Entropy + the Shannon-McMillan-Breiman Theorem + A counting argument

Shannon-McMillan-Breiman Theorem

- third of the three core tools in measurable dynamics
  1) Ergodic Theorem
  2) Rokhlin lemmas
  3) SMB

Theorem For T ergodic and P a finite partition of X, for a.e. $x \in X$

$$\lim_{n \to \infty} \frac{\log_2(\mu(P_n(x)))}{n} = h(T,P)$$

Came from counting

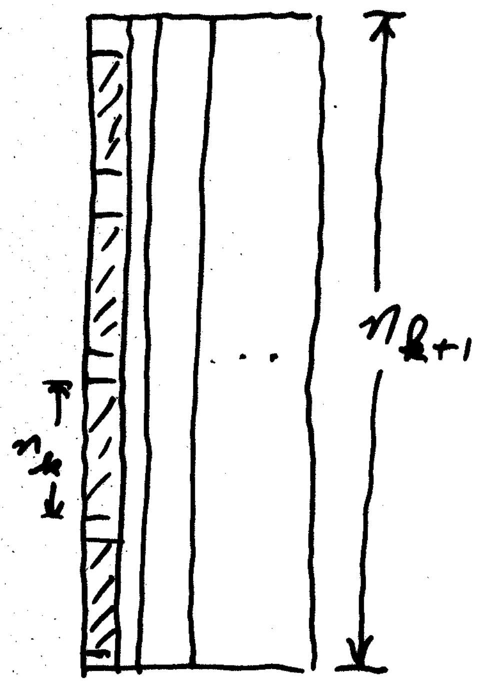
Says for *most* x, when n is large

$$\mu(P_n(x)) = 2^{-(h(T,P) \pm \varepsilon)n}$$

$$= \left(2^{-h(T,P)n}\right)\underbrace{\left(2^{\pm\varepsilon}\right)^n}_{\text{close to } 1}$$

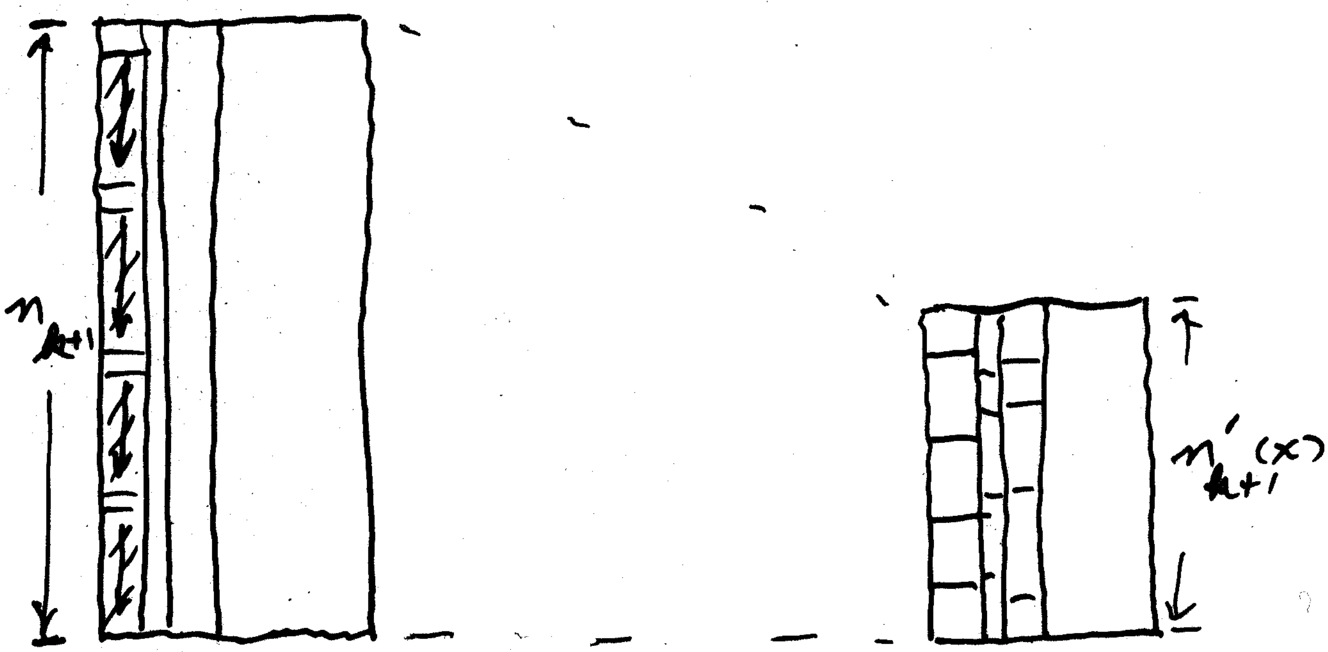
Watch out!!

Reminder of the basic move in Dye's Theorem



Rearrange the blocks of length n_k by permuting them + then modify names a small fraction.

A little prep work:
Collapse the blocks from the previous stage to be just <u>one</u> level and label it with the T, P, n_k-name of the block



New height is variable but approximately constant

$$n'_{k+1} = \underbrace{\varepsilon_k n_{k+1}}_{\text{outside blocks}} + \underbrace{(1-\varepsilon_k)\frac{n_{k+1}}{n_k}}_{\text{collapsed blocks}} \pm \underbrace{\varepsilon_{k+1} n_{k+1}}_{\text{ergodic Thm variability}}$$

$$= \left(\varepsilon_k + \frac{(1-\varepsilon_k)}{n_k} \pm \varepsilon_{k+1}\right) n_{k+1} = \delta_k n_{k+1}$$

small essentially constant value

How many permutations <u>might</u> we use?

$$(\delta_k n_{k+1})! \simeq 2^{\delta_k n_{k+1} \log_2(\delta_k n_{k+1})}$$

$\boxed{\text{superexponential in } n_{k+1}}$

How many names are there and how big are they?

$$\left.\begin{array}{l} \# \text{ is } 2^{(h(T,P) \pm \varepsilon_{k+1})n} \\ \text{size is } 2^{-(h(T,P) \pm \varepsilon_{k+1})n} \end{array}\right\} \text{off subset of } \varepsilon_{k+1} \text{ of base}$$

More prep — Modify P on a small part near bottom to be one name so the names are fat.

$$\# \text{ is } 2^{(h(T,P) - 2\varepsilon_{k+1} \pm \varepsilon_{k+1})n_{k+1}}$$

$$\text{size is } 2^{-(h(T,P) - 2\varepsilon_{k+1} \pm \varepsilon_{k+1})n_{k+1}}$$

Do a combinatorial argument:
Simplified form but expresses the core idea of how one can control the # of permutations one uses in Dye's theorem.

Fix a (large) value n, and let $k_1, k_2, \ldots, k_t$ be given $\sum k_i = n$.

Consider the set of words $\mathcal{L}$ of length n in the symbols $\{1, 2, \ldots, t\}$ where symbol i occurs k_i times.

there are $N = \binom{n}{k_1, k_2 \cdots k_t} = \dfrac{n!}{k_1! \cdots k_t!}$

such names:

Two facts

1) $N = 2^{h(\alpha) n}$, $\alpha = \left\{ \frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_t}{n} \right\}$

+ $h(\alpha) \approx -\sum_{i=1}^{t} \alpha_i \log_2 \alpha_i$.

2) If you take one word $\eta \in \mathcal{L}$ and act on it by S^n by permuting the order of the symbols, the image under S^n of η will cover $\mathcal{L}$ uniformly.

Write down two lists of N words from $\mathcal{L}$

$$L_1 = \{\eta_1, \eta_2, \cdots, \eta_N\} = \mathcal{L} \quad \text{is } 1\text{-}1$$

$$L_2 = \{\eta_1', \eta_2' \cdots \eta_N'\} \quad \text{allows multiplicities}$$

We look for lists of permutations $\pi_i : i = 1 \cdots N$ in S^n and a bijection $H : \{1 \cdots N\} \circlearrowleft$ so that

$$\pi_i(\eta_i') = \eta_{H(i)}$$

$$\begin{array}{c} \{\eta_1, \eta_2, \cdots, \eta_N\} \\ \uparrow H \\ \{\eta_1', \eta_2', \cdots, \eta_N'\} \end{array}$$

Call π_i, H a "matching" of L_2 to L_1.

<u>Theorem</u> Suppose words in L_2 occur with maximum multiplicity K and $\varepsilon > 0$. One can find a matching π_i, H of L_2 to L_1 so that all but a fraction at most ε of $1, \cdots, n$ are covered by fewer than $\frac{K}{\varepsilon^2}$ distinct π_i.

In our case, what will K and ε be?

Our names are <u>fat</u>, so ~~first~~ we can modify P so that all symbols $1, \dots, t$ occur the same number of times $k_1, \dots, k_t$ and

$$h(\alpha) = h(T,P) \pm \varepsilon_{k+1}$$

<u>so</u> $$N = 2^{(h(T,P) \pm \varepsilon_{k+1})\, n_{k+1}}$$

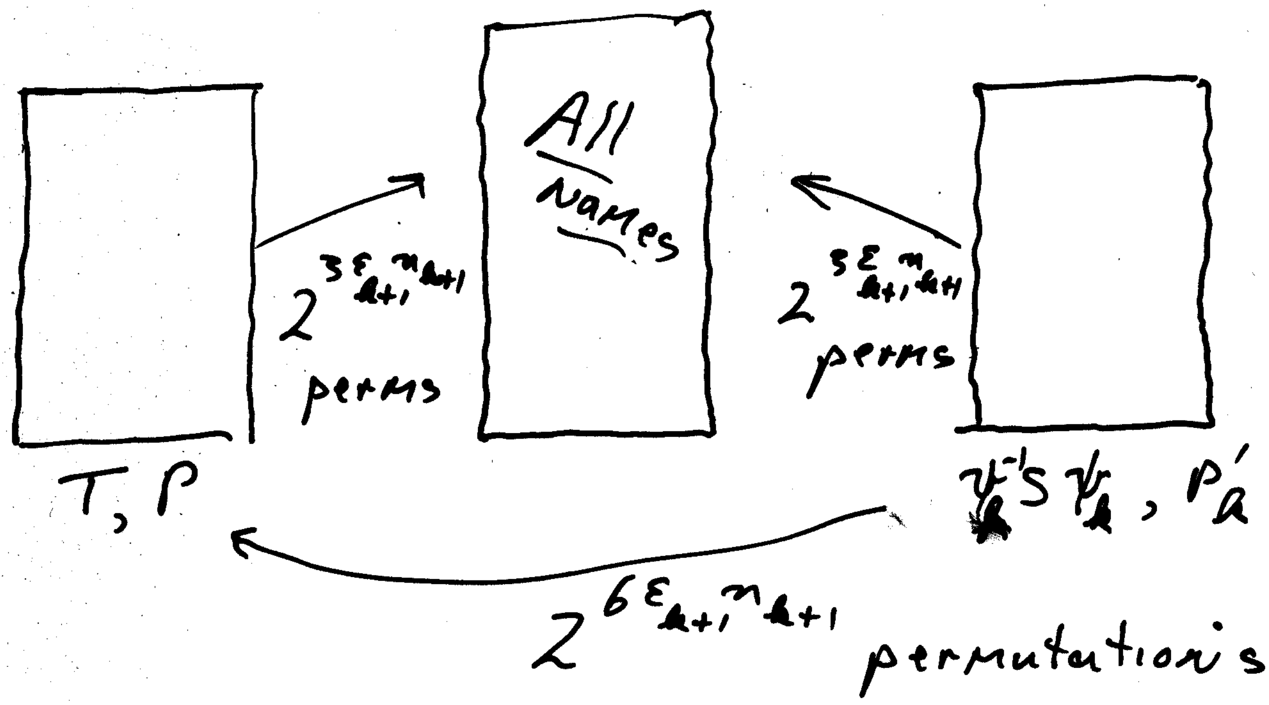
and cut a name in tower into pieces of size $\frac{1}{N}$, you get at least $2^{\varepsilon_{k+1} n_{k+1}}$ of them and at <u>most</u> $2^{2\varepsilon_{k+1} n_{k+1}}$ of them. So <u>remainder</u>, the <u>bit</u> left is a fraction at most $2^{-\varepsilon_{k+1} n_{k+1}}$ (tiny) of the name. Modify P again so cuts are perfect + we're ready to apply the Theorem.

What is K? $\quad K \leq 2^{2\varepsilon_{k+1} n_{k+1}}$

so # of permutations used is

$$\leq \frac{2^{2\varepsilon_{k+1} n_{k+1}}}{2\varepsilon_{k+1}} \leq 2^{3\varepsilon_{k+1} n_{k+1}}$$

if n_{k+1} is large



Comments:

1) Really about discrete amenable group actions - As is Dye's Theorem. Is there a group-independant version?

2) At least for some cases can a more explicit algorithm for selecting the small # of permutations be given?

3) Can a topological version for homeomorphisms of Cantor sets be given?