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Decomposable Compositions and Ribbon Schur Functions

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Abstract. We describe recent results, obtained in collaboration with Hugh Thomas and Stephanie van Willigenburg [**1**], which provide a complete description of when two ribbon Schur functions are identical.

Résumé. Nous présentons des résultats récents, obtenus en collaboration avec Hugh Thomas et Stephanie van Willigenburg [**1**], qui permettent de déterminer légalité de deux fonctions Schur ruban.

Extended Abstract

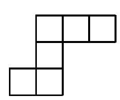
An important basis for the space of symmetric functions of degree n is the set of classical Schur functions s_λ, where λ runs over all partitions of n. For example, the skew Schur functions $s_{\lambda/\mu}$ can be expressed in terms of these by means of the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ by

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

These same coefficients give the expressions for the product of two Schur functions, as well as the multiplicity of irreducible representations of the symmetric group in the tensor product of two irreducibles. Thus there is some interest in determining relations among the $c^\lambda_{\mu\nu}$.

A particular type of skew Schur functions are those corresponding to *ribbon* or *border strip* shapes λ/μ. These are connected shapes with no 2×2 square. The resulting skew Schur functions $s_{\lambda/\mu}$ are called *ribbon Schur functions.*

Ribbons of size n are in one-to-one correspondence with compositions β of size n by setting β_i equal to the number of boxes in the i-th row from the bottom. For example, the skew diagram 422/11



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is a ribbon, corresponding to the composition 213. We will henceforth indicate ribbon Schur functions by means of the compositions corresponding to their ribbon shapes. Thus $s_{422/11}$ will be denoted s_{213}.

We address the question of when two ribbon Schur functions are identical; that is, for compositions β and γ of n, when is it true that $s_\beta = s_\gamma$? When equality holds, we automatically get Littlewood-Richardson coefficient identities of the form

$$c^{\lambda}_{\mu,\nu} = c^{\rho}_{\eta,\nu}$$

for all partitions ν of n, whenever ribbon skew shapes λ/μ and ρ/η correspond to β and γ.

To this end, we define an equivalence relation on the set of compositions of n. For $\beta \vDash n$, let F_β denote the fundamental quasisymmetric function indexed by β. (In [**2**], these functions are denoted L_β.) Since the F_β, $\beta \vDash n$, form a basis for the quasisymmetric functions of degree n, any *symmetric* function, in particular, can be written $F = \sum_{\beta \vDash n} c_\beta F_\beta$.

Definition 1.1. For compositions $\beta, \gamma \vDash n$, we say β and γ are *equivalent*, denoted $\beta \sim \gamma$, if for all symmetric functions $F = \sum c_\alpha F_\alpha$, $c_\beta = c_\gamma$.

For a composition α, we denote by $\lambda(\alpha)$ the unique partition whose parts are the components of β in weakly decreasing order. We write $\alpha \geq \beta$ if α is a *coarsening* of β, that is, α is obtained from β by adding consecutive components. If $\beta \vDash n$ then always $n \geq \beta$. Let $\mathcal{M}(\beta)$ be the *multiset* of partitions determined by all coarsenings of β, that is,

$$\mathcal{M}(\beta) = \{\lambda(\alpha) \mid \alpha \geq \beta\}.$$

It is easy to see that $\mathcal{M}(\beta) = \mathcal{M}(\beta^*)$, where β^* is the reversal of the composition β.

Finally, we define a way of composing two compositions as follows. If $\alpha \vDash n$ and $\beta \vDash m$, then we wish to define $\alpha \circ \beta \vDash nm$. Let $\beta = \beta_1\beta_2\cdots\beta_k$. For $\alpha = n$, then $\alpha \circ \beta = n \circ \beta$ is the composition

$$\underbrace{\beta_1 \cdots \beta_{k-1}(\beta_k + \beta_1)\beta_2 \cdots (\beta_k + \beta_1)\beta_2 \cdots \beta_k}_{n \text{ times}}$$

which is nearly the concatenation of n copies of the composition β, except each pair of adjacent terms $\beta_k\beta_1$ are added. For $\alpha = \alpha_1\alpha_2\cdots\alpha_l$, then $\alpha \circ \beta$ is the usual concatenation of the l compositions $\alpha_i \circ \beta$:

$$\alpha \circ \beta = \alpha_1 \circ \beta \cdot \alpha_2 \circ \beta \cdots \alpha_l \circ \beta.$$

For example $12 \circ 12 = 12132$.

Our main result is

Theorem 1.2. *The following are equivalent for a pair of compositions $\beta, \gamma \vDash n$:*

(1) $s_\beta = s_\gamma$,
(2) $\beta \sim \gamma$
(3) $\mathcal{M}(\beta) = \mathcal{M}(\gamma)$,
(4) *for some k,*

$$\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k \quad \text{and} \quad \gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k,$$

and, for each i, either $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$.

Thus, for example, $s_{12132} = s_{13212} = s_{23121} = s_{21231}$, and these four equal no others.

The last condition shows that the size of the equivalence class of β is 2^r, where r is the number of nonpalindromic factors in the unique nontrivial irreducible factorization of β. We always have, for example, that $s_\beta = s_{\beta^*}$.

References

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