse diagram of S_4, while they are in covering relation in $Invol(S_4)$.

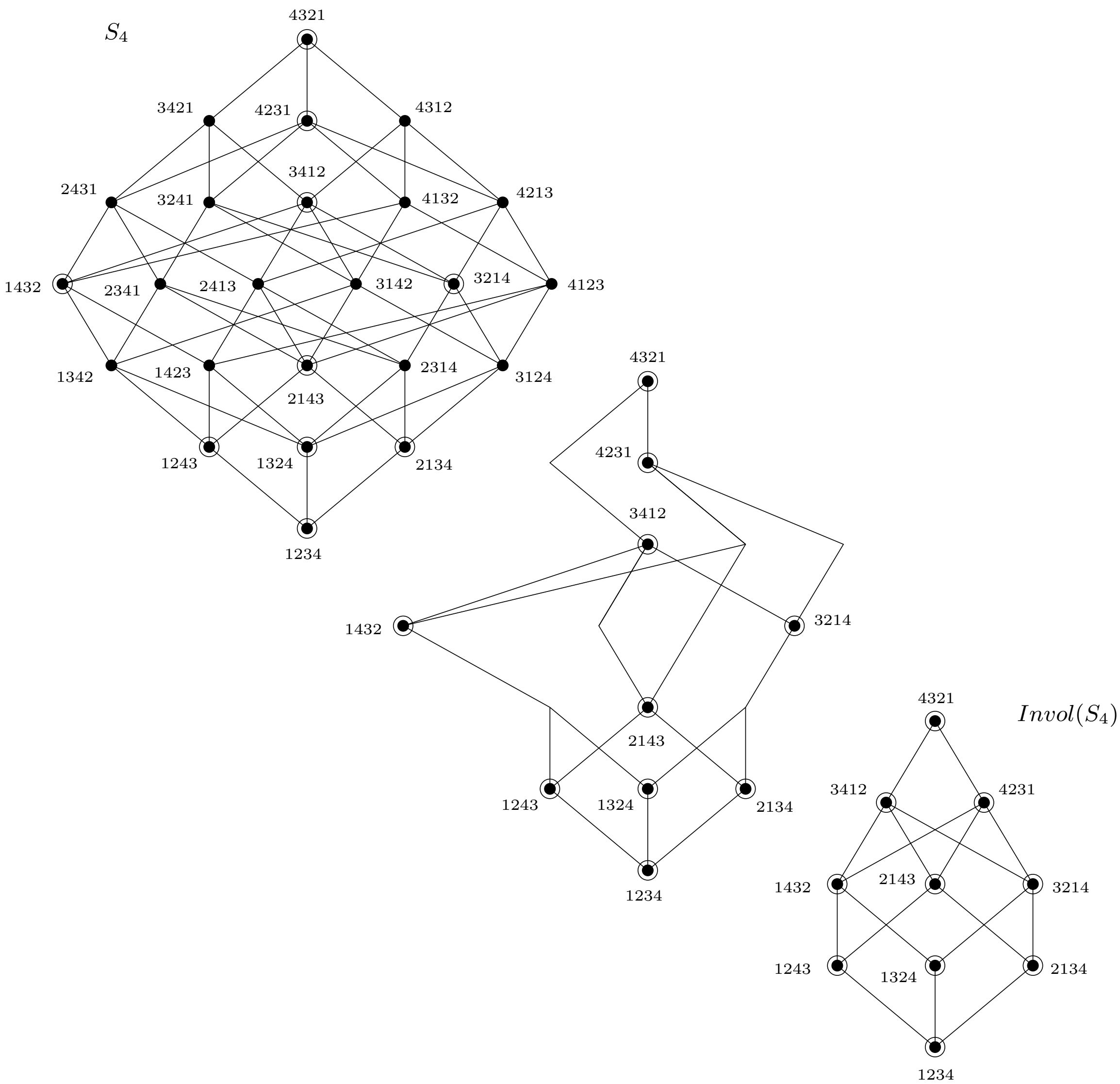


FIGURE 3. From S_4 to $Invol(S_4)$.

3.3. The main result. The following is the main result of this work.

Theorem 3.1. *Let W be a classical Weyl group. The poset $Invol(W)$ is*

1. *graded, with rank function given by*

$$\rho(w) = \frac{l(w) + al(w)}{2},$$

for every $w \in Invol(W)$;

2. *EL-shellable, hence Cohen-Macaulay;*
3. *Eulerian.*

We will give a sketch of the proof in Section 5.

4. Preliminary results

In this section we discuss some new results, which play a crucial role in the proof of the main result of this work. Precisely, we describe the covering relation in the groups B_n and D_n, and we give a combinatorial description of the absolute length of the involutions in classical Weyl groups.

4.1. Covering relation in the Bruhat order of B_n and D_n.

DEFINITION 4.1. Let $\sigma \in B_n$. A rise (i,j) of σ is *central* if

$$(0,0) \in [i,j] \times [\sigma(i), \sigma(j)].$$

A central rise (i,j) of σ is *symmetric* if $j = -i$.

The characterization of the covering relation in B_n is then the following.

THEOREM 4.2. Let $\sigma, \tau \in B_n$. Then $\sigma \lhd \tau$ in B_n if and only if either

1. $\tau = \sigma(i,j)(-i,-j)$, where (i,j) is a not central free rise of σ, or
2. $\tau = \sigma(i,-i)$, where $(i,-i)$ is a central symmetric free rise of σ.

Theorem 4.2 is illustrated in Figure 4, where black dots and white dots denote respectively σ and τ, inside the gray areas there are no other dots of σ and τ than those indicated, and the diagrams of the two permutations are supposed to be the same anywhere else.

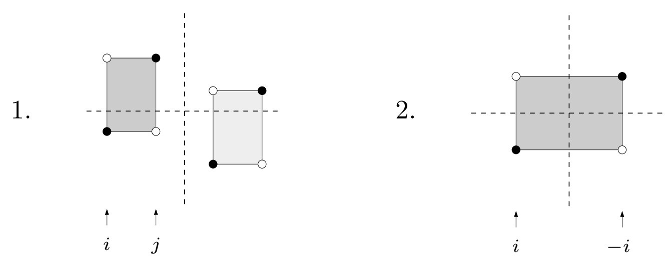


FIGURE 4. Covering relation in B_n.

For the even-signed permutation group we introduce the following definition.

DEFINITION 4.3. Let $\sigma \in D_n$. A central rise (i,j) is *semifree* if

$$\{k \in [i,j] : \sigma(k) \in [\sigma(i), \sigma(j)]\} = \{i, -j, j\}.$$

An example of central semifree rise is illustrated in Figure 5 (3).

THEOREM 4.4. Let $\sigma, \tau \in D_n$. Then $\sigma \lhd \tau$ in D_n if and only if

$$\tau = \sigma(i,j)(-i,-j),$$

where (i,j) is either

1. a not central free rise of σ, or
2. a central not symmetric free rise of σ, or
3. a central semifree rise of σ.

Theorem 4.4 is illustrated in Figure 5, with the same notation as in Figure 4.

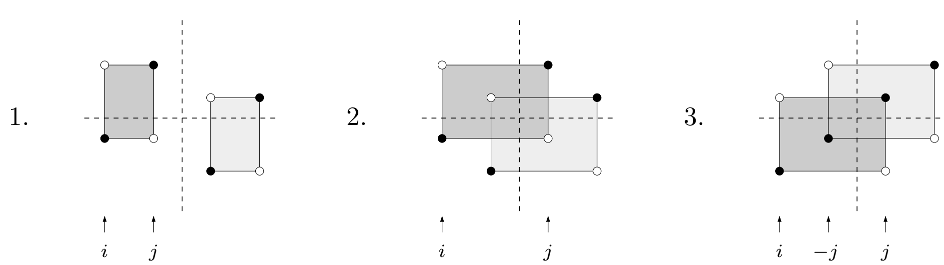


FIGURE 5. Covering relation in D_n.

4.2. Absolute length of involutions in classical Weyl groups. In classical Weyl groups there is a nice combinatorial description for the absolute length of the involutions. In the symmetric group it is simply given by the number of excedances. Note that an involution of S_n has the diagram symmetric with respect to the diagonal.

PROPOSITION 4.5. Let $\sigma \in Invol(S_n)$. Then

$$al(\sigma) = exc(\sigma),$$

where

$$exc(\sigma) = |\{i \in [n] : \sigma(i) > i\}|$$

is the number of *excedances* of σ.

For example, for $\sigma = 32154 \in Invol(5)$, we have $al(\sigma) = exc(\sigma) = 2$. In fact

$$\sigma = \underbrace{(1,3)}_{t_1} \cdot \underbrace{(4,5)}_{t_2}$$

is a minimal decomposition of σ as a product of reflections of S_5.

We now define a new statistic on a signed permutation σ. Note that an involution of B_n has the diagram symmetric with respect to both the diagonals.

DEFINITION 4.6. Let $\sigma \in B_n$. The number of *deficiencies-not-antideficiencies* of σ is

$$dna(\sigma) = |\{i \in [n] : -i \leq \sigma(i) < i\}|.$$

For example, consider $\sigma = 4\,\underline{7}\,\underline{3}\,1\,5\,\underline{6}\,\underline{2} \in B_7$, whose diagram is shown in Figure 6. Looking at the picture, $dna(\sigma)$ is the number of dots which lie in the gray area. In this case

$$dna(\sigma) = 4.$$

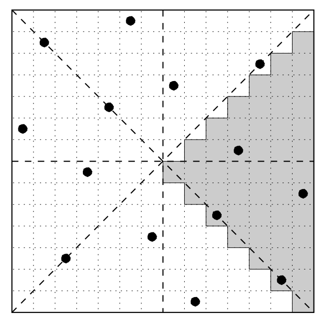


FIGURE 6. The *dna* statistic.

A surprising fact is that in the hyperoctahedral group and in the even-signed permutation group, the combinatorial description for the absolute length of an involution is exactly the same: in both cases it is given by the *dna* statistic. But the reasons are different.

PROPOSITION 4.7. Let $\sigma \in Invol(B_n)$. Then

$$al_B(\sigma) = dna(\sigma).$$

For example, for the involution of Figure 6, we have $al_B(\sigma) = dna(\sigma) = 4$. In fact

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,-3)}_{t_3} \cdot \underbrace{(6,-6)}_{t_4} \tag{4.1}$$

is a minimal decomposition of σ as a product of reflections of B_7.

PROPOSITION 4.8. Let $\sigma \in Invol(D_n)$. Then

$$al_D(\sigma) = dna(\sigma).$$

For example, for the involution of Figure 6, which is also in $Invol(D_7)$, we have $al_D(\sigma) = dna(\sigma) = 4$. Note that the decomposition in (4.1) does not work in D_7, since $(3,-3)$ and $(6,-6)$ are not elements of D_7. But in general an involution σ of D_n necessarily has an even number of antifixed points (that is, indices $i > 0$ such that $\sigma(i) = -i$), so we can consider them in pairs. In the example, σ has the two antifixed points 3 and 6 and

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,6)(-3,-6)}_{t_3} \cdot \underbrace{(3,-6)(-3,6)}_{t_4}$$

is a minimal decomposition of σ as a product of reflections of D_7.

5. Sketch of proofs

5.1. Gradedness. To prove that the posets $Invol(S_n)$, $Invol(B_n)$ and $Invol(D_n)$ are graded with rank function ρ we follow two steps:

1. we first give a characterization of the covering relation in the poset (this is done starting from the description of the covering relation in S_n, B_n and D_n);
2. then we prove that in every covering relation the variation of ρ is 1 (this is done using the combinatorial description of the absolute length of the involutions).

The following are the characterizations of the covering relations in the posets.

THEOREM 5.1. Let $\sigma, \tau \in Invol(S_n)$. Then $\sigma \lhd \tau$ in $Invol(S_n)$ if and only if there exists a rectangle $R = [i,j] \times [\sigma(i), \tau(i)]$ such that σ and τ have the same diagram except for the dots in R, and in its symmetric with respect to the diagonal, for which the situation, depending on the position of R with respect to the diagonal, is described in Figure 7: black dots and white dots denote respectively σ and τ, and the rectangle R (darker gray rectangle) contains no other dots of σ and τ than those indicated.

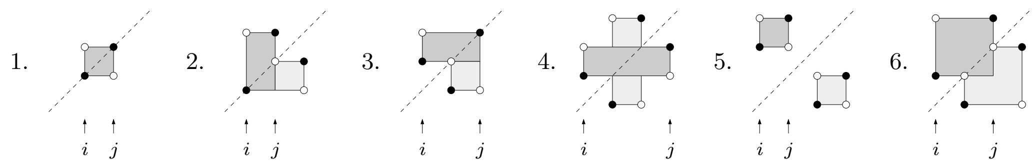


FIGURE 7. Covering relation in $Invol(S_n)$.

Looking at the diagram of a signed permutation, with *orbit* of an object (which can be a dot, a cell or a rectangle of cells), we mean the set made of that object and its symmetric with respect to the main diagonal, to the antidiagonal and to the center.

THEOREM 5.2. Let $\sigma, \tau \in Invol(B_n)$. Then $\sigma \lhd \tau$ in $Invol(B_n)$ if and only if there exists a rectangle $R = [i,j] \times [\sigma(i), \tau(i)]$ such that σ and τ have the same diagram except for the dots in R, and in the rectangles of its orbit, for which the situation, depending on the position of R with respect to the antidiagonal and to the main diagonal, is described in Figure 8, with the same notation as in Figure 7.

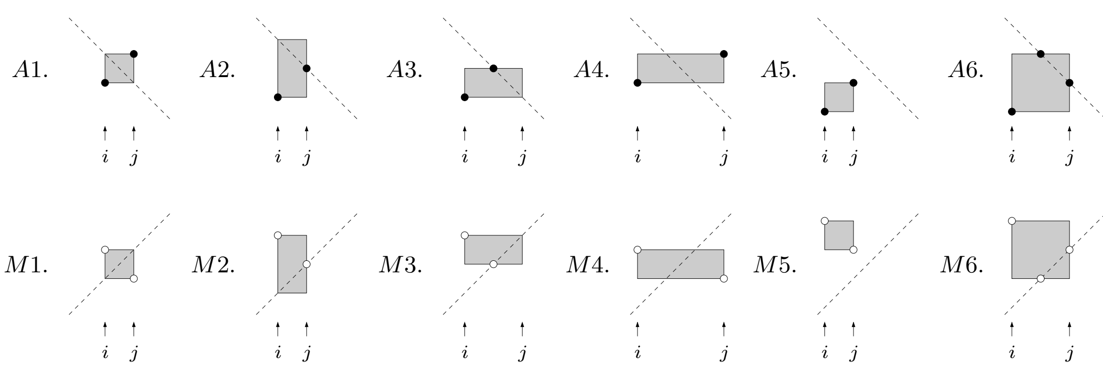
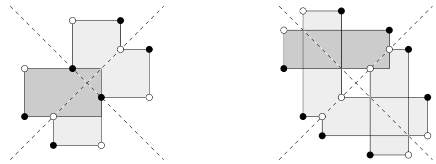


FIGURE 8. Covering relation in $Invol(B_n)$.

The *case* of (σ, τ) is (Ah, Mk), with $h, k \in [6]$, where Ah and Mk refer to the cases of Figure 8. Note that for geometrical reasons not all the 36 pairs are possible cases. In Figure 9 two examples are shown.

THEOREM 5.3. Let $\sigma, \tau \in Invol(D_n)$. Then $\sigma \lhd \tau$ in $Invol(D_n)$ if and only if there exists a rectangle $R = [i,j] \times [\sigma(i), \tau(i)]$, either not central or central not symmetric, such that the same conditions as in Theorem 5.2 are satisfied, with the exceptions, if R is central not symmetric, that:



$(A6, M3)$ $(A4, M6)$

FIGURE 9. Two examples of covering relation in $Invol(B_n)$.

1. in cases $(A6, M1)$ and $(A6, M3)$, picture $A6$ is replaced by picture $A6'$, and in cases $(A1, M6)$ and $(A3, M6)$, picture $M6$ is replaced by picture $M6'$, as shown in Figure 10;

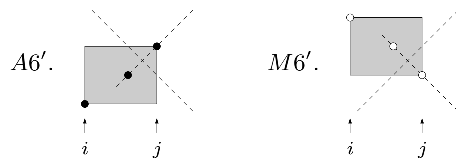
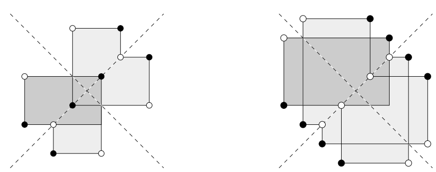


FIGURE 10. Covering relation in $Invol(D_n)$: new cases.

2. in the remaining cases, $(A3, M4)$, $(A4, M3)$, $(A4, M4)$, $(A4, M6)$, $(A6, M4)$, the presence in R of one more dot either of σ or of τ, which is in the orbit of one of those indicated in the pictures, is allowed.

In Figure 11 two examples are shown.



$(A6', M3)$ $(A4, M6)$

FIGURE 11. Two examples of covering relation in $Invol(D_n)$.

In the following the gradedness of the posets is stated.

Theorem 5.4. *The poset $Invol(S_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) + exc(\sigma)}{2},$$

for every $\sigma \in Invol(S_n)$. In particular $Invol(S_n)$ has rank

$$\rho(Invol(S_n)) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 5.5. *The poset $Invol(B_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) + neg(\sigma) + 2dna(\sigma)}{4},$$

for every $\sigma \in Invol(B_n)$. In particular $Invol(B_n)$ has rank

$$\rho(Invol(B_n)) = \frac{n^2+n}{2}.$$

Theorem 5.6. *The poset $Invol(D_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) - neg(\sigma) + 2dna(\sigma)}{4},$$

for every $\sigma \in Invol(D_n)$. In particular $Invol(D_n)$ has rank

$$\rho(Invol(D_n)) = \left\lfloor \frac{n^2}{2} \right\rfloor.$$

5.2. EL-shellability and Eulerianity. Let P be one of $Invol(S_n)$, $Invol(B_n)$ or $Invol(D_n)$. The characterization of the covering relation gives rise in a natural way to the definition of a "standard labelling" of P. In fact, for every $\sigma, \tau \in P$, with $\sigma \lhd \tau$, we call *main rectangle* of the pair (σ, τ) the rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$, mentioned in each of the Theorems 5.1, 5.2 and 5.3. Note that this rectangle necessarily is unique. Then we can give the following definition.

Definition 5.7. The *standard labelling* of P is the function

$$\lambda : \{(\sigma, \tau) \in P^2 : \sigma \lhd \tau\} \to \{(i, j) \in I^2 : i < j\}$$

(where $I = [n]$ if $P = Invol(S_n)$, and $I = [\pm n]$ otherwise) so defined: for every $\sigma, \tau \in P$, with $\sigma \lhd \tau$, if $R = [i, j] \times [\sigma(i), \tau(i)]$ is the main rectangle of (σ, τ), then we set

$$\lambda(\sigma, \tau) = (i, j).$$

To prove that the poset P is EL-shellable, we show that the standard labelling actually is an EL-labelling. This is proved first describing the lexicographically minimal saturated chains, and then showing that those are the unique with the property of having non decreasing labels.

Theorem 5.8. *The poset P is EL-shellable, hence Cohen-Macaulay.*

To prove that the poset P is Eulerian, we show that the standard labelling satisfies the condition of Theorem 2.2, that is, for every $\sigma, \tau \in P$, with $\sigma < \tau$, there is a unique saturated chain from σ to τ with decreasing labels. This is proved starting from the EL-shellability and considering the lexicographically minimal descending chains.

Theorem 5.9. *The poset P is Eulerian.*

6. Conjecture

It is natural to conjecture that our main result actually holds for every Coxeter group.

Conjecture 7.1. *Let W be a Coxeter group. The poset $Invol(W)$ is*

1. *graded, with rank function given by*
$$\rho(w) = \frac{l(w) + al(w)}{2},$$
for every $w \in Invol(W)$;
2. *EL-shellable, hence Cohen-Macaulay;*
3. *Eulerian.*[1]

After a preliminary investigation on the affine Weyl groups (which also have nice combinatorial descriptions), we feel that our techniques may be applied also to this class of Coxeter groups. There is another class of Coxeter groups, which are not Weyl groups, for which the result is valid: the class of dihedral groups.

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[1] In the infinite cases, we mean that every interval $[\hat{0}, x]$ of the poset has the mentioned properties.