I) Brief description of Restricted orbit Equivalence

II) Rokhlin's Lemmas & Dye's Theorem

Fix $(X, \mathcal{F}, \mu)$ - a nonatomic standard probability space & an orbit relation

$\mathcal{O} = \mathcal{O}(U) \subseteq X \times X$, U-invertible μ-pres. map

$$\mathcal{O} = \{ (x, U^j(x)) \mid x \in X, j \in \mathbb{Z} \}$$

Let $\mathcal{A} = \{ T \mid T$ has the same orbits as U i.e. $\mathcal{O}(T) = \mathcal{O}(U) = \mathcal{O} \}$

Let Γ = full group $= \{ \varphi \mid \varphi$ is 1-1 onto & $(x, \varphi(x)) \in \mathcal{O} \}$

For $T \in \mathcal{A}$ & $\varphi \in \Gamma$ we say (T, φ) is a "rearrangement pair." φ rearranges (perturbs) the orbit as "arranged" by T.

Define an abstract notion of the "size" of a rearrangement $m(T, \varphi) \in \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$.

Three properties:

1) For $\varepsilon > 0$, $\exists \delta$ so that if $m(T,\varphi) < \delta$ then $\mu(\{x \mid \varphi(T(x)) \neq T(\varphi(x))\}) < \varepsilon$

$$\begin{tikzcd} \cdot \arrow[r, "T"] \arrow[d, "\varphi"'] & \cdot \arrow[d, "\varphi"] \\ \cdot \arrow[r, "T"'] & \cdot \end{tikzcd}$$

2) For $T \in \mathcal{A}$ define

$$m_T(\varphi_1, \varphi_2) = m(\varphi_1^{-1} T \varphi_1, \varphi_1^{-1}\varphi_2)$$

+ this should be a pseudometric on Γ (triangle inequality).

3) For pair (T,φ), $\varphi(x) = T^{h(x)}(x)$

$g = g^{T,\varphi} : X \to \mathbb{Z}^{\mathbb{Z}}$, $g(j) = h(T^j(x))$

$g_*\mu$ is a $\mathbb{T}$-invariant measure on $\mathbb{Z}^{\mathbb{Z}}$. Now $\mathbb{Z}^{\mathbb{Z}}$ is Polish so $\mathcal{M}(\mathbb{Z}^{\mathbb{Z}})$ - the space of $\mathbb{T}$-invariant Borel measures on $\mathbb{Z}^{\mathbb{Z}}$ is Polish - weak*

$\vdash$ For all (T,φ) and $\varepsilon > 0$ there is a nghd O of $g_*\mu$ in $\mathcal{M}(\mathbb{Z}^{\mathbb{Z}})$ so that if $g_*^{T',\varphi'}(\mu') \in O$ then

$$m(T', \varphi') < m(T,\varphi) + \varepsilon$$

Call this: "weak* upper semi-Continuity"

Lots of examples - here are two

Ex $\quad m_0(T,\varphi) = \mu(\{x \mid T\varphi(x) \neq \varphi T(x)\})$

Ex $\quad \mathcal{E}(T,\varphi)$ satisfies 2) + 3) but not 1) so set

$$m^e(T,\varphi) = \mathcal{E}(T,\varphi) + m_0(T,\varphi)$$

Abstract ~~non~~sense about m.

Fix T, set Λ_T - m_T closure of $\Gamma/_{m_T=0}$ - Λ_T is a complete seperable metric space

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Γ still acts on Λ_T - in fact isometrically

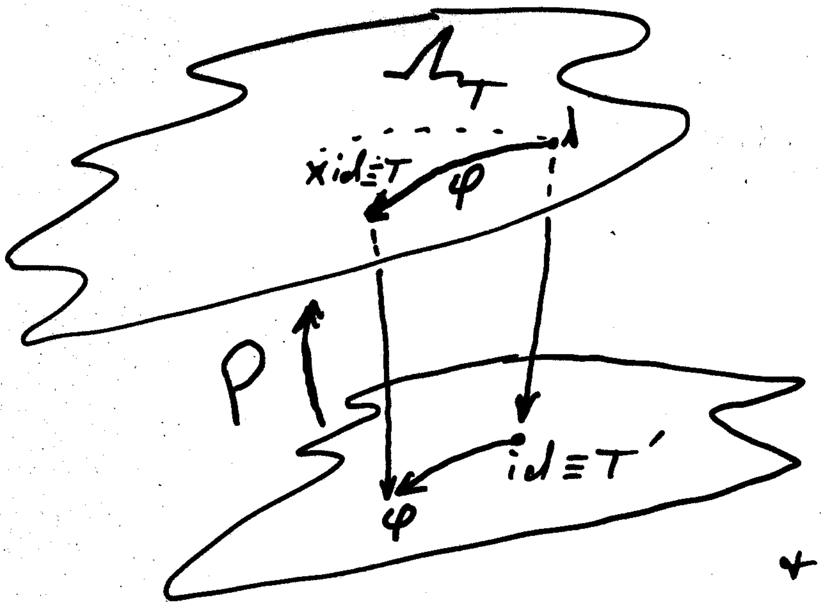
$$\left.\begin{array}{l} \varphi_1^{-1}T\varphi_1 \longrightarrow \varphi_2^{-1}T\varphi_2 \\ \varphi^{-1}\varphi_1^{-1}T\varphi_1\varphi \longrightarrow \varphi^{-1}\varphi_2^{-1}T\varphi_2\varphi \end{array}\right\}$$ These two rearrangements are identical weak*

$\parallel$

$(\varphi^{-1}\varphi_1\varphi)^{-1}(\varphi^{-1}T\varphi)(\varphi^{-1}\varphi_1\varphi)$

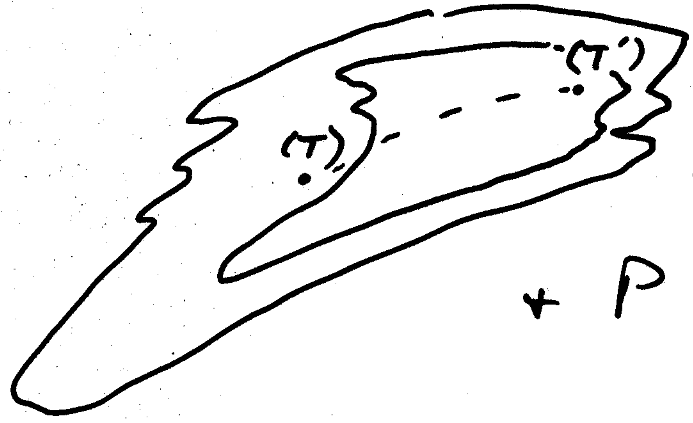
From 1), if φ_i are m_T-cauchy then $\varphi_i^{-1}T\varphi_i$ converges pointwise to some $T' \in \mathcal{A}$.

Repeat the construction for T'



The map, $\lambda \longleftrightarrow T'$ extends to a map $\Gamma/m_{T'} = 0$ back into Λ_T + this is a <u>contraction</u> $m_{T'}$ to m_T we call P (this is 3))

So it extends to a contraction $\Lambda_{T'} \longrightarrow \Lambda_T$



Suppose id $(\equiv T)$ is in $P(\Lambda_{T'})$. Then $P(\Lambda_{T'}) = \Lambda_T$ + P is an isometry.

In this case we say $T, T' \in \mathcal{A}$ are m-equivalent – true of a <u>residual</u> subset of Λ_T.

We say T + S are m-equivalent if $S \cong T'$ + T' is in $\mathcal{A}$, m-equiv. to T.

What *is* the equivalence relation?

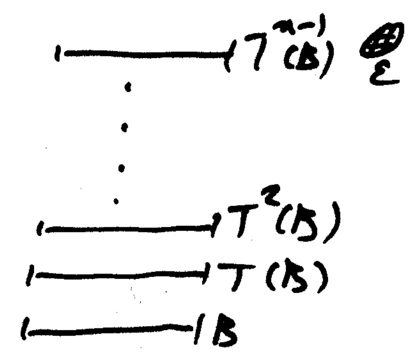
Dye's Theorem + $\mathcal{M}_0$.

Rokhlin's Lemma's

Lemma: For T an ergodic map on a nonatomic space, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$, $\exists$ a set B with

$B, T(B), \ldots, T^{n-1}(B)$ all disjoint

+ $\mu\left(\bigcup_{i=0}^{n-1} T^i(B)\right) = 1 - \varepsilon$



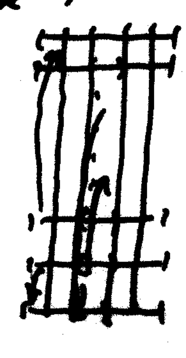
Here's a way to construct maps $\varphi \in \mathcal{M}$.

Cut B into subsets $B_1, B_2, \ldots, B_t$ + pick permutations $\pi_1, \ldots, \pi_t \in S^n$.

For $x = T^j(x_0)$, $x_0 \in B_k$, $0 \le j < n$

Set $\varphi(x) = T^{\pi_k(j)}(x_0)$.

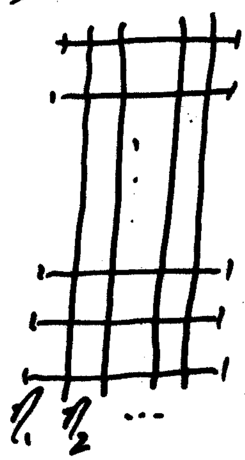
$\varphi(x) = x$ otherwise.



Strong Rokhlin lemma

For P a finite partition, $\varepsilon > 0$, $n \in \mathbb{N}$ the set B in the lemma can be chosen with

$$B \perp \bigvee_{i=0}^{n-1} T^{-i}(P)$$

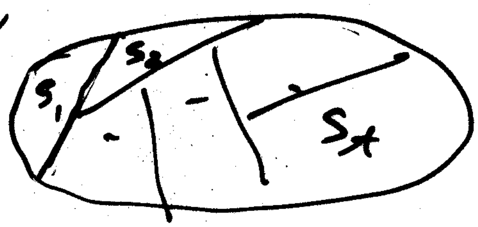


Cut according to T, P, n-names

distribution of T, P, n-names on $B \equiv$ distribution of T, P, n-names on X.

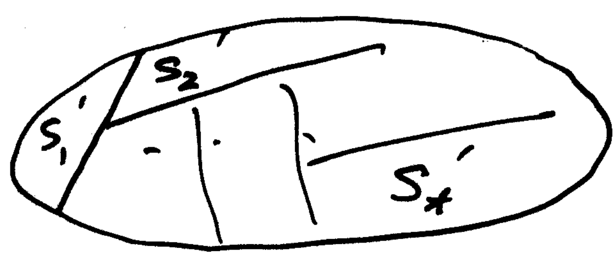
Let consider now how to "model" one system on another T on $(X, \mathcal{F}, \mu)$, S on $(Y, \mathcal{G}, \nu)$

X



Partition X by some partion P.

Y

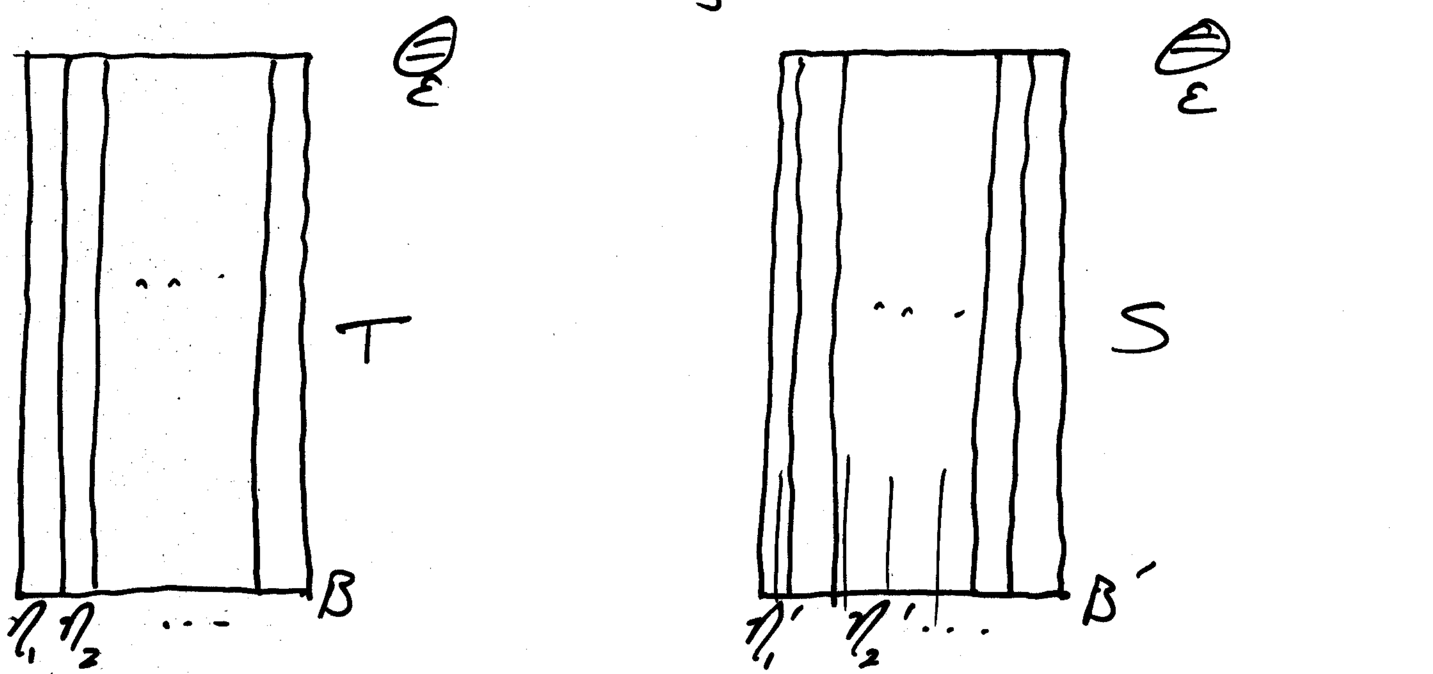


On Y I can find a partion P' with sets of the same measure as those of P.

This is a "0'th order" model.
On <u>names</u> of some length $n >>> 1$, look very different - There is <u>one</u> similarity from ergodic theorem. On <u>most</u> names, the fractions occupied by corresponding symbols will be close to the same.

For $\varepsilon > 0$, $\exists n$ so that all but ε of the T, P, n-names + ε of the S, P', n-names give densities to corresponding symbols i within a fraction ε of $\mu(S_i) = \nu(S_i')$.

Use the <u>Strong</u> Rokhlin lemma



Cut B' into subsets of relative measures $\mu(\eta_1), \mu(\eta_2) \cdots$. Call these $B_1', B_2' \ldots$

For all but a fraction 2ε of the $y \in B'$, the S, P', n-name of y can be permuted by some π_y to agree, all but ε proportion of the time with the T, P, n-name of η_k where $y \in B'_k$.

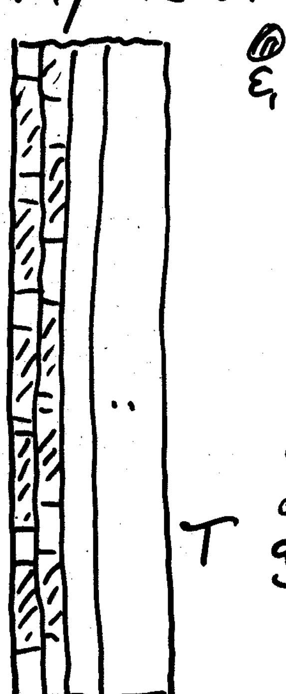
a) Construct the $\varphi_1 \in \Gamma$ you get, + $\varphi_1^{-1} S \varphi_1, P', n$-names on B' are "ε-close" to the T, P, n-name

so

b) change P' to P_1' on a fraction $< 3\varepsilon$ so the distribution of

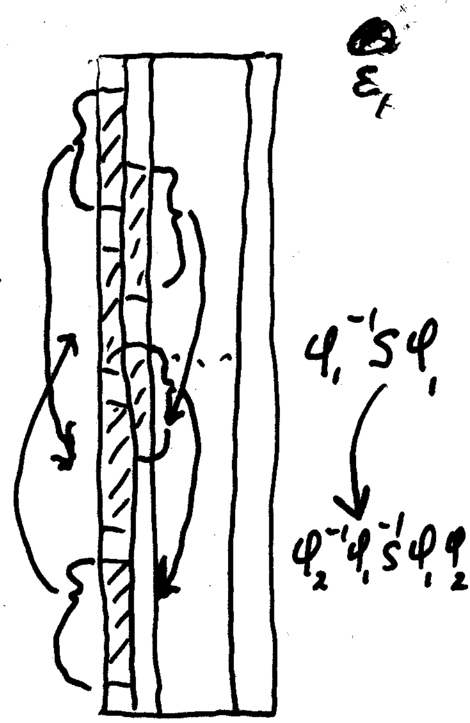
$\varphi^{-1} S \varphi, P_1', n$-names on B' + T, P, n-names on B are $\equiv$.

Now what? – Layer on complications:



New taller towers + density of occurrences of the previous blocks approximately agree.

Now permute contiguous blocks of orbits to get a close match, then modify $P_1' \to P_2'$ a perfect match



Etc.

The maps $\overbrace{(\varphi_1\varphi_2\cdots\varphi_k)^{-1}}^{\psi_k^{-1}} S \overbrace{(\varphi_1\varphi_2\cdots\varphi_k)}^{\psi_k}$

have $m_0(\psi_k^{-1} S \psi_k, \underbrace{\psi_k^{-1}\psi_{k+1}}_{\varphi_{k+1}}) \leq \frac{2}{\text{height of } k\text{'th tower}} + \varepsilon_k$

+ so can be made to converge pointwise to some S'. The partitions P_k' converge to some limiting partition P_∞' + the measures of all S', P_∞', n-names are identical to the measures of the corresponding T, P, n-name

To get Dye's Theorem need 2 more layers of complication

① Need to successively refine P as a sequence of partitions generating $\mathcal{F}$

② Need to also bring in a similar refining sequence of partitions of Y generating $\mathcal{D}$.

Tomorrow want to examine how to control the # of permutations we use in this argument.