A Sequent Calculus for Skeptical Reasoning in Autoepistemic Logic

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Abstract

A sequent calculus for skeptical consequence in infinite autoepistemic theories is presented and proved sound and complete. While skeptical consequence is decidable in the finite case, the move to infinite theories increases the complexity of skeptical reasoning to being Π^1_1-complete. This implies the need for sequent rules with countably many premises, and such rules are employed.

Introduction

Skeptical consequence is a notion common to all forms of nonmonotonic reasoning. Every nonmonotonic formalism permits different world views to be justified using the same set of facts and principles; the skeptical consequences of a framework are the notions common to all world views associated with that framework. Our purpose in this paper is to present a Gentzen-style sequent calculus (incorporating some infinitary rules) which will allow us to deduce the skeptical consequences of a given autoepistemic framework. Such sequent calculi (with purely finite rules) were defined for several types of nonmonotonic systems by Bonatti and Olivetti in (Bonatti & Olivetti 2002), but they restricted their attention to finite propositional systems for which skeptical consequence is decidable. We will adapt and extend their systems to accommodate infinite systems.

We will focus on autoepistemic logic (due to Moore, (Moore 1984)). Sequent calculi have also been developed for stable model logic programming (due to Gelfond and Lifschitz, (Gelfond & Lifschitz 1988)), default logic (due to Reiter, (Reiter 1980)). (See (Milnikel 2003).) When one steps from the finite to the potentially infinite, finding the set of skeptical consequences of a framework goes from being decidable to being Π^1_1-complete, at the same level of the computability hierarchy as true arithmetic. This result was proved for stable model logic programming by Marek, Nerode, and Remmel in (Marek, Nerode, & Remmel 1994), but it translates to autoepistemic logic quite easily. (The reader should be aware that there will be a few computability theoretic ideas and motivations discussed in this introductory section, but that they may be considered "deep background" and will not be a part of the exposition of the main ideas.)

Members of Π^1_1 sets correspond to finite-path computable (or Π^0_1) subtrees of $\omega^{<\omega}$ in a very natural way. See (Cenzer & Remmel 1998) for an excellent exposition. This makes skeptical consequence a natural fit for sequent calculi with infinitary rules, since a sequent proof is, at its core, a finite-path tree. Bonatti and Olivetti also addressed credulous consequence ("Can this notion be a part of *some* world view?") in their paper, but in the cases they were interested in, this question was also decidable. In our more general context, credulous reasoning is Σ^1_1-complete, not a natural type of question to address with trees-as-proofs. (One could write a sequent calculus for credulous reasoning in Π^1_2 logic, but this would take us too far afield.)

Because nonmonotonic logics deal not only with proof but with lack of proof, we will need not only standard monotone sequent calculi, but also rule systems for showing a lack of proof. (We will call these antisequent calculi, using the terminology of Bonatti from (Bonatti 1993).)

In our discussion of autoepistemic logic, we limit ourselves to the propositional case, but with a potentially infinite theory. Predicate autoepistemic logic is discussed in the literature (see, for example, (Konolige 1994)), and the interested reader should not find it difficult to combine the rules specific to predicate logic (from the discussion of default logic in (Milnikel 2003)) with the results about propositional autoepistemic logic.

Section 2 will consist of some preliminary definitions and results about autoepistemic logic, as well as Bonatti's antisequent calculus for propositional logic. Section 3 presents a sequent calculus for skeptical reasoning in infinite propositional autoepistemic theories, and proves soundness and completeness theorems for this calculus. Section 4 looks toward further directions for research.

Preliminaries

We will present some preliminary definitions and results on autopepistemic logic as well as on sequent and antisequent calculi. We assume that the reader is familiar with the modal operator L (also written $\Box$).

Stable Theories and Autoepistemic Logic

Moore's autoepistemic logic ((Moore 1984)) is an approach to nonmonotone reasoning which uses context both positively and negatively to reflect positive and negative introspection. In many ways, this is the most straightforward and intuitive approach one can take to nonmonotone reasoning.

The language for autoepistemic logic will be $\mathcal{L}_L$, a propositional language $\mathcal{L}$ extended by the modal operator L. However, we will not be interested in traditional modal interpretations of $\mathcal{L}_L$, but in a strictly propositional interpretation, in which every formula of the form $L\varphi$ is treated as an independently valued proposition. (Traditional modal logic will come up once, very briefly. The premise of a proposition will be that a theory $T \in \mathcal{L}_L$ is consistent with **S5**. If the reader is not familiar with modal logics in general, this proposition and its use in the midst of the final proof of the paper can safely be skimmed.)

The core definition in autoepistemic logic is that of a *stable expansion*. A stable expansion is a special sort of *stable theory*. Before we define stable expansions, let us define stable theories and list some of their elementary properties. Most of these results are due to Moore ((Moore 1985)). Proofs can be found in (Marek & Truszczyński 1993).

Definition 1. A propositionally deductively closed theory $T \subseteq \mathcal{L}_L$ is called *stable* if it meets the following two criteria:

- For every $\varphi \in \mathcal{L}_L$, if $\varphi \in T$ then $L\varphi \in T$.
- For every $\varphi \in \mathcal{L}_L$, if $\varphi \notin T$ then $\neg L\varphi \in T$.

To state the results we want, we will need several further definitions.

Definition 2.
- The L-*depth* of a formula $\varphi \in \mathcal{L}_L$, denoted $d_L(\varphi)$, is defined recursively.
  - If $\varphi \in \mathcal{L}$, then $d_L(\varphi) = 0$.
  - If $\varphi = \neg\psi$, then $d_L(\varphi) = d_L(\psi)$.
  - If $\varphi = \psi_1 \vee \psi_2$, $\varphi = \psi_1 \wedge \psi_2$, or $\varphi = \psi_1 \rightarrow \psi_2$, then $d_L(\varphi) = \max\{d_L(\psi_1), d_L(\psi_2)\}$.
  - If $\varphi = L\psi$, then $d_L(\varphi) = d_L(\psi) + 1$.
- $\mathcal{L}_{L,n} = \{\varphi \in \mathcal{L}_L | d_L(\varphi) \leq n\}$.
- Given $T \subseteq \mathcal{L}_L$, $[T]_n = T \cap \mathcal{L}_{L,n}$.

Proposition 3. *Let $T \subseteq \mathcal{L}_L$ be stable. For every integer $n \geq 0$, $[T]_{n+1} =$*

$$\mathrm{Th}([T]_n \cup \{L\varphi | \varphi \in [T]_n\} \cup \{\neg L\varphi | \varphi \in \mathcal{L}_{L,n} \backslash [T]_n\}) \cap \mathcal{L}_{L,n+1}.$$

Proposition 4. *Let $U \subseteq \mathcal{L}_L$ be consistent with **S5**. Then there is a unique stable and consistent theory T such that $U \subseteq T$.*

Proposition 5. *If T is a stable consistent theory, then*

$$T = \mathrm{Th}([T]_0 \cup \{L\varphi | \varphi \in T\} \cup \{\neg L\varphi | \varphi \notin T\}).$$

Proposition 6. *If T is a stable consistent theory, then for every $\varphi \in \mathcal{L}_L$, either $L\varphi \in T$ or $\neg L\varphi \in T$, and not both.*

We will now define stable expansions, the main object of study in autoepistemic logic.

Definition 7 (Stable Expansion). A set of formulas $T \subseteq \mathcal{L}_L$ is a *stable expansion of* $A \subseteq \mathcal{L}_L$ if and only if T is a consistent set of formulas for which $T = \mathrm{Th}(A \cup \{L\varphi | \varphi \in T\} \cup \{\neg L\varphi | \varphi \notin T\})$.

If we think of stable expansions as coherent, justified points of view in a framework represented by the rules of the autoepistemic theory, there are two important sets of formulas we want to take note of: those which can be part of some coherent, justified point of view, and those which must be part of any coherent, justified point of view. If you take as your set of conclusions things which are present in at least one stable expansion, you are reasoning *credulously* or *bravely* (both terms are widely used). If, on the other hand, you believe only those facts true in all stable expansions, you are reasoning *skeptically* or *cautiously*. (The standard contrasts are credulous vs. skeptical reasoning and brave vs. cautious reasoning.)

A question arises: What is the set of skeptical consequences of an autoepistemic theory A if A has no stable expansions? One might be tempted to consider the set of skeptical expansions empty, but it is generally accepted that this is not the approach to use. On the contrary, if A has no stable expansions, *every* proposition φ is considered a skeptical consequence of A. The most intuitive argument for this is to look at the sentence "φ is a member of every stable expansion of A". It is vacuously true if there are no stable expansions of A.

Antisequent Calculus

Bonatti in (Bonatti 1993) presented an antisequent calculus for propositional logic, a counterpart to the propositional fragment of Gentzen's sequent calculus **LK**. We assume that the reader is familiar with this or some other propositional sequent calculus.

An *antisequent* is a pair $\langle\Gamma, \Delta\rangle$ of finite sets of formulas, denoted $\Gamma \nvdash \Delta$. We will call $\Gamma \nvdash \Delta$ *true* if there is a model of Γ in which all of the formulas of Δ are false. We have the benefit of the Soundness and Completeness Theorems for **LK**, which tell us that $\Gamma \nvdash \Delta$ is true if and only if $\Gamma \vdash \Delta$ is false if and only if $\Gamma \vdash \Delta$ is not derivable in **LK**.

An antisequent $\Gamma \nvdash \Delta$ will be considered an *axiom* of our antisequent calculus if $\Gamma \cup \Delta$ consists entirely of atomic formulas and $\Gamma \cap \Delta = \emptyset$. The rules for the antisequent calculus can be found in Table 1.

The antisequent calculus in Table 1 was shown to be sound and complete by Bonatti ((Bonatti 1993)), making use of the classical propositional Soundness and Completeness theorems.

Skeptical Sequent Calculus for Autoepistemic Logic

We will use the propositional fragment of **LK** and its antisequent counterpart given in Table 1, extended to the propositional language $\mathcal{L}_L$. In dealing with lack of proof, we will need not only to assert that such-and-such proposition is proved, but to state explicitly *how* it was proved. Were we dealing explicitly with a full nonmonotonic modal logic, we might want to incorporate Artemov's logic of proofs ((Artemov 2001)), but for our purposes, it will be sufficient to build our skeptical sequents out of a combination of formulas and classical sequents.

A *sequent* for skeptical reasoning in autoepistemic logic will be a pair $\langle\Gamma, \Delta\rangle$, usually written $\Gamma {\mid\!\sim} \Delta$, where $\Gamma \subseteq \mathcal{L}_L \cup \{[\Gamma_0 \vdash \Delta_0] | \Gamma_0 \cup \Delta_0 \subseteq \mathcal{L}_L \text{ is finite}\}$ and $\Delta \subseteq \mathcal{L}_L$. Because our sequent calculus rules will incorporate monotone

$(\neg \nvdash)$	$\dfrac{\Gamma \nvdash \Delta, \varphi}{\Gamma, \neg\varphi \nvdash \Delta}$	$(\nvdash \neg)$	$\dfrac{\Gamma, \varphi \nvdash \Delta}{\Gamma \nvdash \Delta, \neg\varphi}$
$(\wedge \nvdash)$	$\dfrac{\Gamma, \varphi, \psi \nvdash \Delta}{\Gamma, \varphi \wedge \psi \nvdash \Delta}$	$(\nvdash \bullet\wedge)$	$\dfrac{\Gamma \nvdash \Delta, \varphi}{\Gamma \nvdash \Delta, \varphi \wedge \psi}$
		$(\nvdash \wedge\bullet)$	$\dfrac{\Gamma \nvdash \Delta, \psi}{\Gamma \nvdash \Delta, \varphi \wedge \psi}$
$(\bullet\vee \nvdash)$	$\dfrac{\Gamma, \varphi \nvdash \Delta}{\Gamma, \varphi \vee \psi \nvdash \Delta}$	$(\nvdash \vee)$	$\dfrac{\Gamma \nvdash \Delta, \varphi, \psi}{\Gamma \nvdash \Delta, \varphi \vee \psi}$
$(\vee\bullet \nvdash)$	$\dfrac{\Gamma, \psi \nvdash \Delta}{\Gamma, \varphi \vee \psi \nvdash \Delta}$		
$(\bullet \rightarrow\nvdash)$	$\dfrac{\Gamma \nvdash \Delta, \varphi}{\Gamma, \varphi \rightarrow \psi \nvdash \Delta}$	$(\nvdash\rightarrow)$	$\dfrac{\Gamma, \varphi \nvdash \Delta, \psi}{\Gamma \nvdash \Delta, \varphi \rightarrow \psi}$
$(\bullet \rightarrow\nvdash)$	$\dfrac{\Gamma, \psi \nvdash \Delta}{\Gamma, \varphi \rightarrow \psi \nvdash \Delta}$		

Table 1: Rules of the Antisequent Calculus

sequent and antisequent rules, we enclose the classical sequents which are elements of a larger sequent in [brackets]. This is not an ideal notational situation, but $[\Gamma_0 \vdash \Delta_0]$ will be true as part of a larger sequent if and only if $\Gamma_0 \vdash \Delta_0$ is true as its own independent sequent, so there should not be any confusion. The meaning of a sequent $\Gamma \mid\!\sim \Delta$ will be: If monotone sequents in Γ are true, then each stable expansion of $\Gamma \cap \mathcal{L}_L$ contains some member of Δ. The other notion that we will need is that of an L-subformula. ψ is an L-subformula of φ if ψ is a subformula of φ of the form $L\varphi'$. We will denote by $LS(\varphi)$ the set of subformulas of φ. We will also define $LS(\Phi) = \bigcup\{LS(\varphi) | \varphi \in \Phi\}$.

A last standard bit of shorthand: $L\Phi = \{L\varphi | \varphi \in \Phi\}$ and $\neg L\Phi = \{\neg L\varphi | \varphi \in \Phi\}$.

Definition 8 (Skeptical Sequent Calculus—Autoepistemic Logic). The axioms of the skeptical sequent calculus for autoepistemic logic are the axioms of the classical sequent calculus **LK** and of its counterpart antisequent calculus, all in the modal propositional language $\mathcal{L}_L$. The rules of the sequent calculus are:

0. The propositional rules of the classical sequent and antisequent calculi.
1. $\dfrac{\Gamma' \vdash \Delta'}{\Gamma \mid\!\sim \Delta}$ where $\Gamma' \subseteq \Gamma \cap \mathcal{L}_L$ is finite and $\Delta' \subseteq \Delta$ is finite.
2. $\dfrac{\neg L\varphi, \Gamma' \vdash \varphi}{\neg L\varphi, \Gamma \mid\!\sim \Delta}$ where $\Gamma' \subseteq \Gamma \cap \mathcal{L}_L$ is finite.
3. $\dfrac{\Gamma_0 \nvdash \varphi}{[\Gamma_0 \vdash \varphi], \Gamma \mid\!\sim \Delta}$ where $\Gamma_0 \subseteq \mathcal{L}_L$ is finite.
4. $\dfrac{\{[\Gamma_0, L\Phi, \neg L\Psi \vdash \theta], L\Phi, \neg L\Psi, \Gamma \mid\!\sim \Delta | \text{ (*) and (**)}\}}{L\theta, \Gamma \mid\!\sim \Delta}$

   (*) $\Gamma_0 \subseteq \Gamma \cap \mathcal{L}_L$ is finite.

   (**) $L\Phi \cup L\Psi \subseteq LS(\Gamma \cup \{L\theta\})$ is finite.
5. $\dfrac{L\varphi, \Gamma \mid\!\sim \Delta \qquad \neg L\varphi, \Gamma \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}$ where $L\varphi \in LS(\Gamma \cup \Delta)$.

It is worth noting that rule 4 will have infinitely many premises exactly if Γ is infinite. All other rules have finitely many preferences, and this formulation yields a decision procedure in the case that Γ is finite. The intuitive meaning of each of the rules should be made clear by the discussions of their soundness.

We will conclude this short paper with the promised soundness and completeness theorems for the skeptical sequent calculus for autoepistemic logic.

Theorem 9. *If autoepistemic logic skeptical reasoning sequent $\Gamma \mid\!\sim \Delta$ is derivable, then it is true.*

Proof. We will establish the soundness of each of our rules:

1. Because any stable expansion T of $\Gamma \cap \mathcal{L}_L$ will contain Γ' and will be closed under propositional provability, T must contain an element of Δ' and therefore of Δ.
2. Any stable model T of $\{\neg L\varphi\} \cup (\Gamma \cap \mathcal{L}_L)$ will contain Γ' and will be closed under propositional provability, and hence will contain φ. Because T is a stable model it will also contain $L\varphi$. Thus, T will contain $L\varphi$ and $\neg L\varphi$, making T inconsistent. Because stable models are, by definition, consistent, there is no such T and the conclusion is vacuously true.
3. If $\Gamma_0 \nvdash \varphi$, then it is not the case that $\Gamma_0 \vdash \varphi$, and the conclusion is immediately true.
4. Suppose that the conclusion of the rule, $L\theta, \Gamma \mid\!\sim \Delta$, were false. That is, assume that there is a stable expansion T of $\Gamma \cap \mathcal{L}_L$ with $T \cap \Delta = \emptyset$ and $\theta \in T$. (We need $L\theta \in T$. Were T stable and $\theta \notin T$, then $\neg L\theta \in T$ and T would be inconsistent.) By the definition of stable model, θ is a propositional consequence of $\Gamma \cap \mathcal{L}_L \cup \{L\varphi | \varphi \in T\} \cup \{\neg L\varphi | \varphi \notin T\}$, so by compactness it must be a propositional consequence of some finite subset thereof. It should be clear to the reader familiar with propositional logic that formulas $L\varphi$ and $\neg L\varphi$ which are not in $LS(\Gamma \cup \{L\theta\})$ will play no part in this propositional proof. Choose a specific proof of θ. Let Φ be the set of $\varphi \in T$ such that $L\varphi$ is used in this proof, let Ψ be the set of $\psi \notin T$ such that $\neg L\psi$ is used in the proof, and let Γ_0 be the formulas from $\Gamma \cap \mathcal{L}_L$ used in the proof. Obviously, $[\Gamma_0, L\Phi, \neg L\Psi \vdash \theta]$ is true by our choice of Φ, Ψ, and Γ_0. T itself will be a witness to the falsehood of $[\Gamma_0, L\Phi, \neg L\Psi \vdash \theta], L\Phi, \neg L\Psi, \Gamma \mid\!\sim \Delta$, because T is a stable model of $\Gamma \cap \mathcal{L}_L \cup \{L\varphi | \varphi \in \Phi\} \cup \{\neg L\psi | \psi \in \Psi\}$ which has empty intersection with Δ. This will be a premise of rule 4 which is false. Thus, if all premises of rule 4 are true, so is the conclusion.

   (In this rule and the next, we don't have to worry about classical sequent $[\Gamma_0' \vdash \theta'] \in \Gamma$ being false, since we assumed the conclusion of rule 4 false, and had any classical sequent in Γ been false, the skeptical sequent would have been vacuously true.)

5. Suppose that the conclusion of the rule, $\Gamma|\!\sim\Delta$, were false. That is, suppose that there is a stable expansion T of $\Gamma \cap \mathcal{L}_L$ with $T \cap \Delta = \emptyset$. Either $\varphi \in T$ or $\varphi \notin T$. If $\varphi \in T$, then T is a stable expansion of $L\varphi, \Gamma$ with empty intersection with Δ. If $\varphi \notin T$, then T is a stable expansion of $\neg L\varphi, \Gamma$ with empty intersection with T. If the conclusion of rule 5 is false, so is one of its two premises. Thus if both premises are true, so is the conclusion.

□

Theorem 10. *An autoepistemic logic skeptical reasoning sequent $\Gamma|\!\sim\Delta$ is true then it is provable.*

Proof. We will show the contrapositive of our completeness theorem. Let us assume that sequent $\Gamma_0|\!\sim\Delta$ has no proof. We will build a failed attempt at a proof which will be guaranteed to have at least one branch not terminating in an axiom. The Γ developed on this branch will be a witness to the falsehood of $\Gamma_0|\!\sim\Delta$.

We will start with $\Gamma_0|\!\sim\Delta$ and build a proof attempt, expanding each sequent with the premises of some rule having that sequent as a conclusion, if possible. We will be constructing a countably branching tree, so we will need to work in some suitable ordering of $\omega^{<\omega}$.

If the sequent $\Gamma|\!\sim\Delta$ under consideration has a proof under rule 1, 2, or 3 from true premises, make $\Gamma|\!\sim\Delta$ the conclusion of that rule and finish out the proof above that point.

If the sequent $\Gamma|\!\sim\Delta$ whose branch we are trying to extend has at least one $L\theta \in \Gamma$ or has at least one $L\varphi \in LS(\Gamma \cup \Delta)$ such that neither $L\varphi$ nor $\neg L\varphi$ is in Γ, we will be guaranteed that there are rules of type 4 or 5 with conclusion $\Gamma|\!\sim\Delta$. If no rules of type 1, 2, or 3 with the appropriate conclusion (and true premises) can be found, extend the proof attempt with either rule 4 or rule 5, if possible. If there are an even number of sequents below the present one (and if it may be applied) use rule 4; if there are an odd number of sequents below the present one (and it may be applied) use rule 5. When using rule 4, work with the $L\theta$ which appeared earliest in the development of the proof attempt. (If more than one appeared at the same time, choose the one whose θ is least in some ordering of $\mathcal{L}_L$.) When using rule 5, work with the φ least in your ordering of $\mathcal{L}_L$.

Of course, if no rule at all has conclusion $\Gamma|\!\sim\Delta$, leave the sequent alone.

Since we assumed that the sequent $\Gamma_0|\!\sim\Delta$ has no proof, there will be at least some branch of our attempted proof tree which was expanded using only rules 4 and 5 and which therefore does not terminate in an axiom. Select one of these branches. (Note that neither rule 4 nor rule 5 affects Δ, so while Γ will expand as we traverse the branch, Δ will remain constant.)

Let us define Γ^* to be the set of all formulas of $\mathcal{L}_L$ which appear in the Γ of a sequent $\Gamma|\!\sim\Delta$ anywhere along the selected branch. We will show that $\mathrm{Th}(\Gamma^*)$ can be extended to a stable theory which is a stable expansion of Γ for each sequent $\Gamma|\!\sim\Delta$ in our selected branch and which excludes all of Δ. Let us select an arbitrary $\Gamma|\!\sim\Delta$.

The first step in this procedure will be to show $\mathrm{Th}(\Gamma^*)$ consistent with **S5**. To accomplish this, we'll use several claims.

- Claim 1: Γ^* is consistent.
  Justification: If Γ^* weren't consistent, then it would have to have been inconsistent by some finite stage in the development of the branch. We could have used rule 1 to terminate the branch at the point where the inconsistency entered.
- Claim 2: If $L\varphi \in \mathrm{Th}(\Gamma^*)$, then $L\varphi \in \Gamma^*$.
  Justification: Γ^* consists of Γ plus some formulas of the form $L\psi$ and $\neg L\psi$. If $L\varphi \in \mathrm{Th}(\Gamma^*)$, but $L\varphi \notin \Gamma^*$, it must have been because some formula of Γ was used to prove $L\varphi$. (Th refers to propositional provability and $L\varphi$ is treated as a propositional atom, so this is the only explanation.) That means that $L\varphi$ is the conclusion of some implication in Γ, and is therefore in $LS(\Gamma)$. Because of the way Γ^* was defined, for every $L\psi \in LS(\Gamma)$, either $L\psi \in \Gamma^*$ or $\neg L\psi \in \Gamma^*$. Because Γ^* is consistent, if $L\varphi \in LS(\Gamma) \cap \mathrm{Th}(\Gamma^*)$, then $L\varphi \in \Gamma^*$.
- Claim 3: If $\neg L\varphi \in \mathrm{Th}(\Gamma^*)$, then $\neg L\varphi \in \Gamma^*$.
  Justification: Same as for claim 2.
- Claim 4: If $L\varphi \in \Gamma^*$, then $\varphi \in \mathrm{Th}(\Gamma^*)$.
  Justification: By rule 4, if $L\varphi \in \Gamma^*$, then so is $[\Gamma' \vdash \varphi]$ for some finite $\Gamma' \subseteq \Gamma^*$. Were $\Gamma' \vdash \varphi$ false, then the branch could have been terminated using rule 3. Since the branch could not be terminated, it must be that $\Gamma' \vdash \varphi$, and hence $\varphi \in \mathrm{Th}(\Gamma^*)$.
- Claim 5: If $\neg L\varphi \in \Gamma^*$, then $\varphi \notin \mathrm{Th}(\Gamma^*)$.
  Justification: If φ were in $\mathrm{Th}(\Gamma^*)$, then for some $\Gamma'|\!\sim\Delta$ on the selected branch, it would be the case that $\Gamma' \vdash \varphi$ and $\neg L\varphi \in \Gamma'$, and the branch could have been terminated using rule 2.
- Claim 6: If $L\varphi \in \mathrm{Th}(\Gamma^*)$, then $\varphi \in \mathrm{Th}(\Gamma^*)$.
  Justification: Immediate from claims 2 and 4.
- Claim 7: If $\neg L\varphi \in \mathrm{Th}(\Gamma^*)$, then $\varphi \notin \mathrm{Th}(\Gamma^*)$.
  Justification: Immediate from claims 3 and 5.

We can now easily show $\mathrm{Th}(\Gamma^*)$ to be consistent with **S5**.

- To be inconsistent with axiom k, we would need $L(\varphi \to \psi)$, $L\varphi$, and $\neg L\psi$ in $\mathrm{Th}(\Gamma^*)$. By claims 6 and 7 above, this would put $\varphi \to \psi$ and φ in $\mathrm{Th}(\Gamma^*)$ and leave ψ out of $\mathrm{Th}(\Gamma^*)$. So $\mathrm{Th}(\Gamma^*)$ is consistent with axiom k.
- To be inconsistent with axiom t, we would need $L\varphi \in \mathrm{Th}(\Gamma^*)$ and $\varphi \notin \mathrm{Th}(\Gamma^*)$, contradicting claim 6.
- To be inconsistent with axiom 4, we would need $L\varphi \in \mathrm{Th}(\Gamma^*)$ and $\neg LL\varphi \in \mathrm{Th}(\Gamma^*)$. Claim 7 tells us that $L\varphi \notin \mathrm{Th}(\Gamma^*)$, which is impossible, so $\mathrm{Th}(\Gamma^*)$ is consistent with axiom 4.
- To be inconsistent with axiom 5, we would need $\neg L\neg L\varphi$ and $\neg L\varphi$ both in $\mathrm{Th}(\Gamma^*)$. The former, with claim 7, tells us that $\neg L\varphi \notin \mathrm{Th}(\Gamma^*)$, so it must be that $\mathrm{Th}(\Gamma^*)$ is consistent with axiom 5.

We can now say by Proposition 4 that $\mathrm{Th}(\Gamma^*)$ can be extended to a unique consistent stable theory T. Finally, we need to show T to be a stable expansion of Γ.

Let us define $S = \mathrm{Th}(\Gamma \cup \{L\varphi | \varphi \in T\} \cup \{\neg L\varphi | \varphi \notin T\})$. If we can show that $S = T$, then we will have shown T to be a stable expansion of Γ.

That $S \subseteq T$ is fairly clear. We know that $\Gamma \subseteq \mathrm{Th}(\Gamma^*) \subseteq T$. Because T is stable, both $\{L\varphi | \varphi \in T\} \subseteq T$ and $\{\neg L\varphi | \varphi \notin T\} \subseteq T$ and T is propositionally closed.

To show $T \subseteq S$, let us first show that for each $L\varphi \in \Gamma^*$, $\varphi \in T$. Because T extends Γ^*, if $L\varphi \in \Gamma^*$, then $L\varphi \in T$. If it were the case that $\varphi \notin T$, then by the stability of T, we would also have $\neg L\varphi \in T$. Because we know T to be consistent, this can not happen. A nearly identical argument shows that for each $\neg L\varphi \in \Gamma^*$, $\varphi \notin T$.

Now, to show $T \subseteq S$, we'll make use of Proposition 5, which says that

$$T = \mathrm{Th}([T]_0 \cup \{L\varphi | \varphi \in T\} \cup \{\neg L\varphi | \varphi \notin T\}).$$

Thus, all we need to show is that $[T]_0 \subseteq S$. Because $\mathrm{Th}(\Gamma^*)$ was consistent with **S5**, so is $[\mathrm{Th}(\Gamma^*)]_0$. Since each $U \subseteq \mathcal{L}_L$ consistent with **S5** is contained in a *unique* stable theory T, it must be that $[T]_0 = [\mathrm{Th}(\Gamma^*)]_0$. So what we need to show is that any propositional formula in $\mathrm{Th}(\Gamma^*)$ can be proved from Γ, $\{L\varphi | \varphi \in T\}$, and $\{\neg L\varphi | \varphi \notin T\}$. We just argued that any L and $\neg L$ formulas in Γ^* are to be found in the latter two sets, and of course any formula of Γ^* which is not of one of these two types must have come from Γ itself. We have shown not only that $[\mathrm{Th}(\Gamma^*)]_0 \subseteq S$, but that $\mathrm{Th}(\Gamma^*) \subseteq S$.

One last step will complete the proof. We need to show that T excludes all formulas of Δ. By rule 5, we know that Γ^* contained either $L\varphi$ or $\neg L\varphi$ for all formulas $L\varphi \in LS(\Delta)$. We also know that $[T]_0 = [\mathrm{Th}(\Gamma^*)]_0$. Were it possible to prove any formula of Δ from T, it would have been possible to prove that formula from Γ^*. Were it possible to prove some formula of Δ from Γ^*, it would have been possible to prove that formula from Γ' for some sequent $\Gamma' \mid\!\sim \Delta$ on our chosen branch. Had this been possible, we could have terminated the branch using rule 1. Since we could not, it must be that $\mathrm{Th}(\Gamma^*) \cap \Delta = \emptyset$, and hence $T \cap \Delta = \emptyset$.

$\square$

Further Directions For Research

We have now seen sequent calculi for autoepistemic logic, and calculi exist for stable model logic programming and predicate default logic as well. One obvious direction to go to extend this work would be to do the same for predicate circumscription. As we noted in the introduction, this would require a Π^1_2 sequent calculus. Just as, we hope, the calculi presented here will aid in the understanding of the subtlety of skeptical reasoning in the logics discussed, a Π^1_2 sequent calculus for circumscription might offer insights into that framework.

One distinguishing feature of the calculus presented above is that the assertion that φ simply *has* a proof is not enough. We must look at all possible proofs of φ. This necessity to take an assertion of the existence of a proof and explicate it with an actual proof fairly calls out for a connection with Artemov's logic of proofs (see (Artemov 2001)). Because the logic of proofs has such a strong connection with modal logic, McDermott and Doyle's nonmonotonic modal logics are the natural candidates for the first study of such a connection.

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