Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

A Four-Parameter Partition Identity

Cilanne E. Boulet

ABSTRACT. We present a new partition identity and give a combinatorial proof of our result. This generalizes a result of Andrews' in which he considers the generating function for partitions with respect to size, number of odd parts, and number of parts of the conjugate.

RÉSUMÉ. Nous présentons une nouvelle identité sur les partitions ainsi qu'une démonstration combinatoire de notre résultat. Ceci généralise un résultat d'Andrews au sujet de la série génératrice des partitions relative à trois statistiques: la somme des parts, le nombre de parts impaires et le nombre de parts impaires de la partition conjuguée.

1. Introduction

In [**A**], Andrews considers partitions with respect to size, number of odd parts, and number of odd parts of the conjugate. He derives the following generating function

$$\sum_{\lambda \in \mathrm{Par}} r^{\theta(\lambda)} s^{\theta(\lambda')} q^{|\lambda|} = \prod_{j=1}^{\infty} \frac{(1+rsq^{2j-1})}{(1-q^{4j})(1-r^2q^{4j-2})(1-s^2q^{4j-2})} \tag{1.1}$$

where Par denotes the set of all partitions, $|\lambda|$ denotes the size (sum of the parts) of λ, $\theta(\lambda)$ denotes the number of odd parts in the partition λ, and $\theta(\lambda')$ denotes the number of odd parts in the conjugate of λ. Combinatorial proofs of Andrews' result have also been found by Sills in [**Si**] and by Yee in [**Y**].

We generalize this result and outline a combinatorial proof of our generalization. This gives a simpler combinatorial proof of (1.1) than the ones found in [**Si**] and [**Y**].

2. Main Result

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n, denoted $\lambda \vdash n$. Consider the following weight functions on the set of all partitions:

$$\begin{aligned}
\alpha(\lambda) &= \sum \lceil \lambda_{2i-1}/2 \rceil \\
\beta(\lambda) &= \sum \lfloor \lambda_{2i-1}/2 \rfloor \\
\gamma(\lambda) &= \sum \lceil \lambda_{2i}/2 \rceil \\
\delta(\lambda) &= \sum \lfloor \lambda_{2i}/2 \rfloor .
\end{aligned}$$

1991 *Mathematics Subject Classification.* Primary 05A17; Secondary 11P81.

Key words and phrases. partitions, bijective proof, partition identity.
The author was supported in part by NSERC.

Also, let a, b, c, d be (commuting) indeterminants, and define

$$w(\lambda) = a^{\alpha(\lambda)} b^{\beta(\lambda)} c^{\gamma(\lambda)} d^{\delta(\lambda)}.$$

For instance, if $\lambda = (5, 4, 4, 3, 2)$ then $\alpha(\lambda)$ is the number of a's in the following diagram for λ, $\beta(\lambda)$ is the number of b's in the diagram, $\gamma(\lambda)$ is the number of c's in the diagram, and $\delta(\lambda)$ is the number of d's in the diagram. Moreover, $w(\lambda)$ is the product of the entries of the diagram.

$$\begin{array}{ccccc} a & b & a & b & a \\ c & d & c & d & \\ a & b & a & b & \\ c & d & c & & \\ a & b & & & \end{array}$$

These weights were first introduced by Stanley in [**St**].

Let $\Phi(a, b, c, d) = \sum w(\lambda)$, where the sum is over all partitions λ, and let $\Psi(a, b, c, d) = \sum w(\lambda)$, where the sum is over all partitions λ with distinct parts. We obtain the following product formulas for $\Phi(a, b, c, d)$ and $\Psi(a, b, c, d)$:

Theorem 2.1.

$$\Phi(a, b, c, d) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}$$

Corollary 2.2.

$$\Psi(a, b, c, d) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

If we transform $\Phi(a, b, c, d)$ by sending $a \mapsto rsq$, $b \mapsto r^{-1}sq$, $c \mapsto rs^{-1}q$, and $d \mapsto r^{-1}s^{-1}q$, a straightforward computation gives Andrews' result (1.1).

Our main result is a generalization of Theorem 2.1 and Corollary 2.2. It is the corresponding product formula in the case where we restrict the the parts to some congruence class (mod k) and we restrict the number of times those parts can occur. Let R be a subset of positive integers congruent to $i(\text{mod } k)$ and let ρ be a map from R to the even positive integers. Let $\text{Par}(i, k; R, \rho)$ be the set of all partitions with parts congruent to $i(\text{mod } k)$ such that if $r \in R$, then r appears as a part less than $\rho(r)$ times. Let $\Phi_{i,k;R,\rho}(a, b, c, d) = \sum_\lambda w(\lambda)$ where the sum is over all partitions in $\text{Par}(i, k; R, \rho)$.

For example, $\text{Par}(1, 1; \emptyset, \rho)$ is Par, the set of all partitions. Also, if we let R be the set of all positive integers and ρ map every positive integer to 2, then $\text{Par}(1, 1; R, \rho)$ is the set of all partitions with distinct parts. These are the two cases found in Theorem 2.1 and Corollary 2.2.

Theorem 2.3.

$$\Phi_{i,k;R,\rho}(a, b, c, d) = ST$$

where

$$S = \prod_{j=1}^{\infty} \frac{(1 + a^{\lceil \frac{(j+1)k+i}{2} \rceil} b^{\lfloor \frac{(j+1)k+i}{2} \rfloor} c^{\lceil \frac{jk+i}{2} \rceil} d^{\lfloor \frac{jk+i}{2} \rfloor})}{(1 - a^{\lceil \frac{jk+i}{2} \rceil} b^{\lfloor \frac{jk+i}{2} \rfloor} c^{\lceil \frac{jk+i}{2} \rceil} d^{\lfloor \frac{jk+i}{2} \rfloor})(1 - a^{jk} b^{(j-1)k} c^{jk} d^{(j-1)k})}$$

and

$$T = \prod_{r \in R} (1 - a^{\lceil \frac{r}{2} \rceil \frac{\rho(r)}{2}} b^{\lfloor \frac{r}{2} \rfloor \frac{\rho(r)}{2}} c^{\lceil \frac{r}{2} \rceil \frac{\rho(r)}{2}} d^{\lfloor \frac{r}{2} \rfloor \frac{\rho(r)}{2}})$$

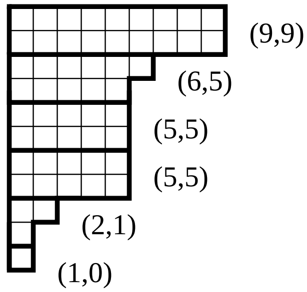

FIGURE 1. $\lambda = (9, 9, 6, 5, 5, 5, 5, 5, 2, 1, 1)$ decomposes into blocks $\{(9, 9), (6, 5), (5, 5), (5, 5), (2, 1), (1, 0)\}$

3. Combinatorial Proof of these Results

The proof of Theorem 2.3 is a slight modification of the proof of Theorem 2.1 and Corollary 2.2. The details of these proofs can be found in the complete version of the paper available at math.CO/0308012.

SKETCHED PROOF OF THEOREM 2.1. Consider the following class of partitions:

$$\mathcal{R} = \{\lambda \in \text{Par} : \lambda_{2i-1} - \lambda_{2i} \leq 1\}.$$

We are restricting the difference between a part of λ which is at an odd level and the following part of λ to be at most 1.

To find the generating function for partitions in $\mathcal{R}$ under weight $w(\lambda)$ we will decompose $\lambda \in \mathcal{R}$ into blocks of height 2. Since the difference of parts is restricted to either 0 or 1 at odd levels, we can only get two types of block: for any $k \geq 1$, we can have a block with two parts of length k, and, for any $k \geq 1$, we can have a block with one part of length k and then other of length $k-1$. Figure 1 shows an example of such a decomposition.

By considering the weights of these parts, we obtain the following generating function:

$$\sum_{\lambda \in \mathcal{R}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}.$$

Let $\mathcal{S}$ be the set of partitions whose conjugates have only odd parts each of which is repeated an even number of times. We give a bijection $f : \mathcal{R} \times \mathcal{S} \to \text{Par}$, such that $\mathcal{S}$ contributes exactly the missing terms. The map f consists of taking the partition whose columns are the union of the columns of the partition from $\mathcal{R}$ and the columns of the partition for $\mathcal{S}$. An example is shown in Figure 2. The weight of the partition from $\mathcal{S}$ does not change when f is applied and contributes

$$\prod_{j=1}^{\infty} \frac{1}{1 - a^j b^j c^{j-1} d^{j-1}},$$

the terms missing in $\sum_{\lambda \in \mathcal{R}} w(\lambda)$.

□

SKETCHED PROOF OF COROLLARY 2.2. To obtain this corollary, consider the following bijection. Let $\mathcal{D}$ denote the set of partitions with distinct parts and let $\mathcal{E}$ denote the set of partitions whose parts appear an even number of times. Then we have a bijection $g : \text{Par} \to \mathcal{D} \times \mathcal{E}$ with $g(\lambda) = (\mu, \nu)$ defined as follows. Suppose λ has k parts equal to i. If k is even then ν has k parts equal to i, and if k is odd then ν has $k-1$ parts equal to i. The parts of λ which were not removed to form ν, at most one of each cardinality, give μ. It is clear that under this bijection, $w(\lambda) = w(\mu)w(\nu)$.

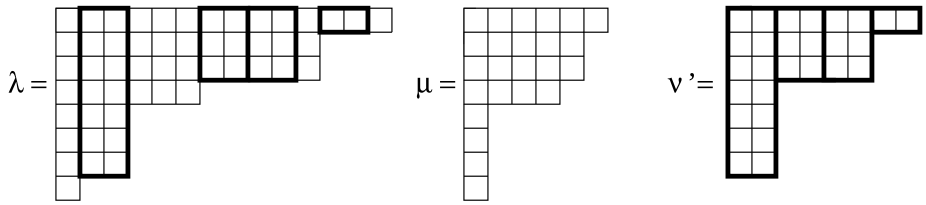

FIGURE 2. $\lambda = (14, 11, 11, 6, 3, 3, 3, 1)$ and $f(\mu, \nu') = \lambda$ where $\nu = (7, 7, 3, 3, 3, 3, 1, 1)$ and $\mu = (6, 5, 5, 4, 1, 1, 1, 1)$

By considering the weights of partition in $\mathcal{E}$ we get that

$$\Phi(a,b,c,d) = \Psi(a,b,c,d) \prod_{j=1}^{\infty} \frac{1}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}$$

and the result follows. □

ACKNOWLEDGEMENT. The author would like to thank Richard Stanley for encouraging her to work on this problem.

References

[A] G. E. Andrews, *On a partition function of Richard Stanley*, to appear in the Electronic Journal of Combinatorics, available from http://www.math.psu.edu/andrews/.

[Si] A. V. Sills, *A combinatorial proof of a partition identity of Andrews and Stanley*, to appear in the International Journal of Mathematics and Mathematical Sciences, available from http://www.math.rutgers.edu/∼asills/.

[St] R. P. Stanley, *Some remarks on sign-balanced and maj-balanced posets*, to appear in Advances in Applied Math., available from http://www-math.mit.edu/∼rstan/.

[Y] A. J. Yee, *On partition functions of Andrews and Stanley*, submitted for publication, available from http://www.math.psu.edu/yee/.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA, 02139
E-mail address: `cilanne@math.mit.edu`