Sharper estimates for the number of permutations avoiding a layered or decomposable pattern

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Abstract

We present two methods that for infinitely many patterns q provide better upper bounds for the number $S_n(q)$ of permutations of length n avoiding the pattern q than the recent general result of Marcus and Tardos. While achieving that, we define an apparently new decomposition of permutations .

Résumé

Nous montrons deux méthodes qui prouvent des bornes supérieurs pour les nombres $S_n(q)$ dénombrant les permutations de longueur n évitant le motif q. Nos méthodes peuvent être appliquées pour un nombre infini des motifs, et les bornes obtenues sont meilleur que celles découlant du résultat récent de Marcus et Tardos. Nous allons également définir une decomposition des permutations qui semble d'être nouvelle.

1 Introduction

Let $S_n(q)$ be the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q. The long-standing Stanley-Wilf conjecture claimed that for any given pattern q, there exist an absolute constant c_q so that $S_n(q) < c_q^n$ for all n. See [3] or [6] for the relevant definitions.

The Stanley-Wilf conjecture was open for more than 20 years. It has recently been proved by a spectacular, yet simple argument [11]. That argument actually proved a stronger conjecture,

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the Füredi-Hajnal conjecture [8] , which was shown to imply the Stanley-Wilf conjecture three years ago in [9].

Perhaps because the Stanley-Wilf conjecture was proved as a special case of a stronger conjecture, the obtained upper bound seems far away from what is thought to be the truth. Indeed, it is proved in [11] (along with another, stronger conjecture from [1]), that if q is a pattern of length k, then

$$S_n(q) \leq c_q^n \qquad \text{where} \qquad c_q \leq 15^{2k^4\binom{k^2}{k}}. \tag{1}$$

For the rest of this paper, k will denote the length of the pattern q. For instance, if $k = 3$, then the above result shows only that $c_q \leq 15^{13608}$, while in fact it is well-known [3] that $c_q = 4$ is sufficient. Therefore, it seems reasonable to think that in the near future significant research will be devoted to the improvement of this upper bound. In fact, R. Arratia [2] conjectures that $c_q \leq (k-1)^2$ for any patterns q. There are several patterns, for instance, monotone patterns, for which $(k-1)^2$ is known [10] to be the smallest possible value of c_q.

In this paper we present two methods that can prove upper bounds for certain patterns from the upper bounds for certain shorter patterns.

For instance, one of our methods will provide upper bounds for all layered patterns, which are patterns consisting of decreasing subsequences that increase among the layers. The other one will work for all decomposable permutations. The arguments will be remarkably simple, compared to previous work on layered patterns. While our upper bounds will still be significantly weaker than the conjectured $(k-1)^{2n}$, they will not be doubly exponential, like the result shown in (1).

We mention that it follows from subsequent work of present author ([4], to be presented at the subsequent Pattern Avoiding Permutations conference) that for any layered pattern q of length k, we have $L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)} \geq (k-1)^2$. In other words, in the asymptotic sense, layered patterns are at least as easy to avoid as the monotone patterns. Present paper complements those results by bounding $L(q)$ from above.

2 The Pattern 1324

We explain our method by demonstrating it on the pattern 1324, but this is only to make our discussion easier to read. The crucial properties of this pattern for our purposes are that it starts with its minimal entry, it ends with its maximal entry, and that if we remove either of these entries, we get a pattern (132 or 213) for which a good exponential upper bound is known.

Our crucial definition is the following.

Definition 2.1 *We will say that an n-permutation $p = p_1 p_2 \cdots p_n$ is* orderly *if $p_1 < p_n$. We will say that p is* dual orderly *if the entry 1 of p precedes the maximal entry n of p.*

It is clear that p is orderly if and only if p^{-1} is dual orderly.

The importance of these permutations for us is explained by the following lemma.

Lemma 2.2 *The number of orderly (resp. dual orderly) 1324-avoiding n-permutations is less than $8^n/4(n+1)$.*

Proof: It suffices to prove the statement for orderly permutations as we can take inverses after that to get the other statement.

The crucial idea is this. Each entry p_i of p has at least one of the following two properties.

(a) $p_i \geq p_1$;

(b) $p_i \leq p_n$.

In words, everything is either larger than the first entry, or smaller than the last, possibly both. This would *not* be the case had we not required that p be orderly.

Define $S = \{i | p_i \geq p_1\}$ and $T = \{i | p_i < p_1\}$. Then S and T are disjoint, $S \cup T = [n]$, and crucially, if $i \in T$, then, in particular, $p_i < p_n$. Recall that for any pattern q of length three, we have $S_n(q) = C_n = \binom{2n}{n}/(n+1)$, and that the numbers C_n are the well-known Catalan numbers [3]. Let $|S| = s$ and $|T| = t$. Then we have C_{s-1} possibilities for the substring p_S of entries belonging to indices in S, and $C_t = C_{n-s}$ possibilities for the substring p_T of entries belonging to indices in S. Indeed, p_S starts with its smallest entry, and then the rest of it must avoid 213, (otherwise, together with p_1, a 1324-pattern is formed) and p_T must avoid 132 (otherwise, together with p_n, a 1324-pattern is formed). Finally, we have $\binom{n-2}{s-2}$ choices for the set of indices that we denoted by S. Once s is known, we have no liberty in choosing the *entries* p_i, ($i \in S$) as they must simply be the s largest entries.

Therefore, the total number of possibilities is

$$\sum_{s=2}^{n} \binom{n-2}{s-2} C_{s-1} C_{n-s} < 2^{n-2} \sum_{s=2}^{n} C_{s-1} C_{n-s} < 2^{n-2} C_n < \frac{8^n}{4(n+1)}.$$

◇

We have seen that it helps in our efforts to limit the number of 1324-avoiding permutations if a large element is preceded by a small one. To make good use of this observation, look at all non-inversions of a generic permutation $p = p_1p_2\cdots p_n$; that is, pairs (i, j) so that $i < j$ and $p_i < p_j$. Find the non-inversion (i, j) for which

$$\max_{(i,j)}(j - i, p_j - p_i) \tag{2}$$

is maximal. If there are several such pairs, take one of them, say the one that is lexicographically first. Call this pair (i, j) the *critical pair* of p.

Recall that an entry of a permutation is called a *left-to-right minimum* if it is smaller than all entries on its left. Similarly, an entry is a *right-to-left maximum* if it is larger than all entries on its right.

The following proposition is obvious, but it will be important in what follows, so we explicitly state it.

Proposition 2.3 *For any permutation $p_1p_2\cdots p_n$, the critical pair (i, j) is always a pair in which p_i is a left-to-right minimum, and p_j is a right-to-left maximum.*

The following definition proved to be useful for treating 1324-avoiding permutations in the past [7].

Definition 2.4 *We say that two permutations are* in the same class *if they have the same left-to-right minima, and the same right-to-left maxima, and they are in the same positions.*

Example 2.5 The permutations 3612745 and 3416725 are in the same class.

Proposition 2.6 *The number of nonempty classes of n-permutations is less than 9^n.*

Proof: Each such class contains exactly one 1234-avoiding permutation, namely the one in which all entries that are not left-to-right minima or right-to-left maxima are written in decreasing order. As it is well-known that $S_n(1234) < 9^n$, the statement is proved. $\diamond$

To achieve our goal, it suffices to find a constant C so that each class contains at most C^n 1324-avoiding n-permutations.

Choose a class A. By Proposition 2.3, we see that the critical pair of any permutation $p \in A$ is the same as it depends only on the left-to-right minima and the right-to-left maxima, and those are the same for all permutations in A.

We will now find an upper bound for the number of 1324-avoiding n-permutations in A.

For symmetry reasons, we can assume that in the critical pair of $p \in A$, we have $j-i \geq p_j - p_i$, in other words, the maximum (2) is attained by $j - i$.

We will now reconstruct p from its critical pair. First, all entries that precede p_i must be larger than p_j. Indeed, if there existed $k < i$ so that $p_k < p_j$, then the pair (j, k) would be a "longer" non-inversion than the pair (i, j), contradicting the critical property of (i, j). Similarly, all entries that are on the right of p_j must be smaller than p_i.

This shows that all entries p_t for which $p_i < p_t < p_j$ must be positioned between p_i and p_j, that is, $i < t < j$ must hold for them. However, if $j - i = p_j - p_i + b$, where b is a *positive* integer, then we can select b additional entries that will be located between p_i and p_j. We will call them *excess entries*; that is, an excess entry is an entry p_u that is located between p_i and p_j, but does *not* satisfy $p_i < p_u < p_j$.

The good news is that we do not have too many choices for the excess entries. No excess entry can be smaller than $p_i - b$. Indeed, if we had $p_u < p_i - b$ for an excess entry, then for the pair (u, j) the value defined by (2) would be larger than for the pair (i, j), contradicting the critical property of (i, j). By the analogous argument, no excess entry can be larger than $p_j + b$. Therefore, the set of b excess entries must be a subset of the at-most-$(2b)$-element set $(\{p_i - b, p_i - b + 1, \cdots, p_i - 1\} \cup \{p_j + 1, p_j + 2, \cdots, p_j + b\}) \cap [n]$. Therefore, we have at most $\binom{2b}{b}$ choices for the set of excess entries, and consequently, we have $\binom{2b}{b}$ choices for the set of $j - i - 1 + b$ elements that are located between p_i and p_j. As $p_i < p_j$, the partial permutation $p_i p_{i+1} \cdots p_j$ is orderly, and certainly 1324-avoiding. Therefore, by Lemma 2.2, we have less than $8^{j-i+1}/4(j - i + 1)$ choices for it once the set of entries has been chosen.

This proves that altogether, we have less than

$$4^b \cdot \frac{8^{j-i+1}}{4(j - i + 1)} < 32^{j-i}$$

possibilities for the string $p_i p_{i+1} \cdots p_j$. We used the fact that $b \leq j - i - 1$ as b counts the excess entries between i and j. Note that we have some room to spare here, so we can say that the above upper bound remains valid even if we include the permutations in which the maximum was attained by (p_i, p_j), and not by (i, j).

We can now remove the entries $p_{i+1} \cdots p_{j-1}$ from our permutations. This will split our permutations into two parts, p_L on the left, and p_R on the right. It is possible that one of them is empty. We know exactly what entries belong to p_L and what entries belong to p_R; indeed each entry of p_L is larger than each entry of p_R. Therefore, we do not loose any information if we relabel the entries in each of p_L and p_R so that they both start at 1 (we call this the *standardization of the strings*). This will not change the location and relative value of the left-to-right minima and right-to-left maxima either. The string $p_{i+1} \cdots p_{j-1}$ should not be standardized, however, as that would result in loss of information.

See Figure 1 for the diagram of a generic permutation, its critical pair, and the strings p_L and p_R.

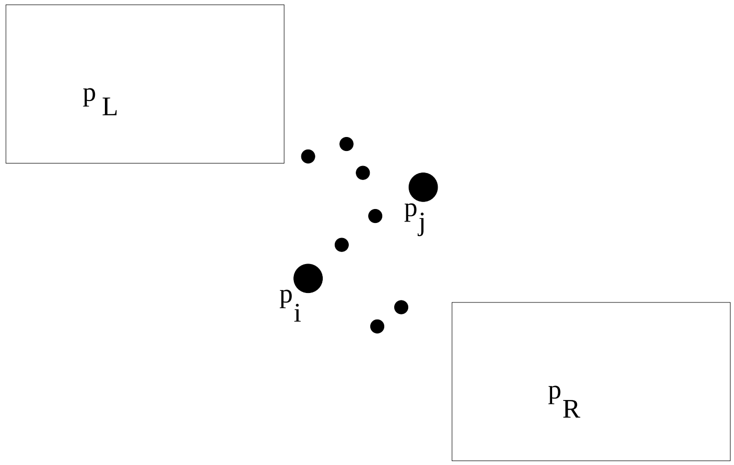

Figure 1: A generic permutation and its critical pair.

Then we iterate our procedure. That is, we find the critical pairs of p_L and p_R, denote them by (i_L, j_L) and (i_R, j_R), and prove, just as above, that there are at most $32^{j_L - i_L}$ possibilities for the string between i_L and j_L, and there are at most $32^{j_R - i_R}$ possibilities for the string between i_R and j_R. Then we remove these strings again, cutting our permutations into more parts, and so on, building a binary tree-like structure of strings. Note that the leaves of this tree will be orderly or dual orderly permutations.

Note that this procedure of decomposing of our permutations is injective. Indeed, given the standardized string p_L, the partial permutation $p_i \cdots p_j$, and the standardized string p_R, we can easily recover p.

Iterating this algorithm until all entries of p that are not left-to-right minima or right-to-left maxima are removed, we prove the following.

Lemma 2.7 *The number of 1324-avoiding n-permutations in any given class A is at most 32^n.*

Proof: The above description of the removal of entries by our method shows that the total number of 1324-avoiding permutations in A is less than

$$32^{\sum_k j_k - i_k}$$

where the summation ranges through all intervals (i_k, j_k) whose endpoints were critical pairs at some point. As these interiors of these intervals are all disjoint, $\sum_k j_k - i_k = n - 1$, and our claim is proved. $\diamond$

Now proving the upper bound for $S_n(1324)$ is a breeze.

Theorem 2.8 *For all positive integers n, we have $S_n(1324) < 288^n$.*

Proof: As there are less than 9^n classes and less than 32^n n-permutations in each class that avoid 1324, the statement is proved. $\diamond$

Note that an alternative way of proving our theorem would have been by induction on n. We could have used the induction hypothesis for the class A' that is obtained from A by making p_i and p_j consecutive entries by omitting all positions between them, and setting their values so that each entry on the left of p_i is larger than each entry after p_j.

Finally, we point out that using specific properties of the pattern 1324, we could have decreased the upper bound a little further, but that is not our goal here. Our goal is to find a method that works for many patterns.

3 Layered Patterns

As a generalization, we look at patterns like 14325, 154326, and so on, that is, patterns that start with 1, end with their maximal entry k, and consist of a decreasing sequence all the way between.

Theorem 3.1 *Let $k \geq 4$, and let $q_k = 1\,k-1\,k-2\cdots 2\,k$. Then for all positive integers n, we have*

$$S_n(q_k) < 72^n(k-2)^{2n} = \left(72(k-2)^2\right)^n.$$

Proof: We again look at orderly permutations first. If p is orderly and avoids q_k, then define S, p_S and T, p_T just as in the proof of Lemma 2.2. Then p_S starts with its smallest entry, and the rest must avoid $q'_k = k-2\cdots 2\,1\,k-1$, whereas p_T must avoid $q''_k = 1\,k-1\,k-2\cdots 2$. It is known that $S_n(q'_k) = S_n(q''_k) = S_n(12\cdots(k-1)) < (k-2)^{2n}$, so it follows, just as in Lemma 2.2 that the number of orderly (resp. dual orderly) n-permutations that avoid q_k is less than $(2(k-2)^2)^n$.

The transition from orderly permutations to generic permutations is identical to what we described in the case of $q_4 = 1324$. $\diamond$

Let us now find an upper bound for all layered patterns. Recall that a permutation is called layered if it is the concatenation of decreasing subsequences $d_1, d_2, \cdots, d_t$ so that each entry of d_i is less than each entry of d_j for all $i < j$. For instance, 321546 is a layered pattern. We will use the following definition and lemma, first used in [7].

Definition 3.2 *Let q be a pattern, and let y be an entry of q. Then* to replace y *by the pattern w is to add $y-1$ to all entries of w, then to delete y and to succesively insert the entries of w at its position.*

Lemma 3.3 *("replacing an element by a pattern") Let q be a pattern and let y be an entry of q so that for any entry x preceding y we have $x < y$ and for any entry z preceded by y we have $y < z$. Suppose that $S_n(q) < K^n$ for some constant K and for all n.*

Let w be a pattern of length m starting with 1 and ending with m so that $S_n(w) < C^n$ holds for all n, for some constant C. Let q' be the pattern obtained by replacing the entry y by the pattern w in q. Then $S_n(q') < (4CK)^n$.

Proof: Take an n-permutation p which avoids q'. Suppose it contains q. Then consider all copies of q in p and consider their entries y. Color these entries blue, that is, and entry is blue if it can play the role of y in a copy of q. Clearly, these entries must form a permutation which does not contain w. For suppose they do, and denote y_1 and y_m the first and last elements of that purported copy of w. Then the initial segment of the copy of q which contains y_1 followed by the y_2 through y_{k-1} and the ending segment of the copy of q which contains y_k would form a copy of q'.

Therefore, if p avoids q', then it either avoids q, or the substring of its blue entries avoids w. As we have at most 2^{n-1} choices for the set of blue entries, and at most 2^{n-1} choices for their positions, this shows that less than $(4C)^{n-1} \cdot K^n + K^n < (4CK)^n$ permutations of length n can avoid q'. $\diamond$

Now let $Q = Q(a_1, a_2, \cdots, a_t)$ be the layered pattern whose layers are of length $a_1, a_2, \cdots, a_t$. It is then clear that Q is contained in the pattern $Q' = Q(1, a_1, 1, a_2, 1, \cdots 1, a_t, 1)$. Therefore,

$$S_n(Q) \leq S_n(Q'). \tag{3}$$

On the other hand, Q' can be obtained if we take $q_{a_1} = Q(1, a_1, 1)$, then replace the last entry of this pattern by the pattern $q_{a_2} = Q(1, a_2, 1)$, then replace the last entry of the obtained pattern by $q_{a_3} = Q(1, a_3, 1)$, and so on.

Then it follows by iterated applications of Theorem 3.1 and Lemma 3.3 that

$$S_n(Q') \leq 4^{tn} \cdot 72^{tn} \prod_{i=1}^{t} a_i^{2n} = 288^{tn} \prod_{i=1}^{t} a_i^{2n}.$$

So by (3), we have

$$S_n(Q) \leq 288^{tn} \prod_{i=1}^{t} a_i^{2n}.$$

For any fixed layered pattern Q, the number of layers t will be fixed, so 288^{tn} is simply exponential. While $\prod_{i=1}^{t} a_i$ can be as large as $3^{k/3}$, which makes c_Q an exponential function of k, it is still not doubly exponential, unlike the general result (1).

4 Further Generalizations

We can find a somewhat more general application of our methodology. For a pattern q, let $1q$ denote the pattern obtained from q by adding one to each of the entries and then writing 1 to the front, and let qm denote the pattern that we obtain from q by simply affixing a new maximal element to the end of q. Finally, let $1qm$ denote the pattern $(1q)m = 1(qm)$. So for example, if $q = 2413$, then $1q = 13524$, and $qm = 24135$, while $1qm = 135246$.

Theorem 4.1 *Let q be a pattern so that there exist constants c_1 and c_2 satisfying $S_n(1q) < c_1^n$ and $S_n(qm) < c_2^n$ for all n. Then for all positive integers n, we have*

$$S_n(1qm) < 72^n \cdot (\max(c_1, c_2))^n.$$

Proof: Similar to the proof of Theorem 3.1. The upper bound for orderly permutations is $2^n \cdot (\max(c_1, c_2))^n$, the number of classes is 9^n, and the remaining 4^n comes from the choices for the excess entries. $\diamond$

This theorem permits a little improvement on the general upper bound (1) for all patterns that start with their minimal entry and end in their maximal entry.

Corollary 4.2 *If $r = 1qm$, then*

$$c_r \leq 72^n \cdot 15^{2(k-1)^4 \binom{(k-1)^2}{k-1}}.$$

Proof: Follows from (1), applied to the patterns $1q$ and qm, and Theorem 4.1. $\diamond$

While this last corollary is not a significant improvement as far as principles are concerned, numerically it still decreases c_r by several orders of magnitude.

Another improvement comes from a variation of Lemma 3.3.

Lemma 4.3 *Let q be as in lemma 3.3 and let y be its* last *entry. Replace y by any pattern w which starts with its smallest entry. Then for the pattern q' obtained this way, we have*

$$S_n(q') < (4CK)^n.$$

Proof: This can be proved exactly as Lemma 3.3. The special values and positions of y obviate the omitted restrictions. $\diamond$

Let us call a permutation v *decomposable* if $v = LR$ so that all entries of L are less than all entries of R, for some nonempty strings L and R. Let $v = LR$ be decomposable, and insert the entry $|L|+1 = h$ immediately after L, increasing all entries of R by one. Call the obtained permutation pattern q'. Then q' is nothing else but the pattern $q = Lh$ in which we replace the entry h by the pattern $1R$. Therefore, Lemma 4.3 applies, and we have

$$S_n(v) \leq S_n(q') < (4c_q c_R)^n.$$

This leads to significant numerical improvements over the general result, particularly if L and R are also decomposable.

References

[1] N. Alon, E. Friedgut, On the number of permutations avoiding a given pattern, *Journal of Combinatorial Theory Series A,* **89** (2000), 133-140.

[2] R. Arratia, On the Stanley-Wilf Conjecture for the Number of Permutations Avoiding a Given Pattern. *Electronic J. Combin.*, 6 (1): N1, 1999.

[3] M. Bóna, A Walk Through Combinatorics, *World Scientific,* 2002.

[4] M. Bóna, The limit of a Stanley-Wilf sequence is not always an integer, and layered patterns beat monotone patterns. *Preprint, arXiv, math.CO/0403502.*

[5] M. Bóna, The Solution of a Conjecture of Stanley and Wilf for all layered patterns, *Journal of Combinatorial Theory Series A,* **85** (1999) 96-104.

[6] M. Bóna, Exact enumeration of 1342-avoiding permutations; A close link with labeled trees and planar maps, *Journal of Combinatorial Theory, Series A,* **80** (1997), 257-272.

[7] M. Bóna, Exact and Asymptotic Enumeration of Permutations with Subsequence Conditions, *PhD thesis, Massachusetts Institute of Technology*, 1997.

[8] Z. Füredi, P. Hajnal, Davenport-Schinzel theory of matrices. *Discrete Math.* 103 (1992), 233-251.

[9] M. Klazar, The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. *Formal Power Series and Algebraic Combinatorics*, Springer, Berlin, 250-255, 2000.

[10] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Advances in Mathematics*, **41** (1981), 115-136.

[11] A. Marcus, G. Tardos, A linear bound on Excluded Permutation matrices, *preprint.*