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A New Representation of Formal Power Series

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ABSTRACT. This paper is dedicated to the genesis arising at the boundary between the theory of formal power series (FPS) and combinatorics.

Similarly to combinatorics where any rational sequence of natural numbers $\{r_k\}_{k\geq 0}$ is representable for all k in the form

(0.1) $$r_{k+n} = \sum_{i=1}^{n} r_{k+n-i} X_{n-i}$$

where X_j – are, generally speaking, complex numbers (Berstel,Reutenauer, [**BR**]), we prove that any rational FPS r is representable in the form (1) where $r_s = \sum_{|w|=s}(r,w)w$, and X_j are elements of some special skew field. As a trivial consequence of such a representation were obtained: 1)truthfulness of Eilenberg's Equality Theorem [**E**], decidability of the equivalence problem of finite multitape deterministic automata (Rabin,Scott [**RS**]) and decidability of problem of whether two given morphisms are equivalent on regular language, (Culik, Salomaa [**CS**]); 2) more simply formulated and proved the results from monographs on FPS (Salomaa, Soittola [**SS**], Berstel, Reutenauer [**BR**], Kuich, Salomaa [**KS**]); 3) solved partial cases of the problem of existence for an inverse element of Hadamard product and others; 4) provided 3 Conjectures and 10 Open problems.

The conclusion contains a complete comparative analysis of the attempts to utilize linear recurrence in theory of FPS by other authors.

RÉSUMÉ.

We use the standard notations from monographs Berstel, Reuteneuer [**BR**] and Cohn [**Coh**]. In particular, it will be assumed that $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_t\}$ is a finite alphabet, $\Sigma^{-1} = \{\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_t^{-1}\}$, ε is empty word and unity in semigroup Σ^* and group G, generated by Σ, $\emptyset$ is empty set and zero in semirings and fields, generated by Σ, $\underline{\varepsilon}, \underline{\sigma}_i, \underline{\sigma}_i^{-1}$ are corresponding characteristic FPS, $\boldsymbol{k}$ is commutative zero-divizor-free semiring embeddable in commutative field $\boldsymbol{K}$ (this includes the semirings $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$).

According to Salomaa, Soittola [**SS**], every FPS $r \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ can be represented as a behaviour of $\boldsymbol{k} - \Sigma^*$-automaton

$$\mathfrak{A} =< \{q_1, q_2, \dots, q_n\}, A, q_1, F >$$

where $A \in \boldsymbol{k}^{n\times n} < \Sigma >$ - transition matrix, q_1-initial state, $F \in \boldsymbol{k}^{n\times 1} < \{\underline{\varepsilon}, \emptyset\} >$ - final states:

$$r_{\mathfrak{A}} = \sum_{i=0}^{\infty} (A^i F)_1.$$

We denote $q_i^{(j)} = (A^j F)_i$, then $r_{\mathfrak{A}} = \sum_{i=0}^{\infty} q_1^{(i)}$ and

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$$q_i^{(j)} = \sum_{s=1}^{n} A_{is} q_s^{(j-1)} \tag{0.3}$$

Let us consider a system of n equations with $(n+1)$ unknowns $X_0, X_1, \ldots, X_n$:

$$\left\{ q_i^{(n)} X_n = \sum_{j=1}^{n} q_i^{(n-j)} X_{n-j},\ i = \overline{1,n} \right. \tag{0.4}$$

and show that it always has a non-zero solution in Malcev-Neumann skew field $\boldsymbol{K}((G))$ of FPS with well-ordered support.

We solve the system (3) following the usual Gauss algorithm by successive excluding unknowns $X_0, X_1, \ldots, X_n$. On step 0 all coefficients of unknowns are $q_i^{(j)} \in \boldsymbol{k} \ll \Sigma^* \gg$ and of course are elements of $\boldsymbol{K}((G))$. Let us assume that on step i an equation for X_i has the form of

$$q_{in} X_n = q_{i(n-1)} X_{n-1} + \ldots + q_{ii} X_i,\ q_{is} \in \boldsymbol{K} \ll G \gg,\ s = \overline{n,i} \tag{0.5}$$

and we can compute a leading term for every q_{is}.

Suppose $q_{ii} \neq \emptyset$ (otherwise we can exchange i-th column with one of $n-1, ..., i+1$; if all $q_{is} = \emptyset$ for $s = \overline{i, n-1}$ then assume $X_n = \emptyset$ and go to an equation for X_{i+1}). Then for all non-zero q_{is} denote $q_{is} = \alpha_{is} + q'_{is}$ where α_{is} is a leading term in q_{is}. It follows that after multiplying both parts of the equation on α_{ii}^{-1} and solving it for X_i we obtain

$$X_i = \left(-\alpha_{ii}^{-1} q'_{ii}\right)^* \left(\alpha_{ii}^{-1}\alpha_{in} + \alpha_{ii}^{-1} q'_{in}\right) X_n - \cdots - (\ldots) X_{i+1} \tag{0.6}$$

where the bracket content $(\ldots)$ is analogous to the coefficient of X_n and is not provided for the sake of simplicity. Substitute equation (5) into the remaining equations for $X_j,\ j = \overline{1, i-1}$:

$$\begin{aligned} (\alpha_{jn} + q'_{jn} \quad &- \quad (\alpha_{ji} + q'_{ji})(-\alpha_{ii}^{-1} q'_{ii})^* (\alpha_{ii}^{-1}\alpha_{in} + \alpha_{ii}^{-1} q'_{in})) X_n = \\ &= \quad (\cdots) X_{n-1} + \cdots + (\cdots) X_{i+1} \end{aligned} \tag{0.7}$$

Since $supp(\alpha_{ii}^{-1} q'_{ii}) > \varepsilon$ then leading term of the coefficient of X_n should be searched for in $\alpha_{jn} - \alpha_{ji}\alpha_{ii}^{-1}\alpha_{in}$. If the coefficient equals $\emptyset$ then we take next in ascending order elements from $supp(q'_{jn}), supp(q'_{ji}), supp(\alpha_{ii}^{-1} q'_{in})$ and so on. This process is constructive (see Lewin [**L**], Cohn [**Coh**]), so the inductive hypothesis holds true – at the beginning of next step of Gauss algorithm all coefficients of unknowns $X_n, X_{n-1}, \ldots X_{i+1}$ will be again from $\boldsymbol{K}((G))$ with known leading terms.

At the last step for the equation $q_{nn} X_n = q_{n(n-1)} X_{n-1}$ we have:

(i) if $q_{n(n-1)} \neq \emptyset$ that is $q_{n(n-1)} = \alpha_{n(n-1)} + q'_{n(n-1)}$ then assume
$X_n = \underline{\varepsilon}, X_{n-1} = \left(-\alpha_{n(n-1)}^{-1} q'_{n(n-1)}\right)^* \alpha_{n(n-1)}^{-1} q_{nn}$;

(ii) if $q_{n(n-1)} = \emptyset$, then assume $X_n = \emptyset, X_{n-1} = \underline{\varepsilon}$. So we proved

Theorem 1. A solution of the system (3) in $\boldsymbol{K}((G))$ exists always in the form:

$$(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_p, \underline{\varepsilon}, \emptyset, \ldots, \emptyset),\ 1 \leq p \leq n-1 \tag{0.8}$$

while some $\tilde{X}_i$ also can be $\emptyset$. ■

We prove the main theorem of the paper.

THEOREM 2. *For all* $k \in \mathbf{N}$ *holds*

$$q_i^{(n+k)} = \sum_{j=1}^{n} q_i^{(n+k-j)} \tilde{X}_{n-j},\ i = \overline{1,n} \tag{0.9}$$

Proof. For $k = 0$ the statement is proved - suppose it is true for k. Then
$q_i^{(n+k+1)} \overset{(2)}{=} \sum_{j=1}^{n} A_{ij} q_j^{(n+k)} = \sum_{j=1}^{n} A_{ij} \sum_{l=1}^{n} q_j^{(n+k-l)} \tilde{X}_{n-l} =$
$= \sum_{l=1}^{n} \left(\sum_{j=1}^{n} A_{ij} q_j^{(n+k-l)} \right) \tilde{X}_{n-l} \overset{(2)}{=} \sum_{l=1}^{n} q_i^{(n+k+1-l)} \tilde{X}_{n-l}$.■

DEFINITION 1. *We call a representation of FPS* q_i *in the form (8)* a linear recurrence representation *(further referred shortly as LRR), vector-solution (7)* - a stencil, $q_i^{(j)}$ - *j-th* layer *of FPS* q_i. ■

EXAMPLE 1. Following the considerations about the solving of system (3) one can find that FPS $s = (\underline{a}^2(\underline{ab})^*\underline{b}^2(\underline{ab})^*)^*$ has LRR

$s_4 = \underline{a}^2\underline{b}^2, s_3 = s_2 = s_1 = \emptyset, s_0 = \underline{\varepsilon},$

$s_{n+5} = s_{n+4} \cdot \emptyset + s_{n+3}(\underline{b}^{-2}\underline{ab}^3 + \underline{ab}) + s_{n+2} \cdot \emptyset + s_{n+1}(\underline{a}^2\underline{b}^2 - \underline{ab}^{-1}\underline{ab}^3) + s_n \cdot \emptyset$.■

Having analyzed the process of solving system (3) it is not difficult to prove:

THEOREM 3. *There exists a stencil with coefficients from the set* $\{-1, 0, 1\}$ *for a characteristic series of an arbitrary given rational languages.* ■

The opposite is intresting:

OPEN PROBLEM 1. ("Fatou extension") *If : 1) stensil of FPS r has all the coefficients from the set* $\{-1, 0, 1\}$*, 2) layers of r have coefficients from the set* $\{0, 1\}$*, then: r is* $\boldsymbol{N}$*-rational? And* $\boldsymbol{Z}$*-rational? And K-algebraic?*

(we point out to the relationship of this problem with the counter-example Reutenauer [**R**]). ■

COROLLARY 1 (Eilenberg's Equality Theorem [**E**]). Let $\mathfrak{A} =< \{q_1, \ldots, q_n\}, A, q_1, F_1 >$and $\mathfrak{B} =< \{p_1, \ldots, p_m\}, B, p_1, F_2 >$ be $\boldsymbol{k} - \Sigma^*$-automata. Then $r_{\mathfrak{A}} = r_{\mathfrak{B}}$ iff $(r_{\mathfrak{A}}, w) = (r_{\mathfrak{B}}, w)$ for all $w \in \Sigma^*$ of length at most (n+m-1).

Proof. Consider a system of equations:

$$\begin{cases} q_i^{n+m} = \sum_{j=1}^{n+m} q_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1,n} \\ p_i^{n+m} = \sum_{j=1}^{n+m} p_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1,m} \quad ■ \end{cases} \tag{9}$$

DEFINITION 2. *We call the solution of the system (9) and the system itself* a common stencil for automata $\mathfrak{A}$ *and* $\mathfrak{B}$. *Common stencil exists for any finite number of* $\boldsymbol{k} - \Sigma^*$*-automata.* ■

COROLLARY 2. (Equivalence Problem for Multitape Deterministic Finite Autumata, Rabin,Scott [**RS**]) *Two automata* $\mathfrak{A}_1 =< \{q_1, \ldots, q_n\}, \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_k, \delta_1, q_1, F_1 >$ *and* $\mathfrak{A}_2 =< \{p_1, \ldots, p_m\}, \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_k, \delta_2, p_1, F_2 >$ *are equivalent iff the sets of their acceptable words of length at most* $(n + m - 1)$ *are equal.*

Proof. Consider common stencil for automata $\mathfrak{A}$ and $\mathfrak{B}$. As the direct product of fully ordered groups equipped with lexicographic order $\Sigma_1 < \Sigma_2 < \ldots < \Sigma_k$ is still fully ordered group G_k (Passman [**Pa**]), hence this common stencil exists in form of solution for system (9) in $\boldsymbol{Z}((G_k))$. ■

REMARK 1. Harju, Karhumaki [**HK**] result is also in checking up of all words of length at most $(n+m-1)$. To check the equivalence of two finite multitape deterministic automata an exponential time, therefore, is required. At the same time there exist polynomial algorithms for the checking of the equivalence of $\boldsymbol{k} - \Sigma^*$-automata ($O(n^4)$ – Tzeng [**T**], $O(n^3)$ – Archangelsky [**A1**]). This provides a hint that should exist a polynomial algorithm. Indeed not every initial set of layers should be checked up because not all of them in combination with stencil would generate only 'clean' noncommutative polynomials – the ones without σ_i^{-1}. ■

Open problem 2. *How many 'clean' tuples of layers there exist for a given stencil?* ■

Remark 2. Corollary 2 could have been proven more simpler by leaving out the process of finding of stencil and the proof of its existence. According to Hebish, Weinert [**HW**] the semiring of FPS on partially commutative monoids over $\boldsymbol{Z}$ is zero-divisor-free and additivelly- cancellative and multiplicately-left-cancellative. This means that a solution of the system (9) exists over some partially commutative skew field. ■

Corollary 3. (Equivalence Problem for Morphisms on Regular Languages, Culik, Salomaa [**CS**]) *Let L be a regular language, defined by the minimal deterministic automaton $\mathfrak{A} =< \{q_1, \ldots, q_n\}$, $\Sigma, \delta, q_1, F >$, and $h, g : \Sigma^* \to \Delta^*$ be morphisms. If $h(w) = g(w)$ for all $w \in L$ of length at most $(2n-1)$, then $h(w) = g(w)$ for all $w \in L$.*

Proof. One may assume that $\Sigma \cap \Delta = \emptyset$ and letters from Σ and Δ commute. We define the transition function δ_h in 2-tape automaton $\mathfrak{A}_h =< \{q_1, \ldots, q_n\}, \Sigma \cup \Delta, \delta_h, q_1, F >$ as follows $\delta_h(q_i, \sigma_j h(\sigma_j)) = q_k \Leftrightarrow \delta(q_i, \sigma_j) = q_k$. Similarly define δ_g and $\mathfrak{A}_g$. Until common stencil of $\mathfrak{A}_h$ and $\mathfrak{A}_g$ is being built we assume for convenience each $\sigma_j h(\sigma_j)$ and $\sigma_j g(\sigma_j)$ to be one unique letter. Thus the length of common stencil will be $2n$. ■

Remark 3. Proof of Corollary 3 does not use unlike Karhumaki [**K**] an Eihrengeucht's conjecture. Actually we have proven a more stronger result – the decidability of morphism equivalence on regular language with multiplicities of words are taken into consideration.■

Corollary 4. Let $r \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ and p be number of first nonzero element in stencil of r, i.e. $\tilde{X}_0 = ... = \tilde{X}_{p-1} = \emptyset, \tilde{X}_p \neq \emptyset$, $0 \leq p \leq n$. Then

(i) r is identecically zero iff $r_i = \emptyset$ for all $i = \overline{0, (n-1)}$;

(ii) r is polynomial iff $r_i = \emptyset$ for all $i = \overline{p, (n-1)}$;

(iii) r is ultimately constant iff $r_i = c\underline{\Sigma}^i$ for all $i = \overline{p, (n-1)}$;

(iv) r is identically constant iff $r_i = c\underline{\Sigma}^i$ for all $i = \overline{0, (n-1)}$.

Proof. Trivial combinatorical considerations.■

Remark 4. Proof of Corollary 4 does not use, unlike Salomaa, Soittola [**SS**], Kuich, Salomaa [**KS**] Hadamard product and morphisms. ■

Let us investigate more scrupulously how of the summands with negative powers of letters in $\sum r_i \tilde{X}_i$ annihilate. In the first approximation it can be done by tracing down step-by-step how only 'clean' non-commutative polynomials are left in the following examples.

Example 2 (Berstel, Reutenauer [**BR**]). *FPS $s = \sum_w |w|_a w = \underline{\Sigma}^* \underline{a} \underline{\Sigma}^*$ has the follows LRR:*

$$s_0 = \emptyset, s_1 = \underline{a}$$

$$s_{n+2} = s_{n+1}(2\underline{a} + \underline{b} + \underline{a}^{-1}\underline{b}\ \underline{a}) + s_n(-\underline{a}^2 - 2\underline{b}\ \underline{a} - \underline{b}\ \underline{a}^{-1}\ \underline{b}\ \underline{a})$$ ■

Example 3 (Berstel, Reutenauer [**BR**]). *FPS*
$s = \sum_w (|w|_a - |w|_b) w = \underline{\Sigma}^*(\underline{a} - \underline{b})\underline{\Sigma}^*$ has the follows LRR:

$$s_0 = \emptyset, s_1 = \underline{a} - \underline{b},$$

$$s_{n+2} = s_{n+1} \cdot 2(\underline{a}^{-1}\underline{b})^*(\underline{a} - \underline{a}^{-1}\underline{b}^2) + s_n(\underline{a} + \underline{b})(\underline{a} + \underline{b} - 2(\underline{a}^{-1}\underline{b})^*(\underline{a} - \underline{a}^{-1}\underline{b}^2))$$ ■

Example 4 (Reutenauer [**R**]). *FPS*

$s = \sum_w \left(\alpha^{2(|w|_x - |w|_y)} + \alpha^{2(|w|_y - |w|_x)}\right) w = \left(\alpha^2 \underline{x} + \alpha^{-2} \underline{y}\right)^* + \left(\alpha^{-2} \underline{x} + \alpha^2 \underline{y}\right)^*$
$\alpha = \frac{1}{2}(\sqrt{5}+1)$,

has the follows LRR: $s_0 = 2\underline{\varepsilon}, s_1 = 3\underline{x} + 3\underline{y}$,

$s_{n+2} = s_{n+1} \cdot 3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2) + s_n \left(\alpha^{-2}\underline{x} + \alpha^2 \underline{y}\right)\left(\alpha^{-2}\underline{x} + \alpha^2 \underline{y} - \right.$
$\left. -3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2)\right)$ ■

Let us consider arbitrary sequential n-tuple of layers of $r \in \boldsymbol{k}^{rat} << \Sigma^* >>$. One can say that they are n-inert in ring $\boldsymbol{K}((G))$ in several weak sense because $r_{k+n-i} \in \boldsymbol{k} < \Sigma^* >$ (and of course, $r_{k+n-i} \in \boldsymbol{K}((G))$), $\tilde{X}_{n-i} \in \boldsymbol{K}((G))$, but $\sum_{i=1}^n r_{k+n-i}\tilde{X}_{n-i} \in \boldsymbol{k} < \Sigma^* >$. And while the inertia theorem is proved (Bergman [**Ber**], Cohn [**Coh**]) also for ring $\boldsymbol{k} < \Sigma^* >$ in ring $\boldsymbol{K} \ll \Sigma^* \gg$, but not in ring $\boldsymbol{K}((G))$, the following analogue seems to be the case.

CONJECTURE 1. *$\boldsymbol{k} < \Sigma^* >$ is inert in the $\boldsymbol{K}((G))$.* ■

CONJECTURE 2. *Assuming Conjecture 1 is true - would matrix-trivializer M exist such that $M, M^{-1} \in \boldsymbol{K}^{n\times n} \ll \Sigma^* \gg$?* ■

Formulae (8) implies the following formulae for computing the coefficients in LRR:

$$(r_{k+n}, w) = \sum_{\substack{(1)\ 1\le i\le n \\ (2) w_{is}\tilde{w}_{is} = w,\ w_{is}\in\Sigma^*,\ \tilde{w}_{is}\in G}} (r_{k+n-i}, w_{is})(\tilde{X}_{n-i}, \tilde{w}_{is}) \tag{10}$$

The second condition of summing means that $w_{is} = \alpha_{is}\beta_{is},\ \beta_{is}^{-1}\gamma_{is} = \tilde{w}_{is},\ \alpha_{is}, \beta_{is}, \gamma_{is} \in \Sigma^*$. Therefore $|\beta_{is}| \le |w_{is}| = k+n-i$ and number of summands in (10) is limited.

CONJECTURE 3. *Would the length of canceling suffixes and prefixes (like a β_{is}) be limited too for each LRR?*■

OPEN PROBLEM 3 (Archangelsky [**A2**]). *For a given $\tilde{r} \in \boldsymbol{K}((G))$ determine whether the lengths of all negative subwords of words in supp $(\tilde{r})$ are limited (i.e., subwords in alphabet Σ^{-1} only).*■

We apply rule (10) for examining coefficients in the inverse element of Hadamard product.We mean FPS p is *Hadamard inverse* of FPS r iff $r \odot p = \sum_w 1 \cdot w = \underline{\Sigma}^*$. The problem of existence of such an element is still open. All papers on the issue either study FPS on cyclic/commutative semigroups (Cori [**Cor**], Benzaghou [**Ben1, Ben2**],Benzaghou, Bezivin [**BB**], Anselmo, Bertoni [**AB**], Poorten [**Po**]) or simple samples of inversable FPS on $\Sigma^*, |\Sigma| \ge 2$ (Gerardin [**G**]).

THEOREM 4. *Let Σ be alphabet, $|\Sigma| \ge 2$, $\mathfrak{A}_r, \mathfrak{A}_p$ be $\boldsymbol{Q}^+ - \Sigma^*$ – automata which behaviours are FPS r, p and let the coefficients in the common stencil of automata*

$$\begin{cases} \mathfrak{A}_r \\ \mathfrak{A}_p \\ q = \underline{\Sigma} q + \underline{\varepsilon} \end{cases} \tag{11}$$

are in $\mathbf{Q}^+$. Then $r \odot p = \underline{\Sigma}^*$ implies both r, p have a finite image.

Proof.Consider common stencil of automata (11) (t is the sum of states for automata $\mathfrak{A}_r$ and $\mathfrak{A}_p$ plus 1):

$$\begin{cases} r_{n+t} = \Sigma_{i=1}^t r_{n+t-i}\tilde{X}_{t-i} \\ p_{n+t} = \Sigma_{i=1}^t p_{n+t-i}\tilde{X}_{t-i} \\ \underline{\Sigma}^{n+t} = \Sigma_{i=1}^t \underline{\Sigma}^{n+t-i}\tilde{X}_{t-i}, \qquad n \ge 0 \end{cases} \tag{12}$$

According to (12) and (10) a coefficient of the word $w \in supp(r_{n+t})$ in r_{n+t} satisfies the follows:

$$\alpha = \sum_{s\in S} \alpha_s x_s \tag{13}$$

where α_s – coefficients of $supp(r_{n+t-i})$ and x_s – coefficients of $supp(\tilde{X}_{t-i})$, and $|S|$ is finite. Respectively,

$$\frac{1}{\alpha} = \sum_{s \in S} \frac{1}{\alpha_s} x_s \tag{14}$$

$$1 = \sum_{s \in S} 1 \cdot x_s \tag{15}$$

Multiplying (13) and (14) we obtain

$$\alpha \cdot \frac{1}{\alpha} = 1 = \left(\sum_{s\in S} \alpha_s x_s\right)\left(\sum_{s\in S} \frac{1}{\alpha_s} x_s\right) =$$
$$= \sum_{s\in S} x_s^2 + \sum_{i\neq j; i,j\in S}\left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\right) x_i x_j \geq \sum_{s\in S} x_s^2 + 2\sum_{i\neq j;i,j\in S} x_i x_j =$$
$$= \left(\sum_{s\in S} x_s\right)^2 = 1,$$

that is why all $\alpha_i = \alpha_j = \alpha$, i.e. new coefficients do not appear in r_{n+t}. ■

Open problem 4. *Positiveness of all coefficients in all stencils and layers is an essential part of the proof of Theorem 3. In general case this limitation would not exist – therefore one would require to solve (or describe the set of solutions for) the system of Diophantine equations* $\{(13),(14),(15)\}$ $(\alpha_i \in \boldsymbol{N}^+, x_i \in \boldsymbol{Q})$. *For small numbers of unknowns the system above indeed has only trivial solutions. It seems like the class of invertable by Hadamard rational FPS is very narrow.* ■

Method of Theorem 4 may be implememted for the obtaining a necessary condition for the solution of following

Open problem 5 (Restivo,Reutenauer [**RR**]). *Let s be a FPS with integer coefficients and p a prime number; if $\sum_w p^{(s,w)} w$ is rational, then so are s and $\sum_w p^{-(s,w)}$.*■

Corollary 5. If $s \in \boldsymbol{Q}^{rat} \ll \Sigma^* \gg$, $p \in \boldsymbol{N}$, $s_1 = \sum_w p^{(s,w)}$, $s_2 = \sum_w p^{-(s,w)}$, $s_1, s_2 \in (\boldsymbol{Q}^+)^{rat} \ll \Sigma^* \gg$ and the coefficients of the common stencil of automata

$$\begin{cases} \mathfrak{A}_{s_1} \\ \mathfrak{A}_{s_2} \\ q = \underline{\Sigma} q + \underline{\varepsilon} \end{cases}$$

are in $\mathbf{R}^+$, then s, s_1, s_2 have a finite image.■

Let us try for a given LRR build a FPS, a representation of which the former is:

$$r = \sum_{i=0}^{n-1} r_i + \sum_{i=n}^{\infty} r_i = \sum_{i=0}^{n-1} r_i + \sum_{i=n}^{\infty}\sum_{j=1}^{n} r_{i-j}\tilde{X}_{n-j} =$$
$$= \sum_{i=0}^{n} r_i + \sum_{j=1}^{n}\sum_{i=j-1}^{\infty} r_i \tilde{X}_{j-1} =$$
$$= \sum_{i=0}^{n-1} - \sum_{j=1}^{n-1}\sum_{s=0}^{j-1} r_s \tilde{X}_j + \sum_{j=1}^{n}\sum_{i=0}^{\infty} r_i \tilde{X}_{j-1} =$$
$$= r_0 + \sum_{i=1}^{n-1}(r_i - \sum_{s=0}^{i-1} r_s \tilde{X}_i) + r\sum_{j=1}^{n}\tilde{X}_{j-1} \tag{16}$$

Solve this equation for r:

$$r = (r_0 + \sum_{i=1}^{n-1}(r_i - \sum_{s=0}^{i-1}\tilde{X}_i))(\sum_{j=1}^{n}\tilde{X}_{j-1})^* \tag{17}$$

Unarguably we took too much liberty when applying limit to both parts of identity (16). It still needs to be proved that the obtained expression is indeed the sum of r_i and only them. Because of size limit we would not do that but do illustrate using Example 2 that it is true:

$$(\underline{a}+\underline{b})^*\underline{a}(\underline{a}+\underline{b})^* = s \overset{(17)}{=} (s_1 + s_0 - s_0\tilde{X}_1)(\tilde{X}_1 + \tilde{X}_2)^* =$$
$$= \underline{a}(2\underline{a} + \underline{b} + \underline{a}^{-1}\underline{ba} - \underline{a}^2 - 2\underline{ba} - \underline{ba}^{-1}\underline{ba})^* = ((\underline{\varepsilon} - 2\underline{a} - \underline{b} - \underline{a}^{-1}\ \underline{b}\ \underline{a} +$$
$$+ \underline{a}^2 + 2\underline{b}\ \underline{a} + \underline{b}\ \underline{a}^{-1}\ \underline{b}\ \underline{a})\underline{a}^{-1})^{-1} = ((\underline{a}^{-1} - \underline{\varepsilon} - \underline{b}\ \underline{a}^{-1})(\underline{\varepsilon} - \underline{a} - \underline{b}))^{-1} =$$

$= ((\underline{\varepsilon} - \underline{a} - \underline{b})\underline{a}^{-1}(\underline{\varepsilon} - \underline{a} - \underline{b}))^{-1} = (\underline{a} + \underline{b})^* \underline{a}(\underline{a} + \underline{b})^*$. ■

Brzozowski, Cohen [**BC**] studied a decompositions of rational languages into star languages : $P = R^*S$.One may ask about such decomposition in $\boldsymbol{K}((G))$. Of course, arbitrary regular language R may be trivially decomposed into star FPS in $\boldsymbol{K}((G))$: $\underline{R} = \underline{P}^*(\underline{\varepsilon} - \underline{P})\underline{R}$, where P is arbitrary regular language too. It is interesting to study a nontrivial case. Consider a common stencil of two arbitrary FPS in the form (17). It implies

THEOREM 5. Each two FPS $r, p \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ have a representation in $\boldsymbol{K}((G))$ with nontrivial common star factor : $r = \tilde{r}_1 \tilde{s}^*, p = \tilde{p}_1 \tilde{s}^*$. ■

Judging by appearance the regular expression (17) does not represent FPS from $\boldsymbol{k} \ll \Sigma^* \gg$, since it contains inverse elements from Σ^{-1} and $\boldsymbol{K}$. The transition matrix for the corresponding $\boldsymbol{K} - (\Sigma \cup \Sigma^{-1})^*$ - automaton would contain elements from Σ^{-1} and $\boldsymbol{K}$ too − although the behavior of this automaton would be exactly FPS r that is without Σ^{-1} and $\boldsymbol{K} \backslash \boldsymbol{k}$.

OPEN PROBLEM 6 ("Fatou extensions"). Let $A \in \boldsymbol{Z}^{n \times n} < \Sigma \cup \Sigma^{-1} >$ but all layers of FPS $r = \sum_{i=0}^{\infty} (A^i)_{1,n}$ are in $\boldsymbol{N} < \Sigma^* >$. Would $r \in \boldsymbol{N}^{rat} \ll \Sigma^* \gg$ be true? And $\boldsymbol{Z}^{rat} \ll \Sigma^* \gg$? And $K^{alg} \ll \Sigma^* \gg$? ■

OPEN PROBLEM 7 (Berstel etc. [**BBCPP**]). *Does a function $n \to r_n$ preserve a rationality? That is if $\{a_n\}_{n \geq 0}$ is a rational sequence of natural numbers, r is rational FPS then would $\sum_{i=0}^{\infty} r_{a_i}$ be rational?* ■

OPEN PROBLEM 8. Based on given LRR of FPS p, q build LRR of : $p^*, p + q, pq$,
$p \odot q, p \sqcup\!\sqcup q$. ■

OPEN PROBLEM 9. *Describe the set of all stencils of given rational FPS.* ■

OPEN PROBLEM 10. *Stencils in their turn are rational FPS. One can be built their LRR and so on. What can be said about the process ?* ■

Conclusion

Many researchers guessed about the existence of a linear dependency between the current value of FPS and a limited number of previous ones, but have failed to express it in a convenient universal form that would allow to obtain trivially results above. Thus for example,

Restivo, Reutenauer [**RR**]: *FPS $s \in K \ll \Sigma^* \gg$ is rational iff for any word x there is a common linear recurrence relation over K satisfied by all the sequences $\{(s, ux^n v)\}_{n \geq 0}, u, v \in \Sigma^*$.*

The below listed authors used for stencil the same ring as for represented FPS , what undercut readability and applications:

Salomaa, Soittola [**SS**]: *Assume $r \in K^{rat} \ll \Sigma^* \gg$ and N is rank of r. Show that if $|w_0| = N$ then there are words $w_1, \dots, w_N$ and elements $c_1, \dots, c_N$ of K such that $|w_i| < N, i = \overline{1, N}$ and for all words w:*

$(r, ww_0) = c_1(r, ww_1) + \dots + c_N(r, ww_N)$

Berstel, Reutenauer [**BR**]: *For any rational series S of rank n there exist a prefix-closed set P of n elements, with an associated prefix set C, and coefficients $\alpha_{c,p} (c \in C, p \in P)$ such that, for all words w and all $c \in C$:*

$$(S, cw) = \sum_{p \in P} \alpha_{c,p}(S, pw).$$

or limited the domain of definition of the linear relation :

Eilenberg [**E**]: $f = \sum a_n z^n$ *is rational iff the following "recursion formula" holds for all t sufficiently large:*

$$a_{t+m} + c_1 a_{t+m} + \ldots + c_m a_t = 0.$$

On the other hand Cohn [**Coh**] did not lost universality and convenience but to achieve that he had to 'maim' previous layers:

A series $r \in K((X; \alpha))$ is rational iff there exist integer m, n_0 and elements $c_1, \dots, c_m \in K$ such that for all $n > n_0$:

$$r_n = r_{n-1}^{\alpha} c_1 + r_{n-2}^{\alpha^2} c_2 + \ldots + r_{n-m}^{\alpha^m} c_m$$

As for Varricchio [**V**] – he did not go beyond the statement of a linear dependency for initial interval of FPS:

Let $s \in K^{rat} \ll \Sigma^* \gg, \Sigma^{[N]}$ *be the set of words of* Σ *whose length is less then or equal to* N*,* μ *be matrix interpretation of* S*. Then one can effectively compute an integer* N*, depending on* S *with the property that for any* $u \in \sum^{[N+1]}$ *there exist a set* $T = \{\sigma_v\}_{v \in \sum^{[N]}} \subseteq K$ *such that* $\mu(u) = \sum_{\sigma_v \in T} \mu(v)$.

It is very strange that author failed to discover the attempts to use linear recurrence in FPS on free commutative monoid $c(\Sigma^*), |\Sigma| \geq 2$. According to Kuich, Salomaa [**KS**] $K^{alg} \ll c(\Sigma^*) \gg = K^{rat} \ll c(\Sigma^*) \gg$. Therefore many K - algebraic FPS can be studied with the help of LRR.

As one see the proposed approach contrary to the predecessors is systematic and handy. As a indirect proof of that fact is a large number of correlations between FPS and combinatorics collected by the author and left out the scope of this work.

References

[BR] BerstelJ., Reutenauer C., *Rational Series and Their Languages*, Springer-Verlag(1988).

[E] Eilenberg S., *Automata, Languages, and Machines*, Vol.A, Acad. Press (1974).

[RS] Rabin M.O., Scott D., *Finite Automata and their decision problems*, IBM J.Res.Develop. 3 (1959), 114-125. 7

[CS] Culik K. II, Salomaa A.,*On the decidability of homomorphism equivalence for languages*, J.Comput.Syst.Sci, 17(1978) , 163-175.

[SS] Salomaa A., Soittola M., *Automata-Theoretic Aspects of Formal Power Series*, Springer-Verlag, 1978.

[KS] Kuich W., Salomaa A.*Semirings, Automata, Languages*, 1985.

[Coh] Cohn P.M.,*Free Rings and Their Relations*, 2nd edn. , Acad. Press (1985).

[L] Lewin J.*Fields of fractions for group algebras of free groups*, Trans.AMS , 192 (1974), 339-346.

[R] Reutenauer C., *On a question of S.Eilenberg*, Theor.Comp.Sci., 5(1977), 219.

[Pa] Passman D.S.,*The Algebraic Structure of Group Rings*, Wiley (1977).

[HK] Harju T., Karhumaki J., *The equivalence problem of multitape finite automata*, Theor.Comp.Sci., 78 (1991) 347-355.

[T] Tzeng W.-G.,*A polynomial-time algorithm the equivalence of probabolistic automata*, FOCS (1989).

[A1] Archangelsky K.V., *Efficient algorithm for checking multiplicity equivalence for the finite zet-sigma-automata*, Lect.NotesComp.Sci., 2450 (2002) ,283-289.

[HW] Hebish U., Weinert H.J., *Generalized semigroup semirings which are zero-divisor-free or multiplicatively left-cancellative*, Theor.Comp.Sci., 92 (1992), 269-289.

[K] Karhumaki J.,*Equations over finite sets of words and equivalence problems in automata theory*, Thepr.Comp.Sci., 106(1993), 103-118.

[Ber] Bergman G.M., *Commuting elements in free algebras and related topics in ring theory*, Thesis, Harvard University, 1967.

[A2] Archangelsky K.V., *The length of homogeneous subwords in rational formal power series*, Bull. Eur. Ass.Theor. Comp. Sci., 71(2000) 253.

[Cor] Cori R.,*Formules d'inversion de series formelles*, Ecole de Printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) 55-67.

[Ben1] Benzaghou B., *Sur les series rationnelles a une variable*, Ecole de Printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) 1-3.

[Ben2] Benzaghou B., *Algebres de Hadamard*, Bull. Soc. Math. France, 98 (1970), 209-252.

[BB] Benzaghou B., Bezivin J.-P., *Propertes algebriques de suites differetielement finies*, Bull.Soc.math.France, 120 (1992), 327-346.

[AB] Anselmo M., Bertoni A., *On 2pfa's and the Hadamard quotient of formal power series*, Bull. Belg.Math.Soc. 1 (1994), 165-173.

[Po] Poorten van der A.J., *Solution de la conjecture de Pisot sur le quotient de Hadamard de deux fractions rationnelles*, C.R.Acad.Sc. Paris 306, serie I (1988), 97-102.

[G] Gerardin K., *Quotient de Hadamard de series rationnelles en variables non commutatives*,Ecole de printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) ,119-134.

[RR] Restivo A.,Reutenauer C., *Rational languagrs and the Burnside problem*, Theor.Comp.Sci., 40(1985),13-30.

[BC] Brzozowski J.A.,Cohen R., *On decomposition of regular events*, J.ACM, 16(1969),132-144.

[BBCPP] Berstel J.,Boasson L.,Carton O.,Petazzoni B.,Pin J.-E.,*Operations preserving recognizable languages*, FCT'03.

[V] Varricchio S.,*Rational series with coefficients in commutative ring*, LITP 91.38,Institut Blaise Pascal, 1991.

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