Ordinal and absolute representations of positive information in possibilistic logic

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Abstract

There are two readings of a possibility distribution, a representation format which is useful for encoding uncertain knowledge or preferences. The negative information reading, based on possibility and necessity measures, emphasizes the fact that some interpretations are impossible, or at least have an upper-bounded possibility level. The positive information reading points out that possibility degrees are lower bounded, and thus that some interpretations have non-zero (guaranteed) possibility degrees. This paper provides technical results for the positive view, showing how sets of absolute, or relative, constraints expressed in terms of guaranteed possibility measures can be represented in terms of a possibility distribution. Using previously obtained results for the "negative interpretation side", it enables us to jointly handle upper and lower logical specifications of a possibility distribution on partitions of the set of interpretations, as pointed out in the conclusion.

Keywords: guaranteed possibility, possibilistic logic, bipolar information, conditionals.

Introduction

Bipolar information is based on the idea of distinguishing between positive information and negative information. Information may express knowledge about the real world or preferences of an agent. In fact, when information stands for knowledge, we may express two kinds of knowledge (Dubois, Prade, & Smets 2001). On the one hand, we express which states of the world are impossible. This is negative knowledge. On the other hand, we may also express what states are guaranteed to be possible, since they have been observed for instance. This is positive knowledge. When information stands for an agent's preferences, negative preferences express what is not acceptable, while positive preferences single out what is really satisfactory (Benferhat *et al.* 2002b). Positive information does not always correspond to the complement of negative information: some worlds (resp. solutions) may not be impossible (resp. not acceptable), although they are not guaranteed to be possible (resp. satisfactory). Negative and positive information are however related by a coherence condition which says that what is positively assessed should be a part of what is not impossible (i.e. acceptable).

Possibility theory offers an adequate representation framework for bipolar information, using the possibility measure Π for representing negative information and the guaranteed possibility measure Δ for representing positive information. Indeed, on the one hand constraints, encoded by necessity measure-based possibilistic logic formulas, of the form $N(\phi) \geq \alpha$ (or equivalently $\Pi(\neg\phi) \leq 1-\alpha$) describe impossibility statements, and thus negative information. On the other hand, constraints of the form $\Delta(\phi) = min_{\omega \models \phi}\pi(\omega) \geq \alpha$ express a lower bound on the possibility degrees of a set of interpretations, which thus corresponds to positive information.

In practice, it is generally difficult to provide a complete possibility distribution on a discrete set of interpretations of exponential size. However, the possibility theory framework offers three compact representation formats of a possibility distribution, namely a *logical representation* by means of a set of prioritized formulas, a *conditional representation* by means of a set of rules and a *graphical representation* by means of a directed acyclic graph, based on conditioning. These three formats apply to both negative and positive information. The representation of negative information has been widely investigated (Benferhat *et al.* 2001) in the three representation formats. Recently, possibilistic logic has been extended to positive information (Benferhat *et al.* 2002b), while in (Benferhat *et al.* 2002a) a preliminary study of the two other formats can be found. In this paper, we will only focus on the conditional and logical representations of positive information and their semantic counterparts in terms of possibility distributions. More precisely, we investigate the conditional representation in greater details in this paper, and also provide translations between the two representation formats.

Background

We consider a propositional language $\mathcal{L}$ over a *finite* alphabet $\mathcal{P}$ of atoms. Ω denotes the set of all classical interpretations (called also solutions). Logical equivalence is denoted by $\equiv$. Classical disjunction and conjunction are respectively represented by $\vee, \wedge$. $[\![\psi]\!]$ denotes the set of all models of the proposition ψ (namely, interpretations satisfying ψ). $\bot$ and

$\top$ denote respectively contradiction and tautology.

Guaranteed possibility and qualitative possibility distributions

From a possibility distribution (Zadeh 1978) π which maps each element ω of a set of interpretations Ω into the unit interval $[0, 1]$, or any linearly ordered scale, one can define

– the *possibility* degree of a formula ϕ, denoted by $\Pi(\phi)$:

$$\Pi(\phi) = max\{\pi(\omega) : \omega \in \Omega \text{ and } \omega \models \phi\}.$$

– the *necessity* degree of a formula ϕ, denoted by $N(\phi)$, using an order-reversing map of the scale:

$$N(\phi) = 1 - \Pi(\neg\phi).$$

$N(\phi) \geq \alpha$ means that any solution violating ϕ is rejected with strength $1 - \alpha$.

– the *guaranteed possibility* degree of a formula, denoted by $\Delta(\phi)$:

$$\Delta(\phi) = min\{\pi(\omega) : \omega \in \Omega \text{ and } \omega \models \phi\}.$$

$\Delta(\phi) \geq \alpha$ means that any solution to desire ϕ has satisfaction level at least α.

Note that Δ is a non-increasing set function and that $\Delta(\phi \vee \psi) = min(\Delta(\phi), \Delta(\psi))$. Thus if a formula ϕ is written in a disjunctive normal form $\phi = \vee_i \phi_i$ where ϕ_i are conjunctions of literals, a constraint of the form $\Delta(\phi) \geq \alpha$ can be equivalently written $\forall i, \Delta(\phi_i) \geq \alpha$.

In the following, it is assumed that Ω is finite and thus that only a finite set of possibility levels is useful. Then, in this paper, a possibility distribution is represented by its well-ordered partition $\pi = (E_1, \cdots, E_n)$ s.t. $E_1 \cup \cdots \cup E_n = \Omega$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, with the following understanding:

- If $\omega \in E_i, \omega' \in E_j$ and $i < j$ then $\pi(\omega) < \pi(\omega')$
- If $\omega \in E_i, \omega' \in E_i$ then $\pi(\omega) = \pi(\omega')$.

Note that the smaller i, the smaller the possibility level of the interpretations in E_i. This choice will be justified later in the paper since we will only focus on Δ measure.

Moreover, this qualitative view of possibility distributions assumes that when considering two distributions, the interpretations having the same rank should be understood as having the same possibility degree. Namely,

$$\forall \omega \in E_i, \omega' \in E'_i \text{ iff } \pi(\omega) = \pi'(\omega'),$$

where $\pi = (E_1, \cdots, E_n)$ and $\pi' = (E'_1, \cdots, E'_m)$.

The following definition gives a way to compare possibility distributions based on the specificity principle (Yager 1993):

Definition 1 *Let $\pi = (E_1, \cdots, E_n)$ and $\pi' = (E'_1, \cdots, E'_m)$ be two possibility distributions. π is said to be more specific than π' if:*

$$\forall \omega, \text{ if } \omega \in E_i \text{ and } \omega \in E'_j \text{ then } i \leq j.$$

π is said to be the most specific possibility distribution among a set of possibility distributions $\mathcal{M}$ if there is no π' in $\mathcal{M}$ s.t. π' is more specific than π.

Δ-based possibilistic bases

The logical representation of a possibility distribution by means a Δ measure is given under the form of a Δ-*based possibilistic base* (or Δ-*possibilistic base* for short) which is a finite set of prioritized formulas of the form $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$, where formulas of Σ_i have priority over formulas in Σ_j for $i > j$, and $\Delta(\phi) = \alpha_i$ for any $\phi \in \Sigma_i$ and $\alpha_i > \alpha_j$ if $i > j$. Thus a Δ-possibilistic base can be viewed as a set of absolute constraints of the form $\Delta(\phi_i) \geq \alpha_i$.

Given a Δ-possibilistic base Σ, we can generate a unique possibility distribution π_Σ, where interpretations are ranked w.r.t. the highest formula that they satisfy, namely (Dubois, Prade, & Smets 2001):

Definition 2 *Let $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ be a Δ-possibilistic base. The possibility distribution associated with Σ, denoted by π_Σ, is $\pi_\Sigma = (E_1, \cdots, E_{n+1})$, s.t. $\forall \omega \in \Omega$:*

- *$\omega \in E_1$ if $\omega \not\models \bigvee_{i=1,n,\phi \in \Sigma_i} \phi$*
- *$\omega \in E_i$ iff $\omega \models \bigvee_{\phi \in \Sigma_{i-1}} \phi$ and $\omega \not\models \bigvee_{j=i,n,\psi \in \Sigma_j} \psi$.*

Example 1 *Let $\Sigma = (\Sigma_1, \Sigma_2)$ with $\Sigma_1 = \{\neg s \wedge t, s \wedge \neg t, h \wedge \neg t\}$ and $\Sigma_2 = \{h \wedge s\}$, where h, s and t are propositional symbols.*
The set of possible worlds is
$\Omega = \{\omega_0 : \neg h \neg s \neg t, \omega_1 : \neg h \neg st, \omega_2 : \neg hs \neg t,$
$\omega_3 : \neg hst, \omega_4 : h \neg s \neg t, \omega_5 : h \neg st, \omega_6 : hs \neg t,$
$\omega_7 : hst\}.$
Then, $\pi_\Sigma = (E_1, E_2, E_3)$ where $E_1 = \{\omega_0, \omega_3\}$,
$E_2 = \{\omega_1, \omega_2, \omega_4, \omega_5\}$ and $E_3 = \{\omega_6, \omega_7\}$.

Let us now give a brief background on Δ-possibilistic bases. For more details, we refer the reader to (Dubois, Prade, & Smets 2001; Benferhat & Kaci 2003).

Definition 3 *Let Σ and Σ' be two Δ-possibilistic bases. Then, Σ and Σ' are said to be semantically equivalent iff they generate the same possibility distribution i.e., $\pi_\Sigma = \pi_{\Sigma'}$.*

The following proposition shows that two formulas having the same rank in Σ can be replaced by their disjunction with also the same rank:

Proposition 1 *Let $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ be a Δ-possibilistic base. Let ϕ and ψ be two formulas in Σ_i, and $\Sigma' = (\Sigma_1, \cdots, \Sigma_{i-1}, \Sigma'_i, \Sigma_{i+1}, \cdots, \Sigma_n)$, where $\Sigma'_i = (\Sigma_i - \{\phi, \phi'\}) \cup \{\phi \vee \phi'\}$. Then, Σ and Σ' are semantically equivalent.*

With Δ-possibilistic bases, subsumed formulas are those that *entail* higher ranked formulas. This comes from the fact that $\Delta(\phi) \leq \Delta(\psi)$ if $\psi \vdash \phi$, and thus $\alpha \leq \Delta(\phi) \leq \Delta(\psi)$ subsumes $\Delta(\psi) \geq \beta$ for $\beta < \alpha$. For instance, declaring that any solution satisfying ψ is nice (with satisfaction degree β) is superseded by a statement that solutions to some less restricted ϕ are actually very nice $(\alpha > \beta)$.

Definition 4 *Let $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ and ϕ be a formula in Σ_i. Then, ϕ is said to be strictly subsumed by Σ if $\phi \vdash \bigvee\{\phi_k : \phi_k \in \Sigma_j$ and $j > i\}$.*

Indeed, we have the following lemma:

Lemma 1 *Let ϕ be a strictly subsumed formula by Σ. Then, Σ and $\Sigma - \{\phi\}$ are semantically equivalent.*

The following lemma shows that contradictions are not useful in Σ since they do not alter the computation of π_Σ, and can be removed without changing π_Σ.

Lemma 2 *Let $\perp$ be a contradictory formula in Σ. Then, Σ and $\Sigma' = \Sigma - \{\perp\}$ are semantically equivalent.*

As can be seen, Δ-based possibilistic logic bases behave in a dual manner (exchanging disjunction and conjunction and reversing the direction of entailment) w.r.t. N-based possibilistic logic bases, as explained in (Dubois, Hajek, & Prade 2000).

Δ-based conditional bases

In the standard representation of conditional information that is based on possibility measure Π, a conditional piece of information of the form "if p then q", denoted by $p \rightsquigarrow q$, means that in the context p (i.e. when p is satisfied), it is preferred to satisfy q rather than to falsify it. This rule expresses a relative constraint based on Π of the form $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$, which compares the best models and the best counter-models of q in the context p.

The "rule" $p \rightarrow q$ based on a Δ measure, corresponding to the constraint $\Delta(p \wedge q) > \Delta(p \wedge \neg q)$ compares the *worst* models and the *worst* counter-models of q, in the context p. It means: in the context p the agent likes q. It implies that if p is true, any model of q is more desired than the worst counter-models of q. From a preference modeling point of view, $\Delta(p \wedge q) > \Delta(p \wedge \neg q)$ corresponds to a pessimistic view since it focuses on worst cases, while $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$ corresponds to an optimistic view by considering only best cases. If $\Delta(p \wedge q) > \Delta(p \wedge \neg q)$ it means that choosing a solution where $p \wedge q$ is true, one is sure to be better off than going with a solution to $p \wedge \neg q$. It is one form of conditional preference for q rather than $\neg q$ in the context p.

In order to select a possibility distribution among a set of constraints of the Δ (resp. Π)-type, a maximal (resp. minimal) specificity principle is used. The maximal specificity principle amounts to saying that anything that is not explicitly desired is considered indifferent. The minimal specificity principle says that anything that is not rejected is acceptable.

Let us now compare the two rules w.r.t. their associated possibility distributions. It has been shown (Benferhat, Dubois, & Prade 1992) that following the *minimal* specificity principle, there is a unique possibility distribution, associated with $p \rightsquigarrow q$ which satisfies the constraint $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$. This possibility distribution is of the form $\pi_\Pi = (E_1, E_2)$, where E_2 is composed of solutions falsifying $p \wedge \neg q$, i.e. the set of models of $\neg p \vee q$, and $E_1 = \Omega - E_2$. It means that the best models are interpretations which satisfy q when p is satisfied, and are also interpretations which falsify p. This confirms the fact that Π is used to represent *negative* information, since the best models are interpretations which are not rejected, those which do not falsify q in the context p.

Now it is easy to check that applying the *maximal* specificity principle, the most specific possibility distribution associated with the rule $p \rightarrow q$ is of the form $\pi_\Delta = (E'_1, E'_2)$, where E'_2 contains models of $p \wedge q$ and $E'_1 = \Omega - E'_2$. Then, the preferred solutions are those which only satisfy q in the context p. This confirms the fact that Δ represents positive information, since E'_2 only contains interpretations that are regarded as more desired in the context where a preference has been expressed. Nothing is known about other contexts, so in the context $\neg p$, preference is neutral by default. Note also that $E'_2 \subseteq E_2$ which confirms the fact that positive information is included in what is not rejected. The complementarity between the Δ and the Π conditionals can be also seen by noticing that the Π conditional is equivalent to $\text{I}\Pi(p \wedge \neg q) > \text{I}\Pi(p \wedge q)$ with $\text{I}\Pi = 1 - \Pi$, which expresses that the models of $p \wedge \neg q$ are more impossible than those of $p \wedge q$, while the Δ conditional means that the models of $p \wedge q$ are guaranteed to be more possible than those of $p \wedge \neg q$.

Besides, note that any inequality of the form $\Delta(r) > \Delta(s)$ can be always put under the form $\Delta(p \wedge q) > \Delta(p \wedge \neg q)$. Indeed $\Delta(r) > \Delta(s)$ is equivalent to the rule $r \vee s \rightarrow r$, since $\Delta((r \vee s) \wedge r) > \Delta((r \vee s) \wedge \neg r) \Leftrightarrow \Delta(r) > \Delta(\neg r \wedge s) \Leftrightarrow min(\Delta(r \wedge s), \Delta(r \wedge \neg s)) > min(\Delta(r \wedge s), \Delta(\neg r \wedge s))$ (indeed $min(x, y) > t \Leftrightarrow min(x, y) > min(x, t)$ since $x < t$ is impossible) $\Leftrightarrow \Delta(r) > \Delta(s)$.

Definition 5 *Let $\mathcal{P}$ be a finite set of Δ-based rules. Then, a possibility distribution π is said to be compatible with $\mathcal{P}$ iff it holds that for each rule $p_i \rightarrow q_i$ in $\mathcal{P}$,*

$$\Delta(p_i \wedge q_i) > \Delta(p_i \wedge \neg q_i).$$

Let $\Delta(\mathcal{P})$ be the set of possibility distributions satisfying all the rules of $\mathcal{P}$ in the sense of Definition 5. In the next section, we will characterize the most specific possibility distribution in $\Delta(\mathcal{P})$.

Characterizing the most specific possibility distribution in $\Delta(\mathcal{P})$

Our aim in this section is to characterize the most specific possibility distribution associated with a set of rules $\mathcal{P}$. The rationale behind maximal specificity here is as follows: if a solution is not pointed out as a desired one, assume the agent is indifferent about it. It leads to minimize degrees of guaranteed possibility. We first show that this most specific possibility distribution can be computed using an appropriate conjunctive aggregation operator, and then that this possibility distribution is unique. Lastly, this possibility distribution is proved to be the one computed by means of an algorithm already given in (Benferhat *et al.* 2002b).

Let us first define the $\mathcal{MIN}$ operator to aggregate two possibility distributions:

Definition 6 *Let $\pi = (E_1, \cdots, E_n)$ and $\pi = (E'_1, \cdots, E'_m)$ be two possibility distributions s.t. $n \geq m$ and $E'_j = \emptyset$ for $m < j \leq n$. Then, $\mathcal{MIN}(\pi, \pi') = (E''_1, \cdots, E''_n)$ is defined as follows:*

- $E''_1 = E_1 \cup E'_1$,
- *for* $i = 2, \cdots, m$, $E''_i = (E_i \cup E'_i) - (\bigcup_{j=1,\cdots,i-1} E''_j)$.
- *remove* E''_i *which are empty by renumbering the non-empty ones from bottom to top.*

Example 2 *Let a and b two propositional symbols and $\pi = (E_1, E_2, E_3)$ and $\pi' = (E'_1, E'_2, E'_3)$ s.t. $E_1 = \{a\neg b\}$, $E_2 = \{\neg a\neg b\}$, $E_3 = \{ab, \neg ab\}$ and $E'_1 = \{\neg a\neg b\}$, $E'_2 = \{a\neg b\}$, $E'_3 = \{ab, \neg ab\}$. Then, $\pi'' = (E''_1, E''_2, E''_3)$ where $E''_1 = \{a\neg b, \neg a\neg b\}$, $E''_2 = \emptyset$ and $E''_3 = \{ab, \neg ab\}$. Now we remove E''_2 and renumber the E''_i, we get: $\pi'' = (E''_1, E''_2)$ where $E''_1 = \{a\neg b, \neg a\neg b\}$ and $E''_2 = \{ab, \neg ab\}$.*

The following proposition first shows that $\Delta(\mathcal{P})$ is closed under $\mathcal{MIN}$ operation i.e., combining two possibility distributions in $\Delta(\mathcal{P})$ provides a possibility distribution which also belongs to $\Delta(\mathcal{P})$. Then, we show that this operator computes the most specific possibility distribution in $\Delta(\mathcal{P})$.

Proposition 2 *Let π and π' be two elements of $\Delta(\mathcal{P})$. Then,*

1. $\mathcal{MIN}(\pi, \pi') \in \Delta(\mathcal{P})$.
2. $\mathcal{MIN}(\pi, \pi')$ *is more specific than π and π'.*

Proof
Let $\pi = (E_1, \cdots, E_n)$ and $\pi' = (E'_1, \cdots, E'_m)$ be two elements of $\Delta(\mathcal{P})$. Let $\pi'' = \mathcal{MIN}(\pi, \pi') = (E''_1, \cdots, E''_{min(n,m)})$. In a first step, we consider π'' without removing the empty strata i.e., only applying the two first points of Definition 6.

1. Let $p \to q \in \mathcal{P}$. Let $[\![p \wedge q]\!]_\pi$ (resp. $[\![p \wedge \neg q]\!]_\pi$) be the set of models of $p \wedge q$ (resp. $p \wedge \neg q$) having the least priority in π.
   π satisfies all the constraints induced by $\mathcal{P}$ means that if $[\![p \wedge q]\!]_\pi \subseteq E_i$ then $[\![p \wedge \neg q]\!]_\pi \subseteq E_j$ s.t. $i > j$.
   Also, π' satisfies all the constraints induced by $\mathcal{P}$ means that if $[\![p \wedge q]\!]_{\pi'} \subseteq E'_k$ then $[\![p \wedge \neg q]\!]_{\pi'} \subseteq E'_m$ s.t. $k > m$.
   Following the two first points of Definition 6, we have $[\![p \wedge q]\!]_{\pi''} \subseteq E''_{min(i,k)}$ and $[\![p \wedge \neg q]\!]_{\pi''} \subseteq E''_{min(j,m)}$.
   Now since $i > j$ and $k > m$ then $min(i,k) > min(j,m)$. Then π'' satisfies $p \to q$.
   Now observe that applying the last point in Definition 6, i.e. diminishing some possibility levels in case of empty stratum, leads to an even more specific distribution while preserving the strict ordering on the interpretations. Indeed $\mathcal{MIN}(\pi, \pi')$ satisfies $p \to q$.
2. To show that $\pi'' = \mathcal{MIN}(\pi, \pi')$ is a more specific than π, let us show that $\forall \omega \in \Omega$, if $\omega \in E_i$ then $\omega \in E''_j$ with $j \leq i$.
   Let $\omega \in E'_k$. Following the two first points of Definition 6, we have $\omega \in E''_{min(i,k)}$.
   Also $min(i,k) \leq i$ then π'' is more specific than π.
   Now since applying the last point of Definition 6 leads to a more specific distribution then $\mathcal{MIN}(\pi, \pi')$ is a more specific than π.

■

Let us now show that the $\mathcal{MIN}$ operator leads to compute the most specific possibility distribution in $\Delta(\mathcal{P})$ which is unique.

Proposition 3 *Let $\mathcal{P}$ be a Δ-based conditional base and $\Delta(\mathcal{P})$ be the set of possibility distributions satisfying all the constraints induced by $\mathcal{P}$. Then there exists a unique possibility distribution in $\Delta(\mathcal{P})$ which is the most specific one, denoted by $\pi_{\Delta\ m.spec}$, and computed in the following way:*

$$\pi_{\Delta\ m.spec} = \mathcal{MIN}\{\pi : \pi \in \Delta(\mathcal{P})\}.$$

Proof
Indeed, following the first point of Proposition 2, $\pi_{\Delta\ m.spec}$ belongs to $\Delta(\mathcal{P})$.
Suppose now that there exists a possibility distribution π^* in $\Delta(\mathcal{P})$ s.t. $\pi_{\Delta\ m.spec}$ is not more specific than π^*. This contradicts the second point of Proposition 2.
■

In (Benferhat *et al.* 2002b), an algorithm has been given for computing a possibility distribution associated with a set of Δ-based constraints.
Let $\mathcal{P} = \{p_k \to q_k : k = 1, \cdots, K\}$ be a set of Δ-based rules. Let us denote by $\mathcal{C} = \{C_k : (L(r_k), R(r_k)), r_k : p_k \to q_k \in \mathcal{P}\}$ the set of Δ-based comparative constraints induced by $\mathcal{P}$, where $L(r_k) = \{\omega : \omega \models p_k \wedge q_k$ and $r_k : p_k \to q_k \in \mathcal{P}\}$ and $R(r_k) = \{\omega : \omega \models p_k \wedge \neg q_k$ and $r_k : p_k \to q_k \in \mathcal{P}\}$.
Algorithm 1 provides a possibility distribution satisfying $\mathcal{P}$, denoted by $\pi_\mathcal{P}$. The idea of the algorithm consists in assigning to each interpretation the lowest possibility degree. At each step i, we put in E_i the interpretations which are not in the left-hand part of any remaining Δ-based comparative constraint $(L(r_k), R(r_k))$ (otherwise such a constraint will be falsified). For instance, the least preferred (or plausible) interpretations are those which do not verify any rule (namely, are not in any $L(r_k)$).

Algorithm 1:

Data: $\mathcal{P} = \{r_k : p_k \to q_k\}$
Result: $\pi_\mathcal{P} = (E_1, \cdots, E_n)$
begin
  Let $\mathcal{C} = \{C_k = (L(r_k), R(r_k)) : r_k \in \mathcal{P}\}$;
  $i \leftarrow 0$;
  while $\Omega \neq \emptyset$ **do**
    $i \leftarrow i + 1$;
    $E_i = \{\omega : \nexists C_k \in \mathcal{C}$ s.t. $\omega \in L(r_k)\}$;
    if $E_i = \emptyset$ **then**
      $\mathcal{P}$ is inconsistent
    - Remove from Ω elements of E_i;
    - Remove from $\mathcal{C}$ constraints C_k s.t. $E_i \cap R(r_k) \neq \emptyset$.
  return $(E_1, \cdots, E_i)$.
end

Example 3 *Let $\mathcal{P} = \{r_1 : \neg s \to t, r_2 : s \vee h \to \neg t, r_3 : h \to s\}$ and s, h and t are three propositional symbols which stand respectively for sun, $holidays$ and $town$.*
Thus r_1 means that when there is no sun, the best thing is to remain in town. r_2 means that if there is sun or if one is on

holidays, it is good to be out of town. In holidays, it is better to have sun.
Let $\Omega = \{\omega_0 : \neg h \neg s \neg t, \omega_1 : \neg h \neg st,\ \omega_2 : \neg hs \neg t, \omega_3 : \neg hst, \omega_4 : h \neg s \neg t, \omega_5 : h \neg st, \omega_6 : hs \neg t, \omega_7 : hst\}$.
Let us now apply Algorithm 1. We have:

$$\mathcal{C} = \{C_1 : (L(r_1), R(r_1)), C_2 : (L(r_2), R(r_2)), C_3 : (L(r_3), R(r_3))\}$$
$$= \{C_1 : (\{\omega_1, \omega_5\}, \{\omega_0, \omega_4\}), C_2 : (\{\omega_2, \omega_4, \omega_6\}, \{\omega_3, \omega_5, \omega_7\}), C_3 : (\{\omega_6, \omega_7\}, \{\omega_4, \omega_5\})\}.$$

We put in E_1 *interpretations which do not belong to any left-hand part of the desires in* $\mathcal{C}$*, namely* $E_1 = \{\omega_0, \omega_3\}$.
Then, we remove from $\mathcal{C}$ *the constraints* C_1 *and* C_2 *since* $R(C_1) \cap E_1 \neq \emptyset$ *and* $R(C_2) \cap E_1 \neq \emptyset$*. We get* $\mathcal{C} = \{C_3 : (L(r_3), R(r_3))\}$.
Similarly, we get $E_2 = \{\omega_1, \omega_2, \omega_4, \omega_5\}$ *and* $E_3 = \{\omega_6, \omega_7\}$*. Then,* $\pi_{\mathcal{P}} = (E_1, E_2, E_3)$.
This reveals that what is preferred is to be in holidays with sun (hs)*, while the least preferred is to be at work* $(\neg h)$ *with either sun in town or no sun out of town.*

Note that the possibility distribution computed in the above algorithm is the most specific possibility distribution associated with $\mathcal{P}$. To show this, it is sufficient to show that it belongs to the set of most specific possibility distributions in $\Delta(\mathcal{P})$.
The maximal specificity criterion can be checked by construction. Indeed let $\pi = (E_1, \cdots, E_n)$.
Following Algorithm 1, we put in E_i all interpretations which don't satisfy any desire induced by the actual set of rules $\mathcal{P}_i$ i.e., if $\omega \in E_i$ then $\forall p \to q \in \mathcal{P}_i, \omega \not\models p \wedge q$. This means that if $\omega \notin E_i$ then $\exists p \to q \in \mathcal{P}_i, \omega \models p \wedge q$.
Let us now try to put some interpretation $\omega \in E_j$ with $j < i$ in E_i. However this is not possible since $\omega \in E_j$ means that $\omega \notin E_i$ i.e., there exists $p \to q$ in $\mathcal{P}_i$ s.t. $\omega \models p \wedge q$, which is a contradiction.

Bridging ordinal and absolute representations of positive information

In the previous sections, two representation frameworks have been given for representing positive information in possibility theory, namely the logical and conditional representations. In this section, we give a method to translate a set of Δ-based rules into a Δ-possibilistic base and conversely. This is interesting for taking advantage of the benefits of each representation format.

From a Δ-based conditional base to a Δ-possibilistic base

The translation of a set of Δ conditional desires into a Δ-possibilistic base can be achieved by Algorithm 2. This is particularily interesting from an information fusion point of view, when information to be merged may be expressed in heterogeneous formats. The fusion of Δ-possibilistic bases has been developed in (Benferhat & Kaci 2003).

Proposition 4 *Let* $\mathcal{P}$ *be a set of rules. Let* $\pi_{\mathcal{P}}$ *and* π_Σ *be the possibility distributions associated to* $\mathcal{P}$ *following Algorithm 1 and 2 respectively. Then,*

$$\pi_{\mathcal{P}} = \pi_\Sigma.$$

Algorithm 2:

Data: $\mathcal{P} = \{r_k : p_k \to q_k\}$
Result: $\Sigma = (\Sigma_1, \cdots, \Sigma_m)$
begin
  $m \leftarrow 0$;
  while $\mathcal{P} \neq \emptyset$ **do**
    $m \leftarrow m + 1$;
    $A = \{\neg p_k \vee \neg q_k : p_k \to q_k \in \mathcal{P}\}$;
    $\Sigma_m = \{p_k \wedge q_k : p_k \to q_k \in \mathcal{P}$ and $A \cup \{p_k\}$ is consistent$\}$;
    If $\Sigma_m = \emptyset$ then $\mathcal{P}$ is inconsistent;
    $\mathcal{P} = \mathcal{P} - \{p_k \to q_k : p_k \wedge q_k \in \Sigma_m\}$;
  return $\Sigma = (\Sigma_1, \cdots, \Sigma_m)$.
end

Example 4 *Let us consider again the* Δ*-based conditional base given in Example 3 and apply Algorithm 2.*
First we have $A = \{s \vee \neg t, (\neg s \wedge \neg h) \vee t, \neg h \vee \neg s\}$.
Note that each of $A \cup \{\neg s\}$ *and* $A \cup \{s \vee h\}$ *is consistent, however* $A \cup \{h\}$ *is inconsistent.*
Then, $\Sigma_1 = \{\neg s \wedge t, (s \vee h) \wedge \neg t\}$.
We now remove the rules $\neg s \to t$ *and* $s \vee h \to \neg t$ *from* $\mathcal{P}$ *since* $\neg s \wedge t \in \Sigma_1$ *and* $(s \vee h) \wedge \neg t \in \Sigma_1$*, we get* $\mathcal{P} = \{h \to s\}$*. Then we have* $\Sigma_2 = \{h \wedge s\}$*. Hence we get* $\Sigma = (\Sigma_1, \Sigma_2)$.
We can check that π_Σ *is indeed the possibility distribution computed in Example 3.*

From a Δ-possibilistic base to a Δ-based conditional base

This section provides the converse translation namely from a Δ-possibilistic base to a Δ-based conditional base. This is interesting since Δ-based conditional desires are easily understood: they express preferences given some context. Then the agent may for example revise its desires by changing the rules.
Let $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ be a Δ-possibilistic base without subsumed formulas, where each Σ_i $(i = 1, \cdots, n)$ is composed of one formula [1]. We construct from Σ a Δ-based conditional base $\mathcal{P}$ as follows:

$$\mathcal{P}_\Sigma = \{\to \Sigma_1, \Sigma_1 \vee \Sigma_2 \to \Sigma_2, \cdots, \Sigma_{n-1} \vee \Sigma_n \to \Sigma_n\}.$$

The rule $\Sigma_i \vee \Sigma_{i+1} \to \Sigma_i$ means that when either Σ_i or Σ_{i+1} is satisfied, we prefer to satisfy Σ_{i+1} which reflects the priority between strata of Σ. Then, the following proposition can be established:

Proposition 5 *Let* $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ *be a* Δ*-possibilistic base without subsumed formulas and* $\mathcal{P}$ *be the* Δ*-based conditional base constructed from* Σ *as shown above. Let* π_Σ *and* $\pi_{\mathcal{P}}$ *be the possibility distribution associated to* Σ *and* $\mathcal{P}_\Sigma$ *following Definition 2 and Algorithm 1 respectively.*

[1] This hypothesis is not a limitation since following Proposition 1, a set of formulas having the same rank can be replaced by their disjunction with also the same rank.

Then,

$$\pi_{\Sigma} = \pi_{\mathcal{P}_{\Sigma}}.$$

The proof is omitted for the sake of brevity.

Example 5 *Let* $\Sigma = (\Sigma_1, \Sigma_2)$ *where* $\Sigma_1 = \{\neg s \wedge t, s \wedge \neg t, h \wedge \neg t\}$ *which is equivalent to* $\{(\neg s \wedge t) \vee ((s \vee h) \wedge \neg t)\}$ *and* $\Sigma_2 = \{h \wedge s\}$.
Following the construction of $\mathcal{P}_{\Sigma}$*, we get:*
$\mathcal{P}_{\Sigma} = \{\rightarrow (\neg s \wedge t) \vee (s \wedge \neg t) \vee (h \wedge \neg t),$
$(\neg s \wedge t) \vee (s \wedge \neg t) \vee (h \wedge \neg t) \vee (h \wedge s) \rightarrow h \wedge s\}$.
It can be checked that Σ *and* $\mathcal{P}_{\Sigma}$ *generate the same possibility distributions.*

Conclusion

This paper has shown how a possibility distribution expressing more or less desirable states of the worlds can be induced from a set of desires expressed by means of weighted formulas or "rules" stated in terms of a guaranteed possibility measure. The paper has discussed ordinal and absolute representations in terms of this measure. Ordinal representations of the form $\Delta(p \wedge q) > \Delta(p \wedge \neg q)$ are simpler and more natural for expressing pieces of preference. Moreover, this is a suitable format in case of preference revision. However, absolute representation of the form $\Delta(p) \geq \alpha$ are less natural for direct elicitation but turn out to be simpler to handle in the possibilistic logic calculus.

The framework proposed in this paper applies to the representation of preferences. However it may also be used for uncertainty representation purposes. In this case, a Δ-based possibilistic logic base represents a set of observations which are more or less strongly supported by evidence.

Taking into account the result of this paper together with the one of N-based possibilistic logic, we are now in a position for a joint handling of positive and negative preferences respectively expressed in terms of Π and Δ measures, thus inducing upper and lower bounds on the possibility distribution encoding the preference. Then adding new information of this form leads to an iteratively constructed approximation (from below and above) of a possibility distribution that is defined on finer and finer partitions of the set of interpretations. This comes very close to C-calculus (Caianiello & Ventre 1985), where pieces of information about upper bounds and lower bounds of function over a partitioned domain are combined with similar information pertaining to another partition of the same domain. This is also similar to the idea of rough sets (Pawlak 1991), where set characteristic functions are approximated from above and below on a domain quotiented by an equivalence relation.

The handling of preferences in logical, conditional or graphical settings has raised the interest of several AI researchers in the last past years, e.g., (Boutilier, Deans, & Hanks 1999; Benferhat, Dubois, & Prade 2001; Brafman & Domshlak 2002). Graphical representations of preferences are widely studied in literature such as for example CP-Nets (Brafman & Domshlak 2002), which are graphical models for representing conditional preferences. Indeed they are close to rule-based representation of preferences given in this paper however they do not distinguish between positive and negative preferences. A deeper comparison between the different types of preference representations is left to a future work.

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