On Császár's condition in nonmonotonic reasoning

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Abstract

Császár's condition is a well-known property introduced about 50 years ago in the axiomatic theory of conditional probability. In recent years such condition has been reconsidered by some authors, who have studied its role in the coherence-based approach to conditional probability. In this paper we consider the probabilistic entailment of a conditional knowledge base by another one. We represent Loop rule in a generalized way and, using Császár's condition, we give a simple probabilistic interpretation of it. Then, exploiting the rules Cautious Monotonicity and Cut, we obtain some related results on p-entailment by the knowledge base associated with Loop rule. We also determine the best probability bounds for the quasi-conjunction of two conditional events and we give a probabilistic semantics for the *QAND* rule. Finally, we reconsider our results in the setting of conditional objects.

1 Introduction

In nonmonotonic reasoning the inferential process is developed by applying to a given set of *conditional assertions* a suitable set of rules, deriving in this way other conditional assertions. A survey on nonmonotonic logics has been given given in (Benferhat *et al.* 1997). In such field a widely accepted formalism is System P (Kraus *et al.* 1990), which has a probabilistic semantics based on infinitesimal probabilities (Adams 1975, Pearl 1988). Based on *big-stepped probabilities*, i.e. *atomic bound systems* studied in (Snow 1996, 1999), in (Benferhat *et al.* 1999) a probabilistic semantics has been given for System P, without referring to *infinitesimals*. An approach, based on lower probability bounds, has been proposed in (Bourne and Parsons 1998; Parsons and Bourne 2000). Uncertain reasoning based on conditional constraints has been considered in many papers (see, e.g., Amarger *et al.* 1991; Dubois *et al.* 1993, Lukasiewicz 2002). In (Gilio 2000, 2002a) this approach has been developed in the framework of coherent conditional probabilities. The relationship between model-theoretic probabilistic logic and coherence-based probabilistic logic has been examined in (Biazzo *et al.* 2002), where it has been shown that probabilistic entailment under coherence is a generalization of classical default entailment in System P. We recall that a coherent conditional probability satisfies all the axioms of conditional probability; on the contrary a function P which satisfies such axioms may be not coherent. Sufficient conditions for the coherence of a function P defined on $\mathcal{A} \times \mathcal{X}$, where $\mathcal{A}$ is an algebra of events and $\mathcal{X}$ is a subfamily of $\mathcal{A}$ not containing $\emptyset$, have been considered in many papers. In particular, in (Rigo 1988) it has been shown that P is coherent if and only if it satisfies a suitable condition introduced in (Császár 1955). Such condition has been also considered in (Gilio and Spezzaferri 1992, 1995) and appears in (Amarger *et al.* 1991) under the name of *generalized Bayes theorem*. Interestingly, such theoretical condition is related to *Loop* rule, examined in (Kraus *et al.* 1990) within the **CL** logic system. In fact, using Császár's condition, we can provide a probabilistic semantics for Loop rule. Then, exploiting CM and Cut rules, we can obtain related results which concern the probabilistic entailment of some conditional knowledge bases from the conditional knowledge base associated with Loop rule. We also determine the best bounds for the probability of the quasi-conjunction of two conditional events. Then, we obtain a probabilistic semantics for the *QAND* rule and we reconsider our results in the setting of conditional objects studied in (Dubois and Prade 1994).
The paper is organized as follows:

- in Section 2 we recall the notions of conditional probability and coherence; moreover, we illustrate the role of Császár's condition in the setting of coherence;
- in Section 3 we give, in the setting of coherence, the notions of p-consistency and p-entailment.
- in Section 4, using Császár's condition, we give a probabilistic semantics for a generalized version of Loop rule;
- in Section 5 exploiting Cautious Monotonicity and Cut rules, we obtain related results on p-entailment of some knowledge bases by the knowledge base associated with Loop rule;
- in Section 6 we obtain the best bounds for the probability of the quasi-conjunction of two conditional events and then we give a probabilistic semantics for the *QAND* rule;
- in Section 7 we reconsider our results in the setting of conditional objects;
- in Section 8 we examine a simple example;
- in Section 9 we give some conclusions.

2 Some preliminary notions and results

Given two events A, B, we denote their conjunction by AB and their disjunction by $A \vee B$. We recall that, given an algebra of events $\mathcal{A}$ and a non empty subfamily $\mathcal{X} \subseteq \mathcal{A}$, with $\emptyset \notin \mathcal{X}$, a (finitely additive) conditional probability P on $\mathcal{A} \times \mathcal{X}$ is (usually looked at as) a real function defined on $\mathcal{A} \times \mathcal{X}$ satisfying the following properties:
(i) $P(\cdot|H)$ is a finitely additive probability on $\mathcal{A}$, for every $H \in \mathcal{X}$;
(ii) $P(H|H) = 1$, for every $H \in \mathcal{X}$;
(iii) $P(AB|H) = P(B|AH)P(A|H)$, for every A, B, H, with $A \in \mathcal{A}$, $B \in \mathcal{A}$, $H \in \mathcal{X}$, $AH \in \mathcal{X}$.
Given a real function P defined on an arbitrary family of conditional events $\mathcal{F}$, let $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$ be a finite subfamily of $\mathcal{F}$ and $\mathcal{P}_n$ the vector $(p_1, \dots, p_n)$, where $p_i = P(E_i|H_i)$. We use the same symbol to denote an event and its indicator. Moreover, considering the random gain

$$G_n = \sum_{i=1}^{n} s_i H_i (E_i - p_i),$$

with $s_1, \dots, s_n$ arbitrary real numbers, we denote by $G_n|\mathcal{H}_n$ the restriction of G_n to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, based on the *betting scheme*, we have

Definition 1 The function P is said coherent if and only if

$$Max\, G_n|\mathcal{H}_n \geq 0\,,\ \forall\, n \geq 1,\ \forall\, \mathcal{F}_n \subseteq \mathcal{F},\ \forall\, s_1, \dots, s_n \in \boldsymbol{R}.$$

Remark 1 We recall that, if P is coherent, then P satisfies all the axioms of a conditional probability; the converse is not true; for some counterexamples, see (Gilio and Spezzaferri 1992, Sec. 4.1; Gilio 1995, Example 8; Coletti and Scozzafava 2002, Example 13). To remark that a function P defined on $\mathcal{A} \times \mathcal{X}$ and satisfying the axioms $(i) - (iii)$ may be not coherent, in (Coletti and Scozzafava, 2002) P is called a *weak conditional probability*.

We recall that a family of events $\mathcal{X}$ is said an *additive* class if, for every H_1, H_2 in $\mathcal{X}$, it is $H_1 \vee H_2 \in \mathcal{X}$.
Given a (weak) conditional probability P on $\mathcal{A} \times \mathcal{X}$, the family $\mathcal{X}$ is said a $P-$*quasi additive* class (Császár 1955) if, for every H_1, H_2 in $\mathcal{X}$, there exists $K \in \mathcal{X}$ such that:

$$(i)\ H_1 \vee H_2 \subseteq K\,; \quad (ii)\ P(H_1|K) + P(H_2|K) > 0\,.$$

We observe that, for every H_1, H_2, it is

$$P(H_1 \vee H_2 \,|\, H_1 \vee H_2) = 1\,,$$

and

$$P(H_1 \vee H_2 \,|\, H_1 \vee H_2) \leq P(H_1 \,|\, H_1 \vee H_2) + P(H_2 \,|\, H_1 \vee H_2)\,.$$

Therefore

$$P(H_1 \,|\, H_1 \vee H_2) + P(H_2 \,|\, H_1 \vee H_2) \geq 1,\ \forall\, H_1, H_2\,. \quad (1)$$

Then, given a conditional probability P on $\mathcal{A} \times \mathcal{X}$, with $\mathcal{X}$ additive, for every H_1, H_2 in $\mathcal{X}$ it is $H_1 \vee H_2 \in \mathcal{X}$, so that from (1) it follows that $\mathcal{X}$ is $P-$quasi additive. In particular, if $\mathcal{X} \cup \{\emptyset\}$ is a sub-algebra of $\mathcal{A}$, then $\mathcal{X}$ is, of course, additive and hence it is $P-$quasi additive.

Remark 2 Given a conditional probability P defined on $\mathcal{A} \times \mathcal{X}$, in (Császár 1955, Theorem 8.5) the equivalence of the following propositions has been proved:

- the following condition is satisfied:
$$\Pi_{i=1}^{n} P(E_i|H_i) = \Pi_{i=1}^{n} P(E_i|H_{i+1})\,, \quad (2)$$
where $E_i \in \mathcal{A}, H_i \in \mathcal{X}, E_i \subseteq H_i H_{i+1}$, and $H_{n+1} = H_1$;
- there exists an extension of P to P^* defined on $\mathcal{A} \times \mathcal{X}^*$, where $\mathcal{X}^*$ is an additive class containing $\mathcal{X}$;
- there exists an extension of P to P^* defined on $\mathcal{A} \times \mathcal{X}^*$, where $\mathcal{X}^*$ is a $P-$quasi additive class containing $\mathcal{X}$.

As discussed in (Coletti and Scozzafava 2002), the validity of *Császár's condition* (2) is necessary and sufficient for the existence of a *dimensionally ordered* class of measures μ_α, defined on $\mathcal{A}$, apt to *represent* P, i.e., such that, for any $E|H \in \mathcal{A} \times \mathcal{X}$, it is $P(E|H) = \frac{\mu_\alpha(EH)}{\mu_\alpha(H)}$ for a suitable α.
We remark that, when the involved probabilities are positive, (2) reduces to the *generalized Bayes' theorem* considered in (Amarger *et al.* 1991).

Császár's condition plays a relevant role for what concerns coherence of P. In fact, we have (Rigo 1988; see also Gilio and Spezzaferri 1995)

Theorem 1 A conditional probability P defined on $\mathcal{A} \times \mathcal{X}$, where $\mathcal{A}$ is an algebra of events and $\mathcal{X}$ is a non empty subfamily of $\mathcal{A}$, with $\emptyset \notin \mathcal{X}$, is coherent if and only if, for each n, condition (2) is satisfied.

As it follows by Remark 2 and Theorem 1, if $\mathcal{X}$ is $P-$quasi additive (or additive; or, in particular, $\mathcal{X} \cup \{\emptyset\}$ is a subalgebra of $\mathcal{A}$), then P is coherent.
In (Rigo 1988) it is proved that a (weak) conditional probability P defined on $\mathcal{A} \times \mathcal{X}$ can be extended as a *full* conditional probability P^* defined on $\mathcal{A} \times \mathcal{A}^0$, where $\mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$.
A direct proof of the coherence of P when $\mathcal{X}$ is $P-$quasi additive is given in (Gilio 1989).
A result related with Theorem 1 is the following

Corollary 1 Let $\mathcal{P} = (a_i, b_i,\, i = 1, \dots, n)$ a probability assessment on the family of conditional events $\mathcal{F} = \{E_i|H_i,\, E_i|H_{i+1},\, i = 1, 2 \dots, n\}$, with $E_i \subseteq H_i H_{i+1},\, \forall\, i,\, H_{n+1} = H_1$, and with $a_i = \mathcal{P}(E_i|H_i)$, $b_i = \mathcal{P}(E_i|H_{i+1})$. If $\mathcal{P}$ is coherent, then $\Pi_{i=1}^{n} a_i = \Pi_{i=1}^{n} b_i$, i.e. the condition (2) is satisfied.

Remark 3 We look at a conditional event $B|A$ ($A \neq \emptyset$) as a three-valued logical entity, with values *true*, or *false*, or *undetermined*, according to whether A and B are true, or A is true and B is false, or A is false. Then, for every pair of events A, B, with $A \neq \emptyset$, it is $B|A = BA|A$, so that $P(BA|A) = P(B|A)$. Then, given three events E, F, H and applying Corollary 1, with $n = 2$ and with

$$E_1 = EFH,\ H_1 = H_3 = H,\ E_2 = H_2 = FH,$$

the condition $a_1 a_2 = b_1 b_2$, which is necessary for the coherence of the assessment $\mathcal{P} = (a_i, b_i,\, i = 1, 2)$ on $\mathcal{F} = \{E_i|H_i,\, E_i|H_{i+1},\, i = 1, 2\}$, becomes

$$P(EF|H) = P(E|FH)P(F|H)\,;$$

that is, the third axiom of conditional probabilities is a particular case of Császár's condition.

3 Probabilistic entailment of conditional knowledge bases

In this section we give the notions of probabilistic consistency and probabilistic entailment, introduced in (Adams 1975) and adapted to the coherence-based setting in (Gilio 2002a). We recall that in the framework of default reasoning a conditional knowledge base is a set of defaults, or conditional assertions, $H \mid\!\sim E$, which may be read as "generally, if H then E". In (Adams 1975) $A \mid\!\sim B$, is looked at as $P(B|A) \geq 1 - \varepsilon \quad (\forall \varepsilon > 0)$. Given a set of integers J and a conditional knowledge base $\mathcal{K} = \{H_j \mid\!\sim E_j\,,\ j \in J\}$, associated with a family of conditional events $\mathcal{F} = \{E_j|H_j\,,\ j \in J\}$, we give below the definition of p-consistency for $\mathcal{K}$.

Definition 2 The conditional knowledge base $\mathcal{K} = \{H_j \mid\!\sim E_j\,,\ j \in J\}$ is *p-consistent* iff, for every set of lower bounds $\mathcal{L} = \{\alpha_j, j \in J\}$, there exists a coherent conditional probability assessment $P = \{p_j, j \in J\}$ defined on $\mathcal{F}$, with $p_j = P(E_j|H_j)$, such that $p_j \geq \alpha_j$ for every $j \in J$.

Given two families of conditional events $\mathcal{F}_1 = \{E_j|H_j\,,\ j \in J_1\}$ and $\mathcal{F}_2 = \{A_j|K_j\,,\ j \in J_2\}$, and the associated conditional knowledge bases $\mathcal{K}_1 = \{H_j \mid\!\sim E_j\,,\ j \in J_1\}$ and $\mathcal{K}_2 = \{K_j \mid\!\sim A_j\,,\ j \in J_2\}$, we define below the p-entailment of $\mathcal{K}_2$ by $\mathcal{K}_1$.

Definition 3 Given two p-consistent knowledge bases

$$\mathcal{K}_1 = \{H_j \mid\!\sim E_j\,,\ j \in J_1\}\,,\ \mathcal{K}_2 = \{K_j \mid\!\sim A_j\,,\ j \in J_2\}\,,$$

we say that $\mathcal{K}_1$ *p-entails* $\mathcal{K}_2$, denoted $\mathcal{K}_1 \Rightarrow \mathcal{K}_2$, iff there exists $\Gamma = \{H_j \mid\!\sim E_j\,,\ j \in I\} \subseteq \mathcal{K}_1$ such that, for every set of lower bounds $\mathcal{L}_2 = \{\beta_j, j \in J_2\}$ on $\mathcal{F}_2$, with $\beta_j \leq 1\ \forall\, j$, there exists a set of lower bounds $\mathcal{L}_1 = \{\alpha_j, j \in I\}$ on Γ such that, for all coherent conditional probability assessments $P = \{p_j, j \in I \cup J_2\}$ defined on $\Gamma \cup \mathcal{F}_2$, with $p_j = P(E_j|H_j),\ \forall\, j \in I$, and $p_j = P(A_j|K_j),\ \forall\, j \in J_2$, if $p_j \geq \alpha_j$ for every $j \in I$, then $p_j \geq \beta_j$ for every $j \in J_2$.

Remark 4 By Definition 3 one trivially has

$$\mathcal{K}_1 \Rightarrow \Gamma\,,\ \forall \Gamma \subseteq \mathcal{K}_1\,;$$

in particular,

$$\mathcal{K}_1 \Rightarrow H_j \mid\!\sim E_j\,,\ \forall H_j \mid\!\sim E_j \in \mathcal{K}_1. \quad (3)$$

Moreover, given three knowledge bases $\mathcal{K}_1, \Gamma_1, \Gamma_2$, one has

$$\mathcal{K}_1 \Rightarrow \Gamma_1 \cup \Gamma_2 \ \text{ iff } \ \mathcal{K}_1 \Rightarrow \Gamma_1\,, \mathcal{K}_1 \Rightarrow \Gamma_2\,. \quad (4)$$

Therefore,

$$\mathcal{K}_1 \Rightarrow \mathcal{K}_2 \ \text{ iff } \ \mathcal{K}_1 \Rightarrow \mathcal{K}'\,,\ \forall \mathcal{K}' \subseteq \mathcal{K}_2\,.$$

In particular,

$$\mathcal{K}_1 \Rightarrow \mathcal{K}_2 \ \text{ iff } \ \mathcal{K}_1 \Rightarrow K_j \mid\!\sim A_j, \forall K_j \mid\!\sim A_j \in \mathcal{K}_2. \quad (5)$$

4 Probabilistic semantics of Loop rule

In this section, using Császár's condition, we give a probabilistic interpretation of *Loop* rule (Kraus *et al.* 1990). Then, exploiting the *Cautious Monotonicity* and *Cut* rules, we obtain related results on p-entailment by the knowledge base associated with Loop rule. Given $k+1$ logically independent events $A_0,\ A_1,\ \ldots,\ A_k$, Loop rule is the following one:

$$A_0 \mid\!\sim A_1\,,\ A_1 \mid\!\sim A_2\,,\ \cdots\,,\ A_k \mid\!\sim A_0 \implies A_0 \mid\!\sim A_k\,.$$

As remarked in (Kraus *et al.* 1990), it seems that this rule has never been considered in the literature. In Lemma 4.3 of the same paper it is proved that, for every $i, j = 0, 1, \ldots, k$, the following is a derived rule of **CL** system:

$$A_0 \mid\!\sim A_1\,,\ A_1 \mid\!\sim A_2\,,\ \cdots\,,\ A_k \mid\!\sim A_0 \implies A_i \mid\!\sim A_j\,.$$

A probabilistic interpretation of Loop rule has been already given in (Gilio 2002b), where the following result has been proved.

Theorem 2 Given $k+1$ logically independent events $A_0, A_1, \ldots, A_k$, let us consider the conditional probability assessment $\mathcal{P} = (1, 1, \ldots, 1)$ on the family $\mathcal{F} = \{A_1|A_0, A_2|A_1, \ldots, A_0|A_k\}$. Moreover, given the further conditional event $A_k|A_0$, let $\mathcal{P}' = (\mathcal{P}, p)$ a conditional probability assessment on $\mathcal{F} \cup \{A_k|A_0\}$, with $p = P(A_k|A_0)$. Then, we have
1. the assessment $\mathcal{P}$ is coherent;
2. the assessment $\mathcal{P}' = (\mathcal{P}, p)$ is coherent iff $p = 1$.

The proof of Theorem 2 given in (Gilio 2002b) is based on the following formula

$$A_0 A_1 \cdots A_n \vee A_0 A_1^c \vee A_1 A_2^c \vee \cdots \vee A_n A_0^c =$$

$$= A_0 \vee A_1 \vee \cdots \vee A_n\,,$$

which gives an alternative representation for the disjunction of the events $A_0, A_1, \ldots, A_n$.
We represent Loop rule in the following generalized way

$$A_0 \mid\!\sim A_1\,,\ \ A_1 \mid\!\sim A_2\,,\ \ \cdots\,,\ \ A_k \mid\!\sim A_0\,,$$

$$\Downarrow \Uparrow$$

$$A_1 \mid\!\sim A_0\,,\ \ A_2 \mid\!\sim A_1\,,\ \ \cdots\,,\ \ A_0 \mid\!\sim A_k\,.$$

Given a vector of lower bounds

$$\mathcal{L}_1 = (\alpha_0, \alpha_1, \ldots, \alpha_k),$$

we denote by $\mathbf{P}_{1,\,\mathcal{L}_1}$ the set of coherent assessments

$$\mathcal{P}_1 = (p'_0, p'_1, \ldots, p'_k)$$

on the family

$$\mathcal{F}_1 = \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\},$$

where

$$p'_i = P(A_{i+1}|A_i),\ i = 0, 1, \ldots, k,\ \ A_{k+1} = A_0,$$

such that

$$p'_0 \geq \alpha_0\,,\ p'_1 \geq \alpha_1\,,\ \cdots, p'_k \geq \alpha_k\,.$$

Analogously, given a vector of lower bounds

$$\mathcal{L}_2 = (\beta_0, \beta_1, \ldots, \beta_k),$$

we denote by $\mathbf{P}_{2,\,\mathcal{L}_2}$ the set of coherent assessments

$$\mathcal{P}_2 = (p''_0, p''_1, \ldots, p''_k)$$

on the family

$$\mathcal{F}_2 = \{A_0|A_1, A_1|A_2, \ldots, A_{k-1}|A_k, A_k|A_0\},$$

where

$$p''_i = P(A_i|A_{i+1}),\ i = 0, 1, \ldots, k,\ \ A_{k+1} = A_0,$$

such that

$$p''_0 \geq \beta_0\,,\ p''_1 \geq \beta_1\,,\ \cdots, p''_k \geq \beta_k\,.$$

We denote by $\mathbf{P}_{1,2}$ the set of coherent probability assessment on $\mathcal{F}_1 \cup \mathcal{F}_2$. Then, Theorem 2 can be generalized by the following

Theorem 3 Given $k + 1$ logically independent events $A_0, A_1, \ldots, A_k$, let us consider the families of conditional events

$$\begin{aligned}\mathcal{F}_1 &= \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\}\,,\\ \mathcal{F}_2 &= \{A_0|A_1, A_1|A_2, \ldots, A_{k-1}|A_k, A_k|A_0\}\,,\end{aligned} \tag{6}$$

and the associated conditional knowledge bases

$$\mathcal{K}_1 = \{A_0 \mid\!\sim A_1\,,\ A_1 \mid\!\sim A_2\,,\ \cdots,\ A_k \mid\!\sim A_0\}\,,$$

$$\mathcal{K}_2 = \{A_1 \mid\!\sim A_0\,,\ A_2 \mid\!\sim A_1\,,\ \cdots,\ A_0 \mid\!\sim A_k\}\,.$$

Then, we have: $(i)\ \mathcal{K}_1 \Rightarrow \mathcal{K}_2$; $\quad (ii)\ \mathcal{K}_2 \Rightarrow \mathcal{K}_1$.

Proof. We have to prove that:
(i) for every vector $\mathcal{L}_2 = (\beta_0, \beta_1, \ldots, \beta_k)$, with $\beta_i \leq 1\,\forall\, i$, there exists a vector $\mathcal{L}_1 = (\alpha_0, \alpha_1, \ldots, \alpha_k)$ such that

$$\mathcal{P}_1 \in \mathbf{P}_{1,\mathcal{L}_1} \implies \mathcal{P}_2 \in \mathbf{P}_{2,\mathcal{L}_2}\,,\ \forall\,(\mathcal{P}_1, \mathcal{P}_2) \in \mathbf{P}_{1,2}\,;$$

(ii) for every vector $\mathcal{L}_1 = (\alpha_0, \alpha_1, \ldots, \alpha_k)$, with $\alpha_i \leq 1\,\forall\, i$, there exists a vector $\mathcal{L}_2 = (\beta_0, \beta_1, \ldots, \beta_k)$ such that

$$\mathcal{P}_2 \in \mathbf{P}_{2,\mathcal{L}_2} \implies \mathcal{P}_1 \in \mathbf{P}_{1,\mathcal{L}_1}\,,\ \forall\,(\mathcal{P}_1, \mathcal{P}_2) \in \mathbf{P}_{1,2}\,.$$

Recalling Remark 3, $\mathcal{F}_1$ and $\mathcal{F}_2$ can be written as

$$\mathcal{F}_1 = \{A_1A_0|A_0\,,\ A_2A_1|A_1\,,\ \ldots\,,\ A_0A_k|A_k\}\,,$$

$$\mathcal{F}_2 = \{A_1A_0|A_1\,,\ A_2A_1|A_2\,,\ \ldots\,,\ A_0A_k|A_0\}\,.$$

We observe that $A_iA_{i+1} \subseteq A_i$ and $A_iA_{i+1} \subseteq A_{i+1}$ for every i; so that, given any probability assessment $(\mathcal{P}_1, \mathcal{P}_2)$ on $\mathcal{F}_1 \cup \mathcal{F}_2$, from Theorem 2, applying Császár's condition with $H_i = A_i, E_i = A_{i+1}A_i, i = 0, 1, \ldots, k, H_{k+1} = A_0$, it follows that in order $(\mathcal{P}_1, \mathcal{P}_2)$ be coherent it must be

$$\Pi_{i=0}^k P(A_{i+1}|A_i) \ = \ \Pi_{i=0}^k P(A_i|A_{i+1})\,.$$

(i) given a vector of lower bounds $\mathcal{L}_2 = (\beta_0, \beta_1, \ldots, \beta_k)$, with

$$max\,\{\beta_0, \beta_1, \ldots, \beta_k\} = \beta_j\,,$$

let $\mathcal{L}_1 = (\alpha_0, \alpha_1, \ldots, \alpha_k)$ be a vector of lower bounds such that $\Pi_{i=0}^k \alpha_i \ \geq \ \beta_j$ (for example, we could choose $\alpha_i = 1, \forall\, i$).
For each given assessment $\mathcal{P}_1$ on $\mathcal{F}_1$, we denote by $\mathcal{E}_1$ the set of coherent extensions on $\mathcal{F}_2$ of $\mathcal{P}_1$. Then, for every probability assessment $\mathcal{P}_1 = (p'_0, p'_1, \ldots, p'_k)$ on $\mathcal{F}_1$, with $p'_i = P(A_{i+1}A_i|A_i) = P(A_{i+1}|A_i)$, and with $\mathcal{P}_1 \in \mathbf{P}_{1,\mathcal{L}_1}$, it is $\Pi_{i=0}^k p'_i \ \geq \ \Pi_{i=0}^k \alpha_i$, and hence, denoting by $\mathcal{P}_2 = (p''_0, p''_1, \ldots, p''_k)$ a coherent assessment on the family $\mathcal{F}_2$, with $p''_i = P(A_{i+1}A_i|A_{i+1}) = P(A_i|A_{i+1})$, from Corollary 1 it follows

$$\Pi_{i=0}^k p''_i \ = \ \Pi_{i=0}^k p'_i \ \geq \ \Pi_{i=0}^k \alpha_i \ \geq \ \beta_j\,,\ \forall\,\mathcal{P}_2 \in \mathcal{E}_1\,.$$

Then, one has

$$p''_i \ \geq \ \Pi_{i=0}^k \alpha_i \ \geq \ \beta_j \ \geq \ \beta_i\,,\ \forall\, i\,.$$

Hence, for every vector $\mathcal{L}_2$, there exists a vector $\mathcal{L}_1$ such that

$$\mathcal{P}_1 \in \mathbf{P}_{1,\mathcal{L}_1} \implies \mathcal{P}_2 \in \mathbf{P}_{2,\mathcal{L}_2}\,,\ \forall\,(\mathcal{P}_1, \mathcal{P}_2) \in \mathbf{P}_{1,2}\,.$$

(ii) by the same reasoning, for every vector $\mathcal{L}_1$, there exists a vector $\mathcal{L}_2$ such that:

$$\mathcal{P}_2 \in \mathbf{P}_{2,\mathcal{L}_2} \implies \mathcal{P}_1 \in \mathbf{P}_{1,\mathcal{L}_1}\,,\ \forall\,(\mathcal{P}_1, \mathcal{P}_2) \in \mathbf{P}_{1,2}\,;$$

hence the theorem is proved.
Recalling (5), by Theorem 3 in particular it follows:

$$\mathcal{K}_1 \Rightarrow A_{i+1} \mid\!\sim A_i\,,\ \forall\, A_{i+1} \mid\!\sim A_i \in \mathcal{K}_2\,;$$

$$\mathcal{K}_2 \Rightarrow A_i \mid\!\sim A_{i+1}\,,\ \forall\, A_i \mid\!\sim A_{i+1} \in \mathcal{K}_1\,.$$

5 Some related inference rules

We recall below the inference rules *Cautious Monotonicity* and *Cut*.

$$CM: \quad A \mid\!\sim C\,,\ A \mid\!\sim B \implies AB \mid\!\sim C\,, \tag{7}$$

$$Cut: \quad AB \mid\!\sim C\,,\ A \mid\!\sim B \implies A \mid\!\sim C\,. \tag{8}$$

The exact propagation of probability bounds in such rules, from antecedents to consequents, has been examined in (Gilio 2002a). Exploiting CM and Cut rules we can extend Theorem 3 by the following

Theorem 4 Given $k + 1$ logically independent events $A_0, A_1, \ldots, A_k$, let $\mathcal{F}_1$ be the family defined in (6) and $\mathcal{K}_1$ the associated knowledge base. Moreover, let us consider the family $\mathcal{F}_3 = \mathcal{E}'_3 \cup \mathcal{E}''_3$, where

$$\mathcal{E}'_3 = \{A_0|A_iA_{i+1},\ i = 1, \ldots, k-1\}\,,$$

$$\mathcal{E}''_3 = \{A_0|A_j,\ j = 2, \ldots, k-1\}\,,$$

and the associated knowledge base $\mathcal{K}_3 = \Gamma'_3 \cup \Gamma''_3$, where

$$\Gamma'_3 = \{A_iA_{i+1} \mid\!\sim A_0,\ i = 1, \ldots, k-1\}\,,$$

$$\Gamma''_3 = \{A_j \mid\!\sim A_0,\ j = 2, \ldots, k-1\}\,.$$

Then, one has: $\mathcal{K}_1 \Rightarrow \mathcal{K}_3$.

Proof. We apply an iterative procedure.
1. $i = 1, j = 2$: by (3) and Theorem 3, one has

$$\mathcal{K}_1 \Rightarrow \{A_1 \mid\!\sim A_0,\ A_1 \mid\!\sim A_2\}\,.$$

Moreover, applying (7) with $A = A_1,\ B = A_2,\ C = A_0$, we have

$$A_1 \mid\!\sim A_0\,,\ \ A_1 \mid\!\sim A_2 \quad \implies \quad A_1A_2 \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_1A_2 \mid\!\sim A_0$.
Then, as $\mathcal{K}_1 \Rightarrow A_2 \mid\!\sim A_1$, applying (8) with $A = A_2,\ B = A_1,\ C = A_0$, we have

$$A_2A_1 \mid\!\sim A_0\,,\ \ A_2 \mid\!\sim A_1 \quad \implies \quad A_2 \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_2 \mid\!\sim A_0$.

2. $i = 2, j = 3$: one has $\mathcal{K}_1 \Rightarrow \{A_2 \mid\!\sim A_0,\, A_2 \mid\!\sim A_3\}$; moreover, applying (7) with $A = A_2,\, B = A_3,\, C = A_0$, we have

$$A_2 \mid\!\sim A_0\,,\;\; A_2 \mid\!\sim A_3 \quad\Longrightarrow\quad A_2A_3 \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_2A_3 \mid\!\sim A_0$.
Then, as $\mathcal{K}_1 \Rightarrow A_3 \mid\!\sim A_2$, applying (8) with $A = A_3,\, B = A_2,\, C = A_0$, we have

$$A_3A_2 \mid\!\sim A_0\,,\;\; A_3 \mid\!\sim A_2 \quad\Longrightarrow\quad A_3 \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_3 \mid\!\sim A_0$.

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k-2. $i = k-2, j = k-1$: one has

$$\mathcal{K}_1 \Rightarrow \{A_{k-2} \mid\!\sim A_0,\, A_{k-2} \mid\!\sim A_{k-1}\}\,;$$

moreover, applying (7) with

$$A = A_{k-2},\, B = A_{k-1},\, C = A_0\,,$$

we have

$$A_{k-2} \mid\!\sim A_0\,,\;\; A_{k-2} \mid\!\sim A_{k-1} \quad\Longrightarrow\quad A_{k-2}A_{k-1} \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_{k-2}A_{k-1} \mid\!\sim A_0$.
Then, as $\mathcal{K}_1 \Rightarrow A_{k-1} \mid\!\sim A_{k-2}$, applying (8) with

$$A = A_{k-1},\, B = A_{k-2},\, C = A_0\,,$$

we have

$$A_{k-1}A_{k-2} \mid\!\sim A_0\,,\;\; A_{k-1} \mid\!\sim A_{k-2} \quad\Longrightarrow\quad A_{k-1} \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_{k-1} \mid\!\sim A_0$.

k-1. $i = k-1$: one has $\mathcal{K}_1 \Rightarrow \{A_{k-1} \mid\!\sim A_0,\, A_{k-1} \mid\!\sim A_k\}$; moreover, applying (7) with $A = A_{k-1},\, B = A_k,\, C = A_0$, we have

$$A_{k-1} \mid\!\sim A_0\,,\;\; A_{k-1} \mid\!\sim A_k \quad\Longrightarrow\quad A_{k-1}A_k \mid\!\sim A_0\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_{k-1}A_k \mid\!\sim A_0$.

Based on the previous result, we have

Theorem 5 Given $k+1$ logically independent events $A_0, A_1, \dots, A_k$, let $\mathcal{F}_1$ be the family defined in (6) and $\mathcal{K}_1$ the associated knowledge base. Moreover, let us consider the family $\mathcal{F}_4 = \mathcal{E}_4' \cup \mathcal{E}_4''$, where

$$\mathcal{E}_4' = \{A_{i+1}|A_iA_0,\, i = 1, \dots, k-1\}\,,$$

$$\mathcal{E}_4'' = \{A_j|A_0,\, j = 2, \dots, k-1\}\,,$$

and the associated knowledge base $\mathcal{K}_4 = \Gamma_4' \cup \Gamma_4''$, where

$$\Gamma_4' = \{A_iA_0 \mid\!\sim A_{i+1},\, i = 1, \dots, k-1\}\,,$$

$$\Gamma_4'' = \{A_0 \mid\!\sim A_j,\, j = 2, \dots, k-1\}\,.$$

Then, one has: $\mathcal{K}_1 \Rightarrow \mathcal{K}_4$.

Proof. As in Theorem 4, we apply an iterative procedure.
1. $i = 1, j = 2$: by (3) and Theorem 3, one has

$$\mathcal{K}_1 \Rightarrow \{A_1 \mid\!\sim A_2,\, A_1 \mid\!\sim A_0\}\,.$$

Moreover, applying (7) with $A = A_1,\, B = A_0,\, C = A_2$, we have

$$A_1 \mid\!\sim A_2\,,\;\; A_1 \mid\!\sim A_0 \quad\Longrightarrow\quad A_1A_0 \mid\!\sim A_2\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_1A_0 \mid\!\sim A_2$. Then, as $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_1$, applying (8) with $A = A_0,\, B = A_1,\, C = A_2$, we have

$$A_0A_1 \mid\!\sim A_2\,,\;\; A_0 \mid\!\sim A_1 \quad\Longrightarrow\quad A_0 \mid\!\sim A_2\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_2$.

2. $i = 2, j = 3$: by Theorem 4, one has

$$\mathcal{K}_1 \Rightarrow \{A_2 \mid\!\sim A_3,\, A_2 \mid\!\sim A_0\}\,.$$

Moreover, applying (7) with $A = A_2,\, B = A_0,\, C = A_3$, we have

$$A_2 \mid\!\sim A_3\,,\;\; A_2 \mid\!\sim A_0 \quad\Longrightarrow\quad A_2A_0 \mid\!\sim A_3\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_2A_0 \mid\!\sim A_3$. Then, as $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_2$, applying (8) with $A = A_0,\, B = A_2,\, C = A_3$, we have

$$A_0A_2 \mid\!\sim A_3\,,\;\; A_0 \mid\!\sim A_2 \quad\Longrightarrow\quad A_0 \mid\!\sim A_3\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_3$.

....

k-2. $i = k-2, j = k-1$: by Theorem 4, one has

$$\mathcal{K}_1 \Rightarrow \{A_{k-2} \mid\!\sim A_{k-1},\, A_{k-2} \mid\!\sim A_0\}\,.$$

Moreover, applying (7) with

$$A = A_{k-2},\, B = A_0,\, C = A_{k-1}\,,$$

we have

$$A_{k-2} \mid\!\sim A_{k-1}\,,\;\; A_{k-2} \mid\!\sim A_0 \quad\Longrightarrow\quad A_{k-2}A_0 \mid\!\sim A_{k-1}\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_{k-2}A_0 \mid\!\sim A_{k-1}$.
Then, as $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_{k-2}$, applying (8) with

$$A = A_0,\, B = A_{k-2},\, C = A_{k-1}\,,$$

we have

$$A_0A_{k-2} \mid\!\sim A_{k-1}\,,\;\; A_0 \mid\!\sim A_{k-2} \quad\Longrightarrow\quad A_0 \mid\!\sim A_{k-1}\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_0 \mid\!\sim A_{k-1}$.

k-1. $i = k-1$: by Theorem 4, one has

$$\mathcal{K}_1 \Rightarrow \{A_{k-1} \mid\!\sim A_k,\, A_{k-1} \mid\!\sim A_0\}\,.$$

Moreover, applying (7) with $A = A_{k-1},\, B = A_0,\, C = A_k$, we have

$$A_{k-1} \mid\!\sim A_k\,,\;\; A_{k-1} \mid\!\sim A_0 \quad\Longrightarrow\quad A_{k-1}A_0 \mid\!\sim A_k\,,$$

and hence $\mathcal{K}_1 \Rightarrow A_{k-1}A_0 \mid\!\sim A_k$.

Based on Theorem 5, we have

Theorem 6 Given $k+1$ logically independent events $A_0, A_1, \ldots, A_k$, let $\mathcal{F}_1$ be the family defined in (6) and $\mathcal{K}_1$ the associated knowledge base. Moreover, for each $j \in \{1, 2, \ldots, k-2\}$, let us define the knowledge bases

$$\Upsilon'_j = \{A_0 A_j \mathrel{|}\!\sim A_i \,,\; A_0 A_i \mathrel{|}\!\sim A_j \,;\; i = j+2, \ldots, k\}\,,$$

$$\Upsilon''_j = \{A_j \mathrel{|}\!\sim A_i \,,\;\; A_i \mathrel{|}\!\sim A_j \,;\; i = j+2, \ldots, k\}\,.$$

Then, one has:

$$\mathcal{K}_1 \;\Rightarrow\; \bigcup_{j=1}^{k-2} (\Upsilon'_j \cup \Upsilon''_j)\,.$$

Proof. (i) by Theorem 5, one has

$$\mathcal{K}_1 \Rightarrow \{A_0 \mathrel{|}\!\sim A_j \,,\; A_0 \mathrel{|}\!\sim A_i\}\,,$$

$$\forall\, j \in \{2, \ldots, k-2\},\, \forall\, i \in \{j+2, \ldots, k\}.$$

Moreover, applying (7), respectively, with $A = A_0\,,\, B = A_j\,,\, C = A_i$, and with $A = A_0\,,\, B = A_i\,,\, C = A_j$, we have

$$A_0 \mathrel{|}\!\sim A_i \,,\; A_0 \mathrel{|}\!\sim A_j \Longrightarrow A_0 A_j \mathrel{|}\!\sim A_i\,;$$

$$A_0 \mathrel{|}\!\sim A_j \,,\; A_0 \mathrel{|}\!\sim A_i \Longrightarrow A_0 A_i \mathrel{|}\!\sim A_j\,;$$

hence

$$\mathcal{K}_1 \Rightarrow \{A_0 A_j \mathrel{|}\!\sim A_i \,,\; A_0 A_i \mathrel{|}\!\sim A_j\}\,,$$

$$\forall\, j \in \{2, \ldots, k-2\},\, \forall\, i \in \{j+2, \ldots, k\}\,,$$

and, by (4), it follows

$$\mathcal{K}_1 \Rightarrow \bigcup_{j=1}^{k-2} \Upsilon'_j\,. \tag{9}$$

Then, as $\mathcal{K}_1 \;\Rightarrow\; \{A_j \;|\!\sim\; A_0, A_i \;|\!\sim\; A_0\}$, applying (8), respectively, with

$$A = A_j\,,\, B = A_0\,,\, C = A_i\,,$$

and with

$$A = A_i\,,\, B = A_0\,,\, C = A_j\,,$$

we have

$$A_j A_0 \mathrel{|}\!\sim A_i \,,\; A_j \mathrel{|}\!\sim A_0 \Longrightarrow A_j \mathrel{|}\!\sim A_i\,,$$

$$A_i A_0 \mathrel{|}\!\sim A_j \,,\; A_i \mathrel{|}\!\sim A_0 \Longrightarrow A_i \mathrel{|}\!\sim A_j\,;$$

hence

$$\mathcal{K}_1 \Rightarrow \{A_j \mathrel{|}\!\sim A_i \,,\; A_i \mathrel{|}\!\sim A_j\}\,,$$

$$\forall\, j \in \{2, \ldots, k-2\},\, \forall\, i \in \{j+2, \ldots, k\}\,,$$

and, by (4), it follows

$$\mathcal{K}_1 \Rightarrow \bigcup_{j=1}^{k-2} \Upsilon''_j\,. \tag{10}$$

Then, by (4), (9) and (10), we obtain

$$\mathcal{K}_1 \;\Rightarrow\; \bigcup_{j=1}^{k-2} (\Upsilon'_j \cup \Upsilon''_j)\,.$$

We give now a general result, which includes the previous ones as corollaries; it can be obtained, in a simple way, in the setting of conditional objects (see Proposition 6).

Theorem 7 Let be given the following set of *conjunctive* conditional events

$$\mathcal{C}^* = \{A_{i_1} \cdots A_{i_h} \,|\, A_{j_1} \cdots A_{j_t}\}\,, \tag{11}$$

where

$$\{i_1, \ldots, i_h\} \cup \{j_1, \ldots, j_t\} \subseteq \{0, 1, \ldots, k\}\,,\;\; h \geq 1,\, t \geq 1.$$

Then, denoting by $\mathcal{K}^*$ the knowledge base associated with $\mathcal{C}^*$, one has: $\mathcal{K}_1 \;\Rightarrow\; \mathcal{K}^*$.

The class of conjunctive conditional events has been studied in (Lukasiewicz 1997); see also (Biazzo *et al.* 2001).

6 Best bounds for quasi-conjunction

In this section we consider a probability assessment on a pair of conditional events and we determine the precise probability bounds for their quasi-conjunction. Then, we obtain a probabilistic semantics for the *QAND* rule given in (Dubois and Prade 1994). Let A, H, B, K be logically independent events, with $H \neq \emptyset, K \neq \emptyset$. We recall that the quasi-conjunction of two conditional events $A|H$ and $B|K$, as defined in (Adams 1975), is given by

$$A|H \& B|K = (AH \vee H^c) \wedge (BK \vee K^c) | (H \vee K)\,.$$

It can be easily verified that, for every pair (x, y), with $x \in [0,1], y \in [0,1]$, the probability assessment (x, y) on $\{A|H, B|K\}$ is coherent. Moreover, for each given assessment (x, y) on $\{A|H, B|K\}$, the probability assessment $\mathcal{P} = (x, y, z)$ on

$$\mathcal{F} = \{A|H, B|K, A|H\&B|K\}\,,$$

with $z = P(A|H\&B|K)$, is a coherent extension of (x, y) if and only if $z' \leq z \leq z''$, where

$$z' = \begin{cases} 0, & x+y \leq 1, \\ x+y-1, & x+y > 1, \end{cases} \qquad z'' = \tfrac{x+y-2xy}{1-xy}\,.$$

To obtain the values z', z'' we can study the coherence of $\mathcal{P} = (p_1, p_2, p_3) = (x, y, z)$ by a geometrical approach proposed in (Gilio 1995). We denote by $C_1, \ldots, C_m$ the constituents generated by the family

$$\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\} = \{A|H, B|K, A|H\&B|K\}$$

and contained in $H \vee K$. Then, with each C_h we associate a point $Q_h = (q_{h1}, q_{h2}, q_{h3})$, where, for each $i = 1, 2, 3$, it is

$$q_{hi} \;=\; \begin{cases} 1\,, & \text{if } C_h \subseteq E_i H_i\,, \\ 0\,, & \text{if } C_h \subseteq E_i^c H_i\,, \\ p_i\,, & \text{if } C_h \subseteq H_i^c\,. \end{cases}$$

The points Q_h's are

$$Q_1 = (1,1,1)\,,\; Q_2 = (1,y,1)\,,\; Q_3 = (1,0,0)\,,$$

$$Q_4 = (x,1,1)\,,\; Q_5 = (x,0,0)\,,\; Q_6 = (0,1,0)\,,$$

$$Q_7 = (0,y,0)\,,\; Q_8 = (0,0,0)\,,$$

and, in our case, the coherence of $\mathcal{P}$ simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of

the points $Q_1, \dots, Q_8$.
As we can verify, if $x+y \leq 1$, then $\mathcal{P} = (x, y, 0)$ belongs to the triangle $Q_3Q_6Q_8$, so that the condition $\mathcal{P} \in \mathcal{I}$ is verified and hence $z' = 0$.
If $x + y > 1$, denoting by π_1 the plane containing the triangle $T_1 = Q_1Q_3Q_6$ and considering the point (x, y, z^*) belonging to T_1, in order the condition $\mathcal{P} \in \mathcal{I}$ be satisfied, it must be $z \geq z^*$. Then, observing that the equation of π_1 is

$$Z = X + Y - 1\,,$$

it follows: $z' = z^* = x + y - 1$.
Concerning z'', denoting by π_2 the plane containing the triangle $T_2 = Q_2Q_4Q_8$ and considering the point (x, y, z^{**}) belonging to T_2, in order the condition $\mathcal{P} \in \mathcal{I}$ be satisfied, it must be $z \leq z^{**}$ for every $(x, y) \in [0, 1]^2$. Then, observing that the equation of π_2 is

$$Z = \frac{1-y}{1-xy} \cdot X + \frac{1-x}{1-xy} \cdot Y\,,$$

it follows: $z'' = z^{**} = \frac{x+y-2xy}{1-xy}$.
QAND rule can be derived by applying the inference rules of System P (Kraus *et al.* 1990) and says that, given any knowledge base K, the quasi-conjunction $C(K)$ can be deduced by K using the inference rules of System P. To obtain the probabilistic interpretation for *QAND* rule, let (x, y, z) be any coherent probability assessment on $\{A|H, B|K, A|H\&B|K\}$. Then, given any number $\beta \in [0, 1]$, for every pair $(\alpha_1, \alpha_2) \in [\beta, 1] \times [\beta, 1]$ such that $\alpha_1 + \alpha_2 \geq \beta + 1$, one has

$$x \geq \alpha_1\,,\; y \geq \alpha_2 \implies z \geq z' = \alpha_1 + \alpha_2 - 1 \geq \beta\,.$$

Quasi-conjunction plays a key role in the logic of conditional objects (Dubois and Prade 1994), which will be considered in the next section.

7 Relationship with conditional objects

We recall that, based on the three-valued calculus of conditional objects, in (Dubois and Prade 1994) a very simple semantics has been provided for the preferential entailment studied in (Kraus *et al.* 1990). Conditional objects can be seen as the counterpart of the conditional assertions considered in (Kraus *et al.* 1990) and, for what concerns logical operations, we can look at them as conditional events. Given a set of conditional objects K, we denote by $C(K)$ the quasi-conjunction of the conditional objects in K and by $\models$ the logical entailment among conditional objects, i.e. the logical inclusion among conditional events as defined in (Goodman and Nguyen 1988). In the paper of Dubois and Prade the following definition is given

Definition 4 K entails $q|p$, denoted $K \models q|p$, if and only if either there exists a non-empty subset S of K such that $C(S) \models q|p$, or $p \models q$.

We recall that the relationship between probabilistic reasoning under coherence and default reasoning with conditional objects has been examined in (Biazzo *et al.* 2002).
Based on Definition 4, the results given in a probabilistic framework in the Sections 4 and 5 can also be obtained in the setting of conditional objects. We first give a preliminary result on the quasi-conjunction of $A_1|A_0, \dots, A_k|A_{k-1}$.

Proposition 1 Given the set of conditional objects

$$K = \{A_1|A_0, A_2|A_1, \dots, A_k|A_{k-1}\}\,,$$

one has

$$C(K) = (E_0 \vee \dots \vee E_k)|(A_0 \vee \dots \vee A_{k-1})\,,$$

where

$$E_0 = A_0A_1 \cdots A_k\,,\; E_1 = A_0^cA_1 \cdots A_k\,,\; \cdots\,,$$

$$E_{k-1} = A_0^c \cdots A_{k-2}^cA_{k-1}A_k\,,\; E_k = A_0^cA_1^c \cdots A_{k-1}^c\,.$$

Proof. We proceed by induction.
a) as it can be verified, if $K = \{A_1|A_0, A_2|A_1\}$, then

$$C(K) = (A_0A_1A_2 \vee A_0^cA_1A_2 \vee A_0^cA_1^c)|(A_0 \vee A_1)\,;$$

then, the quasi-conjunction of $K \cup \{A_3|A_2\}$, given by

$$(A_0A_1A_2 \vee A_0^cA_1A_2 \vee A_0^cA_1^c) \wedge (A_2A_3 \vee A_2^c)|(A_0 \vee A_1 \vee A_2)\,,$$

can be written as

$$(E_0^* \vee E_1^* \vee E_2^* \vee E_3^*)|(A_0 \vee A_1 \vee A_2)\,,$$

where

$$E_0^* = A_0A_1A_2A_3\,,\; E_1^* = A_0^cA_1A_2A_3\,,$$

$$E_2^* = A_0^cA_1^cA_2A_3\,,\; E_3^* = A_0^cA_1^cA_2^c\,.$$

b) assume that the quasi-conjunction of the set

$$\{A_1|A_0, A_2|A_1, \dots, A_{k-1}|A_{k-2}\}$$

is

$$(E_0^* \vee \dots \vee E_{k-1}^*)|(A_0 \vee \dots \vee A_{k-2})\,,$$

where

$$E_0^* = A_0A_1 \cdots A_{k-1}\,,\; E_1^* = A_0^cA_1 \cdots A_{k-1}\,,\; \cdots\,,$$

$$E_{k-2}^* = A_0^c \cdots A_{k-3}^cA_{k-2}A_{k-1}\,,\; E_{k-1}^* = A_0^cA_1^c \cdots A_{k-2}^c\,.$$

Then, the quasi-conjunction of the set

$$\{A_1|A_0, A_2|A_1, \dots, A_k|A_{k-1}\}$$

is

$$(E_0^* \vee \dots \vee E_{k-1}^*) \wedge (A_{k-1}A_k \vee A_{k-1}^c)|(A_0 \vee \dots \vee A_{k-1})\,,$$

which can be written as

$$(E_0 \vee E_1 \vee \dots \vee E_k)|(A_0 \vee \dots \vee A_{k-1})\,, \qquad (12)$$

where

$$E_0 = A_0A_1 \cdots A_k\,,\; E_1 = A_0^cA_1 \cdots A_k\,,\; \cdots\,,$$

$$E_{k-1} = A_0^cA_1^c \cdots A_{k-2}^cA_{k-1}A_k\,,\; E_k = A_0^cA_1^c \cdots A_{k-1}^c\,;$$

hence the Proposition is proved.
By (12) it follows that the quasi-conjunction of the family

$$K_1 = \{A_1|A_0, A_2|A_1, \dots, A_k|A_{k-1}, A_0|A_k\}$$

is

$$(E_0 \vee E_1 \vee \dots \vee E_k) \wedge (A_0A_k \vee A_k^c)|(A_0 \vee \dots \vee A_k) =$$

$$= \dots = (A_0A_1 \cdots A_k \vee A_0^cA_1^c \cdots A_k^c)|(A_0 \vee \dots \vee A_k) =$$

$$= A_0A_1 \cdots A_k \,|\, (A_0 \vee \dots \vee A_k)\,.$$

By a similar reasoning, for the family

$$K_2 = \{A_0|A_1, A_1|A_2, \dots, A_{k-1}|A_k, A_k|A_0\}$$

one has $C(K_2) = C(K_1)$. Then, we have

Proposition 2 Let be given the sets

$$K_1 = \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\}\,,$$
$$K_2 = \{A_0|A_1, A_1|A_2, \ldots, A_{k-1}|A_k, A_k|A_0\}\,.$$

For every $i = 0, 1, \ldots, k$, one has

$$(a)\;\; C(K_1) \models A_i|A_{i+1}\,, \;\; (b)\;\; C(K_2) \models A_{i+1}|A_i\,,$$

where $A_{k+1} = A_0$.

Proof. We recall that

$$C(K_1) = C(K_2) = A_0A_1 \cdots A_k \,|\, (A_0 \vee \cdots \vee A_k)\,.$$

Moreover, we observe that $A_0A_1 \cdots A_k \,|\, (A_0 \vee \cdots \vee A_k)$ is true (resp., is false) if and only if $A_0A_1 \cdots A_k$ is true (resp., there exist (at least) a pair of subscripts (i, j) such that A_i is true and A_j is false). Then, the assertion (a) follows by observing that $C(K_1)$ true implies $A_i|A_{i+1}$ true, while $A_i|A_{i+1}$ false implies $C(K_1)$ false. The assertion (b) follows by observing that $C(K_2)$ true implies $A_{i+1}|A_i$ true, while $A_{i+1}|A_i$ false implies $C(K_2)$ false.

Recalling Definition 4, by Proposition 2 one has

$$K_1 \models A_i|A_{i+1}\,,\; \forall\, i = 0, 1, \ldots, k\,,$$
$$K_2 \models A_{i+1}|A_i\,,\; \forall\, i = 0, 1, \ldots, k\,,$$

that is

$$(i)\;\; K_1 \models K_2\,; \quad (ii)\;\; K_2 \models K_1\,;$$

hence we get the same conclusion of Theorem 3.

By a similar reasoning we obtain

Proposition 3 Let be given the set

$$K_1 = \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\}\,.$$

For every $i = 1, \ldots, k-1$ and $j = 2, \ldots, k-1$, one has

$$(a)\;\; C(K_1) \models A_0|A_iA_{i+1}\,; \;\; (b)\;\; C(K_1) \models A_0|A_j\,.$$

By Definition 4 and Proposition 3, considering the set $K_3 = K_3' \cup K_3''$, where

$$K_3' = \{A_0|A_iA_{i+1},\, i = 1, \ldots, k-1\}\,,$$

$$K_3'' = \{A_0|A_j,\, j = 2, \ldots, k-1\}\,,$$

it follows

$$K_1 \models A_0|A_iA_{i+1}\,, \quad \forall\, i = 1, \ldots, k-1\,,$$

$$K_1 \models A_0|A_j\,, \quad \forall\, j = 2, \ldots, k-1\,,$$

that is $K_1 \models K_3$; hence we get the same conclusion of Theorem 4.

Proposition 4 Let be given the set

$$K_1 = \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\}\,.$$

For every $i = 1, \ldots, k-1$ and $j = 2, \ldots, k-1$, one has

$$(a)\;\; C(K_1) \models A_{i+1}|A_iA_0\,; \;\; (b)\;\; C(K_1) \models A_j|A_0\,.$$

Then, considering the set $K_4 = K_4' \cup K_4''$, where

$$K_4' = \{A_{i+1}|A_iA_0,\, i = 1, \ldots, k-1\}\,,$$

$$K_4'' = \{A_j|A_0,\, j = 2, \ldots, k-1\}\,,$$

by Proposition 4 it follows $K_1 \models K_4$, which is the counterpart of Theorem 5.

Proposition 5 Let be given the set

$$K_1 = \{A_1|A_0, A_2|A_1, \ldots, A_k|A_{k-1}, A_0|A_k\}\,.$$

For every $j \in \{1, \ldots, k-2\}$ and $i = j+2, \ldots, k$ one has

$$(a)\;\; C(K_1) \models A_i|A_0A_j\,, \quad (b)\;\; C(K_1) \models A_j|A_0A_i\,,$$

$$(c)\;\; C(K_1) \models A_i|A_j\,, \qquad (d)\;\; C(K_1) \models A_j|A_i\,.$$

Then, considering, for each $j \in \{1, \ldots, k-2\}$, the sets

$$\Phi_j' = \{A_i|A_0A_j\,,\; A_j|A_0A_i\,;\; i = j+2, \ldots, k\}\,,$$

$$\Phi_j'' = \{A_i|A_j\,,\;\; A_j|A_i\,;\; i = j+2, \ldots, k\}\,,$$

one has:

$$K_1 \models \bigcup_{j=1}^{k-2} (\Phi_j' \cup \Phi_j'')\,,$$

which is the counterpart of Theorem 6.

By the same reasoning as in Proposition 2, denoting by C^* the set of conditional objects defined as in (11), we have

Proposition 6 for every pair of subsets

$$\{i_1, \ldots, i_h\}\,, \;\; \{j_1, \ldots, j_t\}$$

of the set $\{0, 1, \ldots, k\}$, with $h \geq 1,\, t \geq 1$, one has

$$C(K_1) \models A_{i_1} \cdots A_{i_h} \,|\, A_{j_1} \cdots A_{j_t}\,.$$

Hence: $K_1 \models C^*$.

We remark that Propositions 2, ..., 5 can be simply obtained as corollaries of Proposition 6, which is the counterpart of Theorem 7.

8 An example

Five friends, Linda, Janet, Steve, George, and Peter, have been invited to a party. We denote by $A_0, \ldots, A_4$ the events defined respectively as *"Linda is present at the party"*, ..., *"Janet is present at the party"*.

We assume the following default knowledge:

- *"if Linda goes to the party, then (very probably) Janet will do the same"*;

. . . .

- *"if Peter goes to the party, then (very probably) Linda will do the same"*;

that is, we start with the knowledge base

$$\mathcal{K}_1 = \{A_0 \mid\!\sim A_1\,,\; A_1 \mid\!\sim A_2\,,\; \cdots\,,\; A_4 \mid\!\sim A_0\}\,.$$

Then, by the previous results, we can entail all conjunctive conditional assertions, like

$$A_i \mid\!\sim A_j,\; A_iA_j \mid\!\sim A_h,\; A_j \mid\!\sim A_hA_k,\; A_iA_jA_h \mid\!\sim A_kA_t, \ldots\,.$$

For instance, we can entail the conditional assertions:

"if Peter is present at the party, then (very probably) Janet is present too";

"if Linda and Janet are present at the party, then (very probably) Steve, George, and Peter are present too"; and so on.

9 Conclusions

In this paper we have considered a generalized version of Loop rule and, using Császár's condition, we have given a probabilistic interpretation of it. Then, exploiting CM and Cut rules, we have obtained related results on p-entailment by the conditional knowledge base associated with Loop rule. Moreover, we have considered a probability assessment on a family of two conditional events, determining the best bounds for the probability of their quasi-conjunction and providing a probabilistic semantics for *QAND* rule.

Finally, we have reconsidered our results in the setting of conditional objects.

References

Adams, E. W. 1975. *The Logic of Conditionals*. Dordrecht, Netherlands: Reidel.

Amarger, S.; Dubois, D.; and Prade, H. 1991. Constraint propagation with imprecise conditional probabilities, *Proceedings UAI-91*, 26-34, Morgan Kaufmann.

Benferhat, S.; Dubois, D.; and Prade, H. 1997. Nonmonotonic reasoning, conditional objects and possibility theory, *Artificial Intelligence* 92(1-2): 259-276.

Benferhat, S.; Dubois, D.; and Prade, H. 1999. Possibilistic and Standard Probabilistic Semantics of Conditional Knowledge Bases, *J. Logic Computat.* 9(6): 873-895.

Biazzo V.; Gilio A.; Lukasiewicz T.; and Sanfilippo G. 2001. Probabilistic Logic under Coherence: Complexity and Algorithms, *Proc. of The* 2^{nd} *Int. Symp. on Impr. Prob. and their Appl.* (ISIPTA '01), Ithaca, USA, June 26 - 29, 51-61.

Biazzo, V.; Gilio, A.; Lukasiewicz, T.; and Sanfilippo, G. 2002. Probabilistic Logic under Coherence, Model-Theoretic Probabilistic Logic, and Default Reasoning in System P, *Journal of Applied Non-Classical Logics* Journal of Applied Non-Classical Logics, Volume 12, No. 2, 189-213.

Bourne, R.; and Parsons, S. 1998. Propagating probabilities in System P, *Proceedings of the 11th International FLAIRS Conference*, 440-445.

Coletti, G.; and Scozzafava, R. 2002. Probabilistic logic in a coherent setting, Kluwer Academic Publishers.

Császár, Á. 1955. Sur la structure des espaces de probabilité conditionnelle, *Acta Mathematica Academiae Scientiarum Hungaricae* 46, 337-361.

Dubois, D.; Godo, L.; Màntaras, R. L.; and Prade, H. 1993. Qualitative reasoning with imprecise probabilities, *Journal of Intelligent Information Systems* 2: 319-363.

Dubois, D.; and Prade, H. 1994. Conditional objects as nonmonotonic consequence relationships, *IEEE Trans. Syst. Man* & *Cybernetics*, 24(12), 1724-1739.

Gilio, A. 1989. Classi quasi additive di eventi e coerenza di probabilità condizionate, *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, Vol. XXI, Fasc. I, 22-38.

Gilio, A. 1995. Algorithms for precise and imprecise conditional probability assessments, *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence* (G. Coletti, D. Dubois, and R. Scozzafava, R., editors), Plenum Press, New York.

Gilio, A. 2000. Precise propagation of Upper and Lower Probability Bounds in System P, *Proc. of The* 8^{th} *Intern. Workshop on Non-monotonic Reasoning, Special Session: Uncertainty Frameworks in Non-Monotonic Reasoning*, Breckenridge, Colorado, USA, April 9-11.

Gilio, A. 2002a. Probabilistic reasoning under coherence in System P, *Annals of Mathematics and Artificial Intelligence* 34, 5-34.

Gilio, A. 2002b. Probabilistic logic under coherence and default reasoning, *Preproceedings of the* 1^{st} *Salzburg Workshop on Paradigms of Cognition (SWPC 1/2002)*, Salzburg (Austria), July 18-20.

Gilio, A.; and Spezzaferri, F. 1992. Knowledge integration for conditional probability assessments, *Uncertainty in Artificial Intelligence* (Eds. D. Dubois, M. P. Wellman, B. D'Ambrosio, P. Smets), Morgan Kaufmann Publishers, San Mateo, California, 98-103.

Gilio, A.; and Spezzaferri, F. 1995. Coherence and extensions of stochastic matrices, *Le Matematiche*, Vol. L, Fasc.I, 119-135.

Goodman, I. R.; Nguyen H. T. 1988. Conditional objects and the modeling of uncertainties. In *Fuzzy Computing Theory, Hardware and Applications*, Gupta, M. M.; and Yamakawa, T. eds., 119-138. New York: North-Holland.

Kraus, K.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics, *Artificial Intelligence* 44: 167-207.

Lukasiewicz T. 1997. Efficient Global Probabilistic Deduction from Taxonomic and Probabilistic Knowledge-Bases over Conjunctive Events, *Proc. of the 6th International Conference on Information and Knowledge Management*, ACM Press, 75-82.

Lukasiewicz, T. 2002. Probabilistic Default Reasoning with Conditional Constraints, *Annals of Mathematics and Artificial Intelligence* 34: 35-88.

Parsons, S.; and Bourne, R. A. 2000. On Proofs in System P, *Intern. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 8(2): 203-233.

Pearl, J. 1988. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. San Mateo, CA: Morgan Kaufmann.

Rigo, P. 1988. Un teorema di estensione per probabilitá condizionate finitamente additive, *Atti della XXXIV Riunione Scientifica della S.I.S.*, Vol. 2(1), 27-34.

Snow, P. 1996. Warranting non-demonstrative reasoning upon the common ground of contending uncertainty approaches, *Proc. of the* 6^{th} *Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems* (IPMU-96), Vol. 2, 1381-1386. Granada, Spain.

Snow, P. 1999. Diverse confidence levels in a probabilistic semantics for conditional logics, *Artificial Intelligence*, 113:269-279.