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Equi-distribution over Descent Classes of the Hyperoctahedral Group (Extended Abstract)

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Abstract. A classical result of MacMahon shows that the length function and the major index are equi-distributed over the symmetric group. Foata and Schützenberger gave a remarkable refinement and proved that these parameters are equi-distributed over inverse descent classes, implying bivariate equi-distribution identities. Type B analogues and further refinements and consequences are given in this paper.

Résumé. Un résultat classique de MacMahon montre l'équidistribution de l'indice majeur et de la fonction de longueur sur le groupe symmétrique. Foata et Schützenberger ont donné une amélioration remarquable et montré l'équidistribution sur les classes de descentes inverse, impliquant ainsi des équidistributions bivariante. Les analogues pour le type B et d'autres raffinements et conséquences sont donnés dans cet article.

1. Introduction

Many combinatorial identities on groups are motivated by the fundamental works of MacMahon [**M**]. Let S_n be the symmetric group acting on $1, \ldots, n$. We are interested in a refined enumeration of permutations according to (non-negative, integer valued) combinatorial parameters. Two parameters that have the same generating function are said to be *equi-distributed*. MacMahon [**M**] has shown, about a hundred years ago, that the inversion number and the major index statistics are equi-distributed on S_n (Theorem 2.4 below). In the last three decades MacMahon's theorem has received far-reaching refinements and generalizations. Bivariate distributions were first studied by Carlitz [**C**]. Foata [**F**] gave a bijective proof of MacMahon's theorem; then Foata and Schützenberger [**FS**] applied this bijection to refine MacMahon's identity, proving that the inversion number and the major index are equi-distributed over subsets of S_n with prescribed descent set of the inverse permutation (Theorem 2.5 below). Garsia and Gessel [**GG**] extended the analysis to multivariate distributions. In particular, they gave an independent proof of the Foata-Schützenberger theorem, relying on an explicit and simple generating function (see Theorem 2.8 below). Further refinements and analogues of the Foata-Schützenberger theorem were found recently, involving left-to-right minima and maxima [**RR, FH2**] and pattern-avoiding permutations [**RR, AR2**]. For a representation theoretic application of Theorem 2.5 see [**Roi**].

Since the length and descent parameters may be defined via the Coxeter structure of the symmetric group, it is very natural to look for analogues of the above theorems in other Coxeter groups. This is a

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challenging open problem. In this paper we focus on the hyperoctahedral group B_n, also known as the classical Weyl group of type B.

Despite the fact that an increasing number of enumerative results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [**Br, FH1, Re3, Re4, Sta1**]) and that several "major index" statistics have been introduced and studied for B_n (see, e.g., [**CF1, CF2, CF3, Re1, Re2, Ste, FK**]), no generalization of MacMahon's result to B_n has been found until the recent paper [**AR1**]. There a new statistic, the *flag major index*, defined in terms of Coxeter elements, was introduced and shown to be equi-distributed with length, which is the natural analogue of inversion number from a Coxeter group theoretic point of view. A search was then initiated for a corresponding "descent statistic" that would allow the generalization to B_n of the Carlitz identity for descent number and major index [**C**], a problem first posed by Foata. In [**ABR1**] we introduced and studied two families of statistics on the hyperoctahedral group B_n, and showed that they give two generalizations of the Carlitz identity. Another solution of Foata's problem, also involving the flag major index, was presented most recently by Chow and Gessel [**CG**]. Combinatorial and algebraic properties of the flag major index were further investigated in [**AR1, HLR, AGR**]. In particular, it was shown to play an important role in the study of polynomial algebras, see [**AR1, ABR2, Ba**].

A natural goal now is to find a type B analogue of the Foata-Schützenberger theorem (Theorem 2.5); namely, to prove the equi-distribution of the flag major index and the length function on inverse descent classes of B_n. This will be carried out by finding a type B analogue of the Garsia-Gessel theorem (Theorem 2.8), which expresses the refined enumeration of the classical major index on shuffle permutations in terms of q-binomial coefficients.

The last digit parameter is involved in several closely related identities on S_n, see e.g. [**AR2, AGR, RR**]. Theorems 4.4 and 4.5 below present a refinement involving the last digit. This refinement implies a MacMahon type theorem for the classical Weyl group of type D, which is the same as the one recently proved in [**BC**]. See Subsection 5.1.

2. Background and Notation

2.1. Notation. Let $\mathbf{P} := \{1, 2, 3, \ldots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ the ring of integers. For $n \in \mathbf{P}$ let $[n] := \{1, 2, \ldots, n\}$, and also $[0] := \emptyset$. Given $m, n \in \mathbf{Z}$, $m \le n$, let $[m, n] := \{m, m+1, \ldots, n\}$. For $n \in \mathbf{P}$ denote $[\pm n] := [-n, n] \setminus \{0\}$. For $S \subset \mathbf{N}$ write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \ldots < a_r$. The cardinality of a set A will be denoted by $|A|$.

For $n, k \in \mathbf{N}$ denote

$$\begin{aligned} [n]_q &:= \frac{1-q^n}{1-q}; \\ [n]_q! &:= \prod_{i=1}^{n} [i]_q \qquad (n \ge 1), \qquad [0]_q! := 1; \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]_q!}{[k]_q![n-k]_q!}. \end{aligned}$$

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that a pair $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $inv(\sigma)$ (respectively, $des(\sigma)$) the number of inversions (respectively, descents) of σ. We also let

$$maj(\sigma) := \sum_{\{i:\, a_i > a_{i+1}\}} i$$

and call it the *major index* of σ.

Let $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$. Denote $m_0 := 0$ and $m_{t+1} := n$. A sequence $\sigma = (a_1, \ldots, a_n)$ is an *M-shuffle* if it satisfies: if $m_i < a < b \le m_{i+1}$ for some $0 \le i \le t$, then $\sigma = (\ldots, a, \ldots, b, \ldots)$ (i.e. a appears to the left of b in σ).

2.2. Binomial Identities. In this subsection we recall some binomial identities which will be used in the proof of Theorem 3.3.

LEMMA 2.1. *For every subset $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$*

$$\prod_{j=1}^{n}(1+q^j) \cdot \begin{bmatrix} n \\ m_1 - m_0, m_2 - m_1, \ldots, m_{t+1} - m_t \end{bmatrix}_q = \tag{2.1}$$

$$\sum_{\{(r_0,\ldots,r_t)\,|\,m_i \le r_i \le m_{i+1}\ (\forall i)\}} \begin{bmatrix} n \\ r_0 - m_0, m_1 - r_0, \ldots, r_t - m_t, m_{t+1} - r_t \end{bmatrix}_{q^2} \cdot q^{\sum_i (r_i - m_i)},$$

where $m_0 := 0$ and $m_{t+1} := n$.

For $t = 0$ (i.e., $M = \emptyset$), identity (2.1) is equivalent to a well-known classical result of Euler, comparing partitions into distinct parts with partitions into odd parts [**An**, Corollary 1.2]. The proof of the lemma is obtained by induction on t, see [**ABR3**, Lemma 3.1].

The following "q-binomial theorem" is well-known.

THEOREM 2.2.

$$\prod_{i=1}^{n}(1+q^i x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^k.$$

2.3. The Symmetric Group. Let S_n be the symmetric group on $[n]$. Recall that S_n is a Coxeter group with respect to the Coxeter generators $S := \{s_i \mid 1 \le i \le n-1\}$, where s_i may be interpreted as the adjacent transposition $(i, i+1)$. The classical combinatorial statistics of $\pi \in S_n$, defined by viewing π as a sequence $(\pi(1), \ldots, \pi(n))$, may also be defined via the Coxeter generators.
For $\pi \in S_n$ let $\ell(\pi)$ be the standard *length* of π with respect to the set of generators S. It is well-known that $\ell(\pi) = inv(\pi)$.
Given a permutation π in the symmetric group S_n, the *descent set* of π is

$$Des(\pi) := \{1 \le i < n \,|\, \ell(\pi) > \ell(\pi s_i)\} = \{1 \le i < n \,|\, \pi(i) > \pi(i+1)\}.$$

The *descent number* of $\pi \in S_n$ is $des(\pi) := |Des(\pi)|$.
The *major index*, $maj(\pi)$ is the sum (possibly zero)

$$maj(\pi) := \sum_{i \in Des(\pi)} i.$$

The *inverse descent class* in S_n corresponding to $M \subseteq [n-1]$ is the set $\{\pi \in S_n \,|\, Des(\pi^{-1}) = M\}$. Note the following relation between inverse descent classes and shuffles.

FACT 2.3. *For every $M \subseteq [n-1]$,*

$$\{\pi \in S_n \mid Des(\pi^{-1}) \subseteq M\} = \{\pi \in S_n \mid (\pi(1), \ldots, \pi(n)) \text{ is an } M\text{-shuffle}\}.$$

MacMahon's classical theorem asserts that the length function and the major index are equi-distributed on S_n.

THEOREM 2.4. (MacMahon's Theorem)

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!.$$

Foata [**F**] gave a bijective proof of this theorem. Foata and Schützenberger [**FS**] applied this bijection to prove the following refinement.

Theorem 2.5. (The Foata-Schützenberger Theorem [**FS**, Theorem 1])
For every subset $B \subseteq [n-1]$,

$$\sum_{\{\pi\in S_n \ \mid \ Des(\pi^{-1})=B\}} q^{\ell(\pi)} = \sum_{\{\pi\in S_n \ \mid \ Des(\pi^{-1})=B\}} q^{maj(\pi)}.$$

This theorem implies

Corollary 2.6.

$$\sum_{\pi\in S_n} q^{\ell(\pi)} t^{des(\pi^{-1})} = \sum_{\pi\in S_n} q^{maj(\pi)} t^{des(\pi^{-1})}$$

and

$$\sum_{\pi\in S_n} q^{\ell(\pi)} t^{maj(\pi^{-1})} = \sum_{\pi\in S_n} q^{maj(\pi)} t^{maj(\pi^{-1})}.$$

An alternative proof of Theorem 2.5 may be obtained using the following classical fact [**Sta2**, Prop. 1.3.17].

Fact 2.7. *Let $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$. Denote $m_0 := 0$ and $m_{t+1} := n$. Then*

$$\sum_{\{\pi\in S_n \mid \ Des(\pi^{-1})\subseteq M\}} q^{inv(\pi)} = \begin{bmatrix} n \\ m_1 - m_0, m_2 - m_1, \ldots, m_{t+1} - m_t \end{bmatrix}_q.$$

Garsia and Gessel proved that a similar identity holds for the major index.

Theorem 2.8. [**GG**, Theorem 3.1] *Let $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$. Denote $m_0 = 0$ and $m_{t+1} = n$. Then*

$$\sum_{\{\pi\in S_n \mid \ Des(\pi^{-1})\subseteq M\}} q^{maj(\pi)} = \begin{bmatrix} n \\ m_1 - m_0, m_2 - m_1, \ldots, m_{t+1} - m_t \end{bmatrix}_q.$$

Combining this theorem with Fact 2.7 implies Theorem 2.5.

2.4. The Hyperoctahedral Group. We denote by B_n the group of all bijections σ of the set $[\pm n] := [-n, n] \setminus \{0\}$ onto itself such that

$$\sigma(-a) = -\sigma(a) \qquad (\forall a \in [\pm n]),$$

with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\sigma \in B_n$ then write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(i) = a_i$ for $1 \le i \le n$, and let

$$\begin{aligned} inv(\sigma) &:= inv(a_1, \ldots, a_n), \\ Des_A(\sigma) &:= Des(a_1, \ldots, a_n), \\ des_A(\sigma) &:= des(a_1, \ldots, a_n), \\ maj_A(\sigma) &:= maj(a_1, \ldots, a_n), \\ Neg(\sigma) &:= \{i \in [n] : \ a_i < 0\}, \end{aligned}$$

and

$$neg(\sigma) := |Neg(\sigma)|.$$

It is well-known (see, e.g., [**BB**, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_0 := [-1, 2, \ldots n]$$

and

$$s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots n] \qquad (1 \le i < n).$$

This gives rise to two other natural statistics on B_n (similarly definable for any Coxeter group), namely

$$\ell_B(\sigma) := min\{r \in \mathbf{N} : \ \sigma = s_{i_1} \ldots s_{i_r} \quad \text{for some } \ i_1, \ldots, i_r \in [0, n-1]\}$$

(known as the *length* of σ) and

$$des_B(\sigma) := |Des_B(\sigma)|,$$

where the *B-descent set* $Des_B(\sigma)$ is defined as

$$Des_B(\sigma) := \{i \in [0, n-1] \mid \ell_B(\sigma s_i) < \ell_B(\sigma)\}.$$

REMARK 2.9. *Note that for every* $\sigma \in B_n$

$$Des_A(\sigma) = Des_B(\sigma) \setminus \{0\}.$$

There are well-known direct combinatorial ways to compute the statistics for $\sigma \in B_n$ (see, e.g., [**BB**, Propositions 8.1.1 and 8.1.2] or [**Br**, Proposition 3.1 and Corollary 3.2]), namely

$$\ell_B(\sigma) = inv(\sigma) - \sum_{i \in Neg(\sigma)} \sigma(i)$$

and

$$des_B(\sigma) = |\{i \in [0, n-1] : \ \sigma(i) > \sigma(i+1)\}|,$$

where $\sigma(0) := 0$. For example, if $\sigma = [-3, 1, -6, 2, -4, -5] \in B_6$ then $inv(\sigma) = 9$, $des_A(\sigma) = 3$, $maj_A(\sigma) = 11$, $neg(\sigma) = 4$, $\ell_B(\sigma) = 27$, and $des_B(\sigma) = 4$.

We shall also use the following formula, first observed by Incitti [**I**]:

$$\ell_B(\sigma) = \frac{inv(\overline{\sigma}) + neg(\sigma)}{2} \qquad (\forall \sigma \in B_n), \tag{2.2}$$

where $\overline{\sigma}$ denotes the sequence $(\sigma(-n), \ldots, \sigma(-1), \sigma(1), \ldots, \sigma(n))$. For example, if we take $\sigma = [-3, 5, -7, 1, 2, -4, 6]$ then $inv(\overline{\sigma}) = 35$ and $\ell_B(\sigma) = \frac{35+3}{2} = 19$.

3. Main Results

The *flag major index* of a signed permutation $\sigma \in B_n$ is defined by

$$fmaj(\sigma) := 2 \cdot maj_A(\sigma) + neg(\sigma),$$

where $maj_A(\sigma)$ is the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$ with respect to the natural order $-n < \cdots < -1 < 1 < \cdots < n$.

The following is a type B analogue of the Garsia-Gessel theorem (Theorem 2.8).

THEOREM 3.1. *For every subset* $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$

$$\sum_{\{\sigma \in B_n \mid \ Des_B(\sigma^{-1}) \subseteq M\}} q^{fmaj(\sigma)} = \prod_{i=m_1+1}^{n} (1+q^i) \cdot \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q,$$

where $m_0 := 0$ *and* $m_{t+1} := n$.

The following is a type B analogue of a classical result (Fact 2.7).

THEOREM 3.2. *For every subset* $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$

$$\sum_{\{\sigma \in B_n \mid \ Des_B(\sigma^{-1}) \subseteq M\}} q^{\ell_B(\sigma)} = \prod_{i=m_1+1}^{n} (1+q^i) \cdot \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q,$$

where $m_0 := 0$ *and* $m_{t+1} := n$.

We deduce a Foata-Schützenberger type theorem for B_n.

THEOREM 3.3. *For every subset* $M \subseteq [0, n-1]$

$$\sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) = M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) = M\}} q^{fmaj(\sigma)}.$$

The following result refines Theorem 3.3.

THEOREM 3.4. *For every subset* $M \subseteq [0, n-1]$ *and* $j \in [\pm n]$

$$\sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) = M, \; \sigma(n) = j\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) = M, \; \sigma(n) = j\}} q^{fmaj(\sigma)}.$$

An analogue of MacMahon's theorem for D_n follows; see Corollary 5.2 below.

4. Proof Outlines

OBSERVATION 4.1. *Let* $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$. *(Note:* $0 \notin M$.*) Let* $m_0 := 0$ *and* $m_{t+1} := n$. *For* $\sigma \in B_n$, *if* $Des_A(\sigma^{-1}) = M$ *then there exist* r_i $(0 \le i \le t)$ *such that* $m_i \le r_i \le m_{i+1}$ *and* σ *is a shuffle of the following increasing sequences:*

$$(-r_0, -r_0+1, \ldots, -1\,(= -(m_0+1)\,)),$$
$$(r_0+1, r_0+2, \ldots, m_1),$$
$$(-r_1, -r_1+1, \ldots, -(m_1+1)),$$
$$(r_1+1, r_1+2, \ldots, m_2),$$
$$\vdots$$
$$(-r_t, -r_t+1, \ldots, -(m_t+1))$$

and

$$(r_t+1, r_t+2, \ldots, n\,(= m_{t+1}\,)).$$

For every i, *if* $r_i - m_i = 0$ $(m_{i+1} - r_i = 0)$ *then the sequence* $(-r_i, \ldots, -(m_i+1))$ *(respectively,* $(r_i+1, \ldots, m_{i+1})$*) is understood to be empty. Also, with the above notations:* $0 \in Des_B(\sigma^{-1})$ *if and only if* $r_0 > 0$.

First, we prove the following special cases.

THEOREM 4.2. *For every subset* $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$

$$\sum_{\{\sigma \in B_n \mid \; Des_A(\sigma^{-1}) \subseteq M\}} q^{fmaj(\sigma)} = \sum_{\{\sigma \in B_n \mid \; Des_A(\sigma^{-1}) \subseteq M\}} q^{\ell_B(\sigma)} =$$
$$= \prod_{i=1}^{n} (1+q^i) \cdot \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q,$$

where $m_0 := 0$ *and* $m_{t+1} := n$.

THEOREM 4.3. *For every subset* $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$

$$\sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) \subseteq M\}} q^{fmaj(\sigma)} = \sum_{\{\sigma \in B_n \mid \; Des_B(\sigma^{-1}) \subseteq M\}} q^{\ell_B(\sigma)} =$$
$$= \prod_{i=m_1+1}^{n} (1+q^i) \cdot \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q,$$

where $m_0 := 0$ *and* $m_{t+1} := n$.

The proofs of Theorems 4.2 and 4.3 rely on Theorem 2.8, binomial identities (mentioned in Subsection 2.2), combinatorial properties of the length and descent functions on B_n (mentioned in Subsection 2.4) and Observation 4.1. For detailed proofs see [**ABR3**].

Proof of Theorems 3.1 and 3.2. Combine Theorems 4.2 and 4.3 with Remark 2.9.

□

Proof of Theorem 3.3. Combine Theorems 3.1 and 3.2, and apply the Principle of Inclusion-Exclusion.

□

Theorem 3.4 is an immediate consequence of the following refinements of Theorems 3.1 and 3.2.

Theorem 4.4. *Let $n \in \mathbf{P}$, $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$ and $i \in [\pm n]$. Then*

$$\sum_{\{\sigma \in B_n:\ Des_B(\sigma^{-1}) \subseteq M,\ \sigma(n)=i\}} q^{fmaj(\sigma)} =$$

$$= \begin{cases} \frac{[m_r - m_{r-1}]_q}{[n]_q} \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q \cdot q^{n-m_r} \prod_{j=\tilde{m}_1+1}^{n-1} (1+q^j), & \text{if } i = m_r \text{ for } r \in [t+1]; \\ \frac{[m_{r+1} - m_r]_q}{[n]_q} \begin{bmatrix} n \\ m_1 - m_0, \ldots, m_{t+1} - m_t \end{bmatrix}_q \cdot q^{n+m_r} \prod_{j=m_1+1}^{n-1} (1+q^j), & \text{if } i = -m_r - 1 \text{ for } r \in [t]; \\ 0, & \text{otherwise.} \end{cases}$$

Here $m_0 := 0$, $m_{t+1} := n$, and

$$\tilde{m}_1 := \begin{cases} m_1 - 1, & \text{if } i = m_1; \\ m_1, & \text{otherwise.} \end{cases}$$

Theorem 4.5. *Let $n \in \mathbf{P}$, $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$ and $i \in [\pm n]$. Then $q^{\ell_B(\sigma)}$ satisfies exactly the same formula as does $q^{fmaj(\sigma)}$ in Theorem 4.4.*

The proofs of Theorems 4.4 and 4.5 use case-by-case analysis.

Proof of Theorem 3.4. Combine Theorems 4.4 and 4.5, and apply the Principle of Inclusion-Exclusion.

□

Problem 4.6. *Find combinatorial (bijective) proofs for Theorems 4.4 and 4.5.*

5. Final Remarks

5.1. Classical Weyl Groups of Type D. Let D_n be the classical Weyl group of type D and rank n. For an element $\sigma \in D_n$, let $\ell_D(\sigma)$ be the length of σ with respect to the Coxeter generators of D_n. It is well-known that we may take

$$D_n = \{\sigma \in B_n \mid neg(\sigma) \equiv 0 \bmod 2\}.$$

Let $\sigma = [\sigma(1), \ldots, \sigma(n)] \in D_n$. Biagioli and Caselli [**BC**] introduced a flag major index for D_n:

$$fmaj_D(\sigma) := fmaj(\sigma(1), \ldots, \sigma(n-1), |\sigma(n)|).$$

By definition,

$$\sum_{\sigma \in D_n} q^{fmaj_D(\sigma)} = \sum_{\{\sigma \in B_n \mid\ \sigma(n) > 0\}} q^{fmaj(\sigma)}. \tag{5.1}$$

Proposition 5.1.

$$\sum_{\{\sigma\in B_n \mid \ \sigma(n)>0\}} q^{\ell_B(\sigma)} = \sum_{\sigma\in D_n} q^{\ell_D(\sigma)}.$$

For a proof see [**ABR3**, Proposition 6.1]. We deduce the following type D analogue (first proved in [**BC**]) of MacMahon's theorem.

Corollary 5.2.

$$\sum_{\sigma\in D_n} q^{fmaj_D(\sigma)} = \sum_{\sigma\in D_n} q^{\ell_D(\sigma)}.$$

Proof. Combine identity (5.1) and Proposition 5.1 with Theorem 3.4.

□

Problem 5.3. *Find an analogue of the Foata-Schützenberger theorem for D_n.*

The obvious candidate for such an analogue is false.

5.2. Two Versions of the Flag Major Index. The flag major index of $\sigma \in B_n$was originally defined as the length of a distinguished canonical expression for σ. In [**AR1**] this length was shown to be equal to $2 \cdot maj_A(\sigma) + neg(\sigma)$, where the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$ was taken with respect to the order $-1 < \cdots < -n < 1 < \cdots < n$. In [**ABR1**] we considered a different order: $-n < \cdots < -1 < 1 < \cdots < n$ (i.e., we defined $fmaj$ as in Section 3 above).

While both versions give type B analogues of the MacMahon and Carlitz identities, only the second one gives an analogue of the Foata-Schützenberger theorem. On the other hand, the first one has the alternative natural interpretation as length, as mentioned above, and also produces a natural analogue of the signed Mahonian formula of Gessel and Simion, see [**AGR**]. The relation between these two versions and their (possibly different) algebraic roles requires further study.

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