Entropy + Orbit Equivalence

Outline

I) Measure preserving dynamics
   The ergodic theorem + decomposition
   Entropy

II) Orbit Equivalence
   Dye's Thm.
   Full group as orbit perturbations

III) "Complexity" of a perturbation
   zero asymptotic complexity

References:

"Restricted Orbit Equivalence for discrete amenable group actions"
J. Kammeyer + djr
Cambridge tracts #146

"An entropy preserving Dye's theorem for ergodic actions"
J. d'Analyse to appear
(www.math.umd.edu\~djr)

Apology: I am presenting this today in a form somewhat different + I hope more accessible than that in these references.

Measure preserving transformations of a standard probability space.

$(X, \mathcal{S}, \mu)$

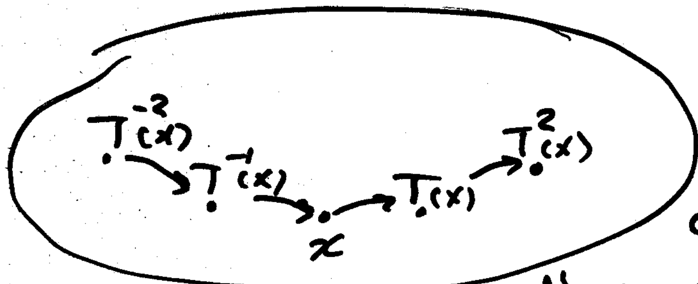
space σ-algebra measure

$T : X \to X$, invertible + measure preserving.

Why? — Dynamics —

State space X

T: state x to state $T(x)$ one "time unit" later.



Quite natural for such to possess an invariant measure and studying behavior "up to probability zero"

~~is quite robust + insightful~~

Standard? Means, up to probability 0, X is compact metric and μ is a Borel measure. This is where a <u>lot</u> of interesting dynamics happens.
unit interval, regions in $\mathbb{R}^n$, Cantor sets
compact manifolds

Isomorphism:

$(X, \mathcal{S}, \mu)$, $(Y, \mathcal{G}, \nu)$ are isomorphic as measure spaces if there is $X_0 \subseteq X$, $Y_0 \subseteq Y$, $\mu(X_0) = \nu(Y_0) = 1$, and a measure preserving + invertible map $\varphi : X_0 \leftrightarrow Y_0$.

Fact: There is only <u>one</u> nonatomic <u>standard probability space</u> (up to iso.)

T on $(X, \mathcal{S}, \mu)$, S on $(Y, \mathcal{G}, \nu)$ are <u>conjugate</u> if φ exists (as above)

$$S\varphi = \varphi T$$

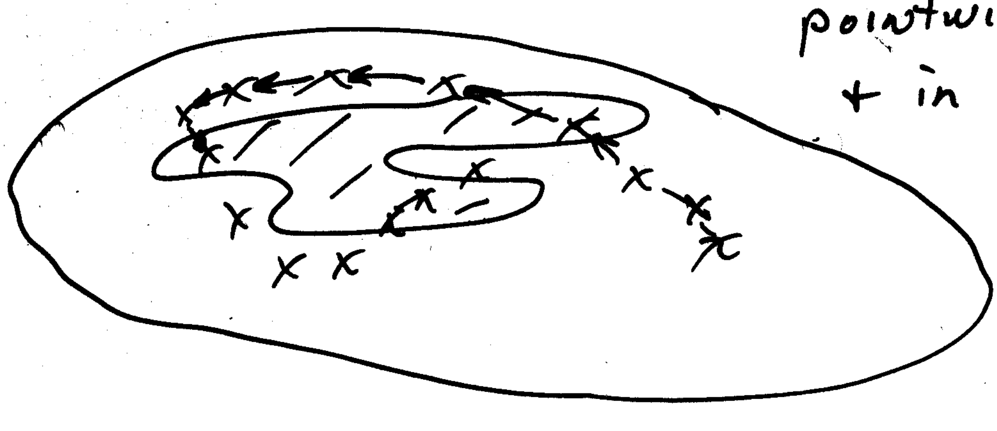
$$\begin{tikzcd}
x \arrow[r, "T"] \arrow[d, "\varphi"'] & x \arrow[r, "T"] \arrow[d, "\varphi"] & x \arrow[r, "S"] \arrow[d, "\varphi"] & {} \\
\circ \arrow[r, "S"'] & \circ \arrow[r, "S"'] & \circ \arrow[r, "S"'] & {}
\end{tikzcd}$$

Ergodic Theorem:

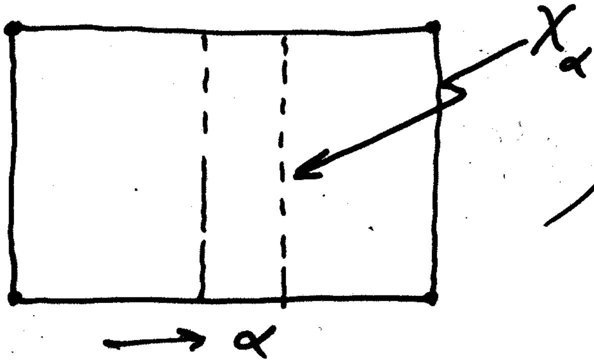
For $f \in L^1(\mu)$, $\frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j$

converges to the projection of f onto the space of T-invariant functions.

pointwise a.s.
+ in L^1.



Ergodicity + ergodic decomposition



$$\mu = \int \mu_\alpha \, d\mu(\alpha)$$

μ_α - supported on X_α — is T-invariant + is "ergodic", any T-invariant set has μ_α-measure 0 or 1.

We assume T from now on is "ergodic"

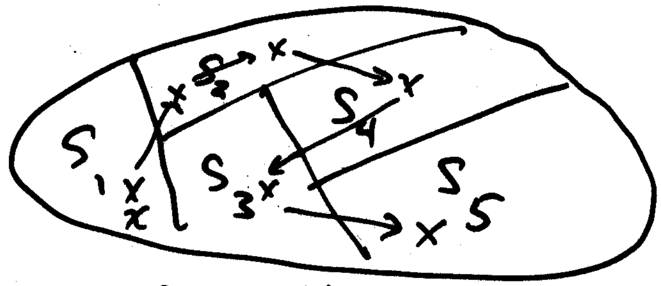
$$\frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j \longrightarrow \int f \, d\mu$$

a.s + in L^1.

Entropy: A quick understanding —

Take P a finite partition of X

$P = \{ S_1, S_2, \cdots, S_k \}$



Set $P(x) = i$ iff $x \in S_i$

and $P_n(x) = \{ P(x), P(T(x)), \cdots, P(T^{n-1}(x)) \}$

the "T, P, n - name of x"

Apology: One regularly lets a "name" $\{ i_0, i_1, \cdots, i_{n-1} \}$ represent the set of points possessing that name.

Let $1 > \varepsilon > 0$ and

$N(T, P, \varepsilon, n) =$

Min # of T, P, n-names it takes to cover all but $\underline{\varepsilon}$ in measure of X.

Expect this # to grow exponentially

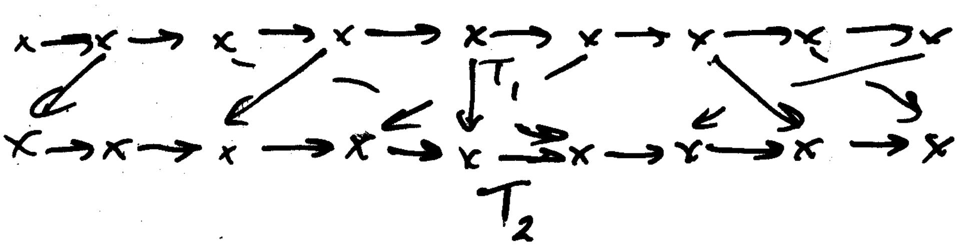
$$h(T,P) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log_2 (N(T, P, \varepsilon, n))}{n}$$

not necessary

$$h(T) = \sup_P h(T,P) - \left\{ \begin{array}{l} \text{Tremendous Theory} \\ \text{follows - we must} \\ \text{move on} \end{array} \right\}$$

Orbit Equivalence

Suppose T_1, T_2 act on $(X, \mathcal{F}, \mu)$.
To say they "have the same orbits"
means $T_2(x) = T_1^{j(x)}(x)$, $T_1(x) = T_2^{k(x)}(x)$ — a.s.



Def'n: We say T and S are orbit equivalent if there is a T_1 with the same orbits as T and $T_1 \cong S$.

AMAZING theorem

Dye, 1952

If T and S are aperiodic and ergodic then T + S are orbit equivalent

For example

Take T an irrat'l rotation of the circle

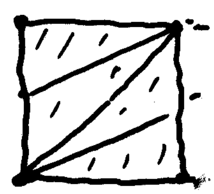
$\alpha \notin \mathbb{Q}$

erg. + aperiodic

+

S a hyperbolic toral automorphism

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ acting on $\mathbb{R}^2/\mathbb{Z}^2$

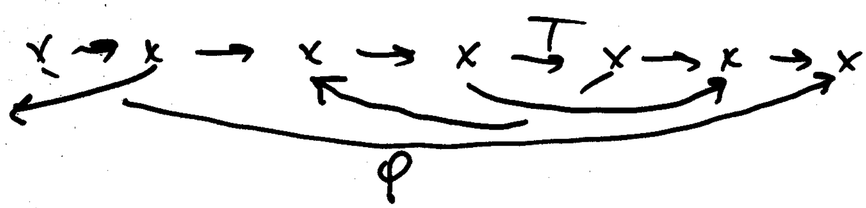


Only one nonatomic space + on it only one set of orbits

Some understanding of what Dye did.

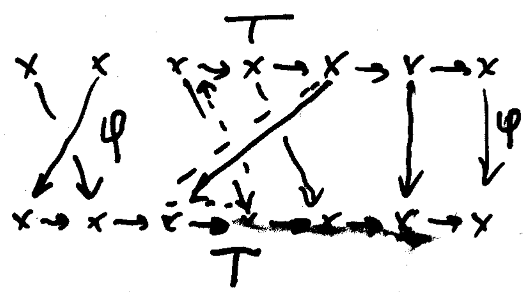
Full group of T:

$$\Gamma(T) = \{\varphi \mid \varphi \text{ is 1-1 onto a.s. } + \ \varphi(x) = T^{h(x)}(x),\ h: X \to \mathbb{Z}\}$$



Can regard elements of $\Gamma(T)$ as perturbations of the action T. rearrangement

$$T \xrightarrow{\text{pert. to}} \varphi^{-1} T \varphi$$



Notice all $\varphi^{-1} T \varphi$ are conjugate to T.
But different maps on same orbits

Def'n: Suppose T_i all have same orbits + $T_i(x)$, $T_i^{-1}x$ become asymptotically constant. i.e. for a.e. x $\exists I$, $i \geq I$, $T_i(x) = T_I(x)$, $T_i^{-1}(x) = T_I^{-1}(x)$ (Halmos)

Then $T_i \to T(x)$, invertible + m.p.

Dye shows its not too difficult to explicitely find $\varphi_i \in \Gamma$ with $\varphi_i^{-1} T \varphi_i$ converging to T'-conjugate to a <u>dyadic</u> <u>adding machine</u> –

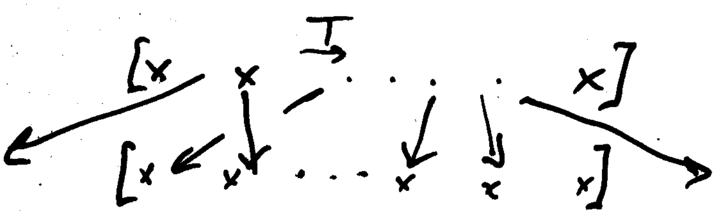
One "models" the "local" orbit <u>structure better + better.</u>

Discuss now the <u>complexity</u> of a perturbation by $\varphi \in \Gamma$ of T.

① For $\varphi \in \Gamma$ + a.e. x

$$\frac{\#\{x, T(x), \dots, T^{n-1}(x)\} \triangle \varphi(\{x, T(x), \dots, T^{n-1}(x)\})}{n} \to 0$$

$\varphi(x) = T^{h(x)}(x)$ –



h is $\mathbb{Z}$ valued + ergodic Thm.

So φ locally on orbit segments is almost a permutation.

② Means for x, n can find a permutation $\pi_{x,n} \in S^n$ +

$$\frac{\#\{i \in \{0, \dots, n-1\} \mid \varphi(T^i(x)) \neq T^{\pi_{x,n}(i)}(x)\}}{n} \to 0$$

③ $\pi_{x,n}$ are not unique, put metrics on S^n, $d(\pi_1, \pi_2) = \#\{i \mid \pi_1(i) \neq \pi_2(i)\}/n$.

Set

$N(T, \varphi, n, \varepsilon) =$

Min # of $\pi \in S^n$ whose ε nghds in d cover all but a set of $\pi_{x,n}$ of measure $< \varepsilon$

(measure of the x's).

Min. # of permutations needed to approximate φ.

although $\# S^n = n!$

we expect $N(T, \varphi, n, \varepsilon)$ to grow at most exponentially.

Set

$$\sigma(T, \varphi) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log_2(N(T, \varphi, n, \varepsilon))}{n}$$

Natural measure of the complexity of perturbation of orbits of T by φ.

Fact: $\sigma(T, \varphi) \leq h(T)$.

Def Suppose $\varphi_i^{-1} T \varphi_i \to T'$ pointwise a.s. We say the perturbations φ_i of orbits of T have "zero asymptotic complexity" if for all $\varepsilon > 0$ $\exists I$ so that for $j > i \geq I$

$$\underline{\mathcal{C}(\varphi_i^{-1} T \varphi_i},\ \varphi_i^{-1}\varphi_j) < \varepsilon.$$

A Cauchy condition:

Thm If the φ_i, as perturbations of the orbits of T, have zero asymptotic complexity then
$$h(T') \geq h(T)$$

(why not just $=$?)

Corollary If $\varphi_i^{-1} T \varphi_i \to T'$ pointwise (then $\varphi_i T' \varphi_i^{-1} \to T$) and we have zero asymptotic complexity in both directions — Then

$$h(T) = h(T')$$

Theorem: If T and S have the same entropy then there are $\varphi_i \in \Gamma$, $\varphi_i^{-1} T \varphi_i \to T'$, of zero asymptotic complexity in both directions and $T' \cong S$

Next 2 lectures indicate why this is true –

2) Ergodic theory background – More on Dye's Theorem

3) How complexity is controlled – lots about entropy.