Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Chromatic Polynomials and Representations of the Symmetric Group

Norman Biggs

The chromatic polynomial $P(G;k)$ is the function which gives the number of ways of colouring a graph G when k colours are available. The fact that it is a polynomial function of k is essentially a consequence of the fact that, when k exceeds the number of vertices of G, not all the colours can be used. Another quite trivial property of the construction is that the names of the k colours are immaterial; in other words, if we are given a colouring, then any permutation of the colours produces another colouring. In this talk I shall outline some theoretical developments, based on these simple facts and some experimental observations about the complex roots of chromatic polynomials of 'bracelets'.

A 'bracelet' $G_n = G_n(B, L)$ is formed by taking n copies of a graph B and joining each copy to the next by a set of links L (with $n+1=1$ by convention). The chromatic polynomial of G_n can be expressed in the form

$$P(G_n;k) \ = \ \sum_{\pi} m_{B,\pi}(k)\,\mathrm{tr}(N_L^{\pi})^n.$$

The sum is taken over all partitions π such that $0 \leq |\pi| \leq b$, where b is the number of vertices of B. The terms $m_{B,\pi}(k)$ are polynomials in k, and they are independent of L. When B is the complete graph K_b the relevant polynomials $m_\pi(k)$ are given by a remarkably simple formula, and when B is incomplete they can be expressed in terms of the $m_\pi(k)$ with $|\pi| \leq b$. The entries of the matrices N_L^{π} are also polynomials in k, but they do depend on L. In order to calculate these entries we construct explicit bases for certain irreducible modules, corresponding to the Specht modules of representation theory.

References

(1) N.L.Biggs: Equimodular curves, *Discrete Math.* 259 (2002) 37-57.
(2) N.L. Biggs, M.H. Klin, P. Reinfeld, Algebraic methods for chromatic polynomials, *European J. Combinatorics* 25 (2004) 147-160.
(3) N.L. Biggs, Specht modules and chromatic polynomials, *CDAM Research Report Series* LSE-CDAM-2003-06.

London School of Economics