CT04 Abstracts

On the Limitations of Birkhoff's (Co)Variety Theorem

Jiří Adámek (Technical University of Braunschweig, Germany)

For large signatures S we prove that Birkhoff's Variety Theorem holds (i.e., equationally presentable collections of S-algebras are precisely those closed under limits, subalgebras, and quotient algebras) iff the universe of small sets is not measurable. Under that limitation Birkhoff's Variety Theorem holds in fact for F-algebras of an arbitrary endofunctor F of the category $Class$ of classes and functions. Modifying an example of J. Reiterman we present two endofunctors of Set for which Birkhoff's Variety Theorem holds, but for the coproduct of which it fails.

In contrast, no limitations are needed for Birkhoff's Covariety Theorem: given an endofunctor F of $Class$, co-equationally presentable classes of coalgebras in the sense of J. Rutten are precisely those closed under coproducts, subcoalgebras, and quotient coalgebras. Moreover, for endofunctors of Set Birkhoff's Covariety Theorem also holds -provided that a coequation is understood to be a subchain of the cofree-coalgebra chain. In contrast, the coequations introduced by Kurz and Rosicky, which turn out to be precisely as expressive as the coequations introduced by H. Porst and the author, are proved to be insufficient for set functors in general.

Advances in Algebraic Set Theory

Steve Awodey (Carnegie Mellon University, USA)

Since its introduction by Joyal and Moerdijk in [2], the algebraic approach to set theory has proven to be a flexible and powerful tool for constructing models of various elementary set theories. Following the approach introduced by Alex Simpson in [3], the theory has been furtheradvanced through recent collaboration between Awodey, Carsten Butz, Thomas Streicher, and Simpson (in [1]).

This talk summarizes these developments, including a natural model consisting of "ideals" over a topos, and a logical completeness theorem for set theory with respect to ideal completions of toposes.

References

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Perfect Maps: An Exact Category in Topology

Michael Barr (McGill University, Canada)

A perfect map of spaces is a continuous map that is closed, compact (the inverse image of a compact set is compact) and satisfies a separation condition that is irrelevant here. Let Perf denote the category of Hausdorff spaces and perfect maps. Then Perf is exact, but not complete (lacks a terminal object), although it has arbitrary fibre products. Then for each space Z, the category Perf$/Z$ is complete and it turns out that the underlying functor Perf$/Z \to$ Perf$/|Z|$ is tripleable and maps in Perf induce adjoint pairs between the corresponding slices, so it is a sort of indexed triple. Of course, Perf$/1$ is just the familiar compact Hausdorff spaces.

Lax-Monoids and Configuration Spaces

Michael Batanin (Macquarie University, Australia)

B. Day and R. Street introduced a notion of lax-monoid in a monoidal bicategory in [1] as a strict monoidal lax-functor from the monoidal category of all ordinals Δ to a monoidal bicategory. It turned out that many well known and important structures are just special cases of lax-monoids. The list includes monoids, monoidal categories, lax-monoidal and promonoidal categories, nonsymmetric operads, multicategories and substitudes.

In the next paper by Day and Street [2] a definition of braided and symmetric lax-monoid was introduced. These are braided (symmetric) strict monoidal lax-functors from $\mathbf{B}\int\Delta$ and $\mathbf{P}\int\Delta$ respectively which are strict functors with respect to the braiding (symmetry). Again, as special cases, we have braided and symmetric analogues of the categorical structures from the list above.

In my talk I will present a notion of n-globular lax-monoid in an n-globular monoidal 2-category V. By definition, this is a strict monoidal lax functor from Ω_n to V where Ω_n is the category of trees introduced in [3].

I will show that the category of n-globular monoids contains the category of lax-monoids (for $n = 1$), of braided lax-monoids ($n = 2$), and of symmetric lax-monoids (for $n > 2$) as full subcategories. Other interesting special cases are n-operads and globular multicategories. For example, we prove that braided operads are special 2-operads in our sense and that symmetric operads are special n-operads for $n > 2$. To prove this we reveal an interesting relationship between the topology of real configuration spaces and the combinatorics of the categories Ω_n.

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The Shape of Linear Logic

Richard Blute (University of Ottawa, Canada)

We examine several monoidal categories from the point of view of the model theory of linear logic. The category of sets and relations and the category of finite-dimensional Hilbert spaces are both compact closed categories and hence models of the geometry of interaction style semantics of linear logic.

But closely related categories, the categories of locally finite relations and arbitrary Hilbert spaces, fail to be compact closed. In both cases, one only has the transposes required in the definition of compact closed category for a limited class of morphisms. For example, in the category of Hilbert spaces, it is the Hilbert-Schmidt maps that are transposable. Hilbert-Schmidt maps exist within the category of Hilbert spaces as a two-sided ideal, and a two-sided ideal satisfying such an adjunction is precisely the notion of a nuclear ideal, as introduced by Abramsky, Blute and Panangaden.

In such categories, the identity map is typically not transposable, but there is an obvious sense in which it is approximated by transposable maps. We claim that shape theory, as introduced by Borsuk, is the appropriate structure for capturing this idea of approximation.

Shape theory in its categorical formulation has two crucial features. First is the idea that the general objects of a category can be approximated by systems constructed from a certain subclass of well-behaved objects. For example in the classical theory, an arbitrary topological space is approximated by a system of polyhedra. Second is the idea that one can extend the original category in question by adding new "idealized" morphisms, which will be systems of morphisms from the base category.

We will show that for our examples, the resulting shape category has not only a monoidal structure, but a polarized version of the notion of compact closed structure. Our notion of polarized compact closed category is based on work of Cockett and Seely on game theory semantics for linear logic.

On the Normal Subobjects in the Category of Topological Groups

Dominique Bourn (Université du Littoral, France)

The category $Gp(Top)$ of topological groups inherits many algebraic aspects of the category Gp of groups on the account of sharing with it the strong algebraic property of being *protomodular*, a conceptual context within which there is, among other things, an intrinsic notion of normal subobject and of centrality. Of course, in this context, any kernel map is a normal subobject, but in general there are far more normal subobjects than kernel maps. The aim of this talk is to detail and to run some aspects of this strong algebraic background. We shall focus our attention on three points, the third of which being the main reason of this work:

1. The characterization of the normal subobjects in $Gp(Top)$.

2. Given (G, T_G) a non abelian topological group, and I a normal subgroup of G, we shall list, at least in the particular case of clopen topological groups, all the topologies T_I on I, such that the inclusion $(I, T_I) \hookrightarrow (G, T_G)$ becomes a normal subobject in $Gp(Top)$. More precisely, we shall show that there are as many topologies as normal subgroups J of G satisfying : $[I, \overline{\{1_G\}}] \subset J \subset I \cap \overline{\{1_G\}}$.

3. Any pointed regular and finitely cocomplete protomodular category C being given, it is possible to develop an intrinsic commutator theory. When moreover C is exact, we have always, for any pair of objects (X, Y) of C, the following inclusion: $[X, Y] \subset X \cap Y$, as in the classical case of the category Gp of groups. We shall exhibit here a counterexample to this property in the regular (but non exact) context of $Gp(Top)$.

When does a Functor Realise a Localization of an Algebraic Category?

Claudia Centazzo[1] (Université Catholique de Louvain, Belgium)

In this talk, we treat both cases of localizations of algebraic categories and of presheaf categories. More explicitly, we give necessary and sufficient conditions for a coproduct-preserving functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, from an algebraic theory (in the sense of Lawvere) $\mathcal{C}$ into a cocomplete, exact category $\mathcal{E}$ with filtered colimits commuting with finite limits, to realise $\mathcal{C}$ as a localization of the category of models on $\mathcal{C}$. On the other hand, we investigate in a similar way the traditional case of localizations of presheaf categories. Namely, we describe necessary and sufficient conditions for

[1] This is a joint work with E. M. Vitale.

a functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, from a small category $\mathcal{C}$ into a cocomplete, exact and extensive category $\mathcal{E}$, to express $\mathcal{C}$ as a localization of the category of presheaves on $\mathcal{C}$.

The point of view is a bit different from the literature: provided a category $\mathcal{E}$ - which satisfies various properties depending on the case we are dealing with - these theorems describe all its realisations as a localization of some appropriate category of set-valued functors. Moreover, the nature of these conditions on the functor k allows us to classify geometric morphisms. We are able, in fact, to distinguish properties relative to fully faithfulness from properties concerning just the left exactness of the left adjoint functor.
To end the contribution, we link our results to the previous renowned ones devoted to this subject, as Giraud theorem for presheaf categories, Vitale theorem for algebraic categories and Gabriel-Popescu theorem in the additive case of Grothendieck categories.

n-categories with duals

Eugenia Cheng (University of Cambridge, UK)

We examine n-categories in which every k-cell is equipped with a specified dual, for all $0 < k < n$, and n-categories in which, in addition, every n-cell has a specified "formal dual". We show that an ω-category with all duals is an ω-groupoid. To avoid questions about the definition of ω-category, we show that this result can be obtained using a bare minimum of ω-categorical structure. The result then holds for any definition of ω-category having this bare minimum.

We study the example of cobordisms of all dimensions; although the full structure of an ω-category of cobordisms has not yet been found, we exhibit enough structure to show that this example fits into the above framework.

Revisiting Cauchy-Completeness

Maria Manuel Clementino (Universidade de Coimbra, Portugal)

We extend Lawvere's notion of (Cauchy-)complete $\mathcal{V}$-category [2] into $(\mathsf{T}, \mathcal{V})$-categories (in the sense of [1]) and explore several examples.

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The Manifold Construction Revised

J. R. B. Cockett (University of Calgary, Canada)

The traditional construction of manifolds using atlases and charts was redescribed categorically by Marco Grandis in the context of e-coherent categories about fifteen years ago. It is a construction which can be applied to any abstract category of partial maps (satisfying certain cocompleteness conditions). The talk will review the construction in the context of abstract categories of partial maps viewed as restriction categories, and describe some connections to synthetic topology, and how the construction is related to the, perhaps, more conventional view of manifolds as sheaves.

Classification of Equivariant Extensions of Categorical Groups

Aurora del Rio (Universidad de Granada, Spain)

In [4], K.-H. Ulbrich proved that the category of Γ-graded categorical groups, for Γ a group, is equivalent to the category of categorical groups supplied with a coherent left action from Γ (the Γ-categorical groups). Using this equivalence and the homotopy classification of graded categorical groups [2], in this talk we show a theory of extensions of Γ-categorical groups [3] This theory both extends the thoery of Breen [1] (the non-equivariant case) and that of equivariant exyensions of groups [2] (the case in which the categorical groups are discrete).

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Covering Theory and the Fundamental Progroupoid of a General Topos

Eduardo Dubuc (Universidad de Buenos Aires, Argentina)

The galois theory of locally constant objects for a locally connected topos is well developed and understood, and as it is, it depends heavily on the local condition of connectedness. Recently topologist working in shape theory have dealt with the theory for a non locally connected base space. In their work, the descent data underneath the notion of covering projection has to be made explicit in one way or another. We realized that we are in face of a situation of classical topological descent as described in the introduction to “Categories Fibrees et Descente”, Expose VI, SGA1.

Consider a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, a cover $\zeta\colon U \to \gamma^* I$, $I \in \mathcal{S}$, $U \in \mathcal{E}$, and a family $S \to I$. We approach the notion of locally constant object X split by U not focusing in the isomorphism $\theta :\ \gamma^* S \times_{\gamma^* I} U \xrightarrow{\cong} X \times U$ but instead focusing in the descent data $\sigma :\ \gamma^* S \times_{\gamma^* I} U \times U \xrightarrow{\cong} \gamma^* S \times_{\gamma^* I} U \times U$ that construct X by descent on $\gamma^* S \times_{\gamma^* I} U$. We introduce a condition on σ that it is vacuous when the topos is locally connected, and that when the topos is spatial, it furnish by descent the notion of covering projection on non locally connected base spaces considered by topologists.

Given an arbitrary topos $\mathcal{E}$, we define the topos $\mathcal{G}_U$ of *covering projections split by* U to be the category of locally constant objects constructed by descent data that satisfies the condition. The key result is that this topos is locally connected even when $\mathcal{E}$ is not. Clearly there is a canonical point (morphism) $\mathcal{S}_{/I} \xrightarrow{f_U} \mathcal{G}_U$ and we have:

Theorem 1 *The topos $\mathcal{G}_U$ is an atomic topos and the point f_U is surjective.*

Theorem 2 *(follows by Joyal-Tierney' theorem 1) There is an equivalence $\theta_U : \mathcal{B}G_U \cong \mathcal{G}_U$ which identifies f_U with the canonical point $\mathcal{S}_{/I} \xrightarrow{p_U} \mathcal{B}G_U$, where G_U is the localic groupoid (with set of objects I) of automorphisms of the point f_U, and $\mathcal{B}G_U$ its classifying topos.*

Let $\mathcal{N}U_\bullet$ be the Cech nerve of $\zeta\colon U \to \gamma^* I$, that is the simplicial set whose n-simplexes are given by $\mathcal{N}U_n = \{(i_0, i_1, \ldots i_n) \mid U_{i_0} \cap U_{i_1} \ldots \cap U_{i_n} \neq \emptyset\}$.

Theorem 3 *If the topos $\mathcal{E}$ is spatial, $\mathcal{G}_U$ is equivalent to the descent topos $\mathcal{S}_{\mathcal{N}U_\bullet} \longrightarrow \mathcal{D}_U$. Furthermore, the point f_U is essential, thus representable by a family $A \to \gamma^* I$, and the localic groupoid is the classical groupoid of automorphisms of $A \to \gamma^* I$.*

The assignments $U \mapsto G_U$ and $U \mapsto \mathcal{G}_U$ are functorial on the bifiltered category of coverings under refinement, and by restriction to the filtering poset of covering sieves we define the *fundamental progroupoid* $p\pi_g(\mathcal{E})$ and the *fundamental protopos* $p\pi_t(\mathcal{E})$. The *fundamental topos* $\mathcal{E} \xrightarrow{\sigma} \pi_t(\mathcal{E})$ is the inverse limit topos corresponding to the protopos $p\pi_t(\mathcal{E})$. It comes equipped with a localic point defined over $Sh(L)$, where L is the inverse limit of the sets I taken in the category of localic spaces.

Theorem 4 *There is an equivalence of topoi $\mathcal{B}p\pi_g(\mathcal{E}) \cong \pi_t(\mathcal{E})$ which identifies the localic points.*

Chu Spaces meet Abstract Homotopy Theory

Jeffrey M. Egger (University of Ottawa, Canada)

We show that it is possible to put a Quillen model structure on certain Chu categories in such a way that a Chu space is separated if and only if it is fibrant, and extensional if and only if it is cofibrant.

The Core of a Category

Peter Freyd (University of Pennsylvania, USA)

The subject generalizes easily to monoidal categories but this abstract will be restricted to categories with products. Given such, its CORE—if it exists—is an object C together with a transformation $X \times C \to X$, natural in X, which transformation is universal among such: that is, any other natural transformation of the form $X \times A \to X$ is uniquely induced by a map $A \to C$.

In an exponential (cartesian-closed) category the core is just the "end" of the bifunctor of exponentiation:

$$\int_X X^X$$

which is one way of seeing that the core is something of an "internalization" of its center: with or without exponentiation the maps from 1 to C are in natural correspondence with the endomorphisms of the identity functor.

Given a category with products that is generated by the subobjects of its terminator, its core is the terminator. In the other direction, we can construct Core($\mathcal{S}^M$) , where M is a monoid and $\mathcal{S}^M$ is the category of right M-sets, as the set of functions, f, from M to M with the property that $\alpha f(\beta\alpha) = (f\beta)\alpha$ for all $\alpha, \beta \in M$. For the action of M on Core($\mathcal{S}^M$) define f^α by stipulating that $f^\alpha(\beta) = f(\alpha\beta)$. The universal transformation $X \times \text{Core}(\mathcal{S}^M) \to X$ sends $\langle x, f\rangle$ to x^{f1}. Given a transformation $t : X \times A \to X$ define $A \to \text{Core}(\mathcal{S}^M)$ to be the function that sends $a \in A$ to f_a where $f_a\beta = t_M\langle 1, a^\beta\rangle$.

It is fairly easy to finesse this construction to one that works for the category of presheaves on an arbitrary small category and it isn't all that hard to further finesse it to a construction that works for the category of sheaves for an arbitrary site. It's easier, though, to use the special adjoint functor theorem to construct cores for arbitrary Grothendieck topoi.

Let C be a core in a category with finite products. C is equipped with a monoid structure, that is, it has a "constant" $1 \to C$ and a binary operation $C \times C \to C$ satisfying the axioms for a monoid. $e : 1 \to C$ is defined as the unique map that such that

$$X \times 1 \xrightarrow{1 \times e} X \times C \xrightarrow{t} X$$

is the canonical natural equivalence (t is the defining transformation for the core). The multiplication $m : C \times C \to C$ is defined as the unique map such that

$$X \times (C \times C) \xrightarrow{1 \times m} X \times C \xrightarrow{t} X$$

is

$$X \times (C \times C) \xrightarrow{a} (X \times C) \times C \xrightarrow{t \times 1} X \times C \xrightarrow{t} X$$

where a is the associativity isomorphism.

If we notate things as if there were elements and denote the values of the canonical transformation at X by $t_X\langle x, c\rangle = x \uparrow c$, then we have defined 1 so that $x \uparrow 1 = x$ and we have defined the product so that $x \uparrow (cd) = (x \uparrow c) \uparrow d$.

Besides the monoid structure on C there is another binary operation on C, to wit, $t_C\langle x, y\rangle = x \uparrow y$. We have an object with a constant and two binary operations satisfying the equations of an (abstract) CORE ALGEBRA:

1, 1′:	$1x = x = x1,$
2:	$x(yz) = (xy)z,$
3, 3′:	$1 \uparrow x = 1,\ x \uparrow 1 = x,$
4:	$(xy) \uparrow z = (x \uparrow z)(y \uparrow z),$
5:	$x \uparrow (yz) = (x \uparrow y) \uparrow z,$
6:	$xy = y(x \uparrow y),$
7:	$(x \uparrow y) \uparrow z = (x \uparrow z) \uparrow (y \uparrow z).$

These equations are complete.

That is, any equation that holds for all categorical cores is a consequence of these axioms. (Indeed, the result holds for any universally quantified first-order sentence.) The proof is via a representation theorem. Let C be an abstract core algebra. View C just as a monoid to form the core algebra Core($\mathcal{S}^C$). Define a map $C \to$ Core($\mathcal{S}^C$) by sending $c \in C$ to the function that sends x to $c \uparrow x$. It is a faithful (by 3′) representation (by 3,4 and 7) of core algebras, hence any abstract core algebra appears as a subalgebra of the core of a category. Any universal sentence that holds for all categorical cores must therefore hold for all abstract cores.

Equations $1', 3'$ and 7 are redundant. When removed, the remaining equations are completely independent. That is:

For any subset of the unprimed equations 1 through 6 there is a model of those equations that fails each equation not in the subset.

The notion of the core of a category is a telling example of how polymorphic definitions are outside the scope of traditional category theory. Let G be any (non-trivial) group: The forgetful functor $\mathcal{S}^G \to \mathcal{S}$ is bi-continuous and a logical functor—indeed, it is of the form $\mathcal{A} \xrightarrow{\Delta} \mathcal{A}/B$. It has both adjoints. It preserves and reflects everything any category theorist ever asked a functor to preserve.

It does not preserve the core. From the construction above we can deduce that Core($\mathcal{S}^G$) is G with the conjugacy G-action. Core($\mathcal{S}$) on the other hand has just one element. From the point of view of the internal language of topoi, the core, therefore, is an example of a structure that necessarily requires more than the internal logic of a topos in its definition.

On a Class of Pullbacks Preserved by the Presheaf Functor

Jonathon Funk (University of Regina, Canada)

The presheaf functor from categories to toposes preserves a pullback square of categories if one pair of opposite sides of the category pullback are discrete opfibrations. This is part of ongoing joint work with M. Bunge, M. Jibladze, and T. Streicher.

Towards a Categorical Quantum Mechanics

Emmanuel Galatoulas (University of Athens, Greece)

Advancements in quite diverse areas of category theory suggest that we are finally in a position to address the issue of a categorical foundation of physical theories, in particulatr of Quantum Mechanics, in earnest. What the most well established developments in Topological Quantum Field theories and string theory indicate is that such a categorical foundation should rather naturally have to facilitate some subtler and more involved structures than say cartesian cloced categories (e.g topoi), more specificaly use of higher order and enriched categories almost forces itself as the framework where both the notion of quantum mechanical system and process can be codified and analysed.

What really appeals in such a foundational approach is that the apparently unnecessary cost in complexity is traded off with a deep gain in understanding the peculiarities of QM, things like entanglement of quantum systems, non-locality and so on which look rather counterintuitive and paradoxical from the point of view of the standard set-theoretical formalism of QM, begin to get a coherent and consistent description in the context of the more implicate higher order categorical structures.

Our pursuit of this alternative foundation for QM evolves in a twofold way. On the one hand by generalising the notion of quantum mechanical system and process by means of "categorification", ie. by unfolding their internal dynamics and recovering their structural features as they organise themselves in higher dimensional entities.

Secondly by generalising the framework of traditional QM built around Hilbert spaces and $\mathcal{C}^*$-alegbras to a much more abstract setting involving structures which not only take over most of the properties and features of say the category of Hilbert spaces but in fact uplift those features to the more generic and intriguing context of arbitrary (higher dimensional) monoidal categories.

For instance there does not seem to be any reason why the lattice of projections of a Hilbert space should have such a priviledged place in the structure of QM, if we can simulate its properties by more generic quantalic structures emerging from monoidal categories other than vector spaces.

We expect this kind of approach not only to enhance our understanding of QM but also to have a reciprocal effect of injecting certain quantum intuitions into our treatment of higher order categorical structures.

Generalised Species of Structures and Analytic Functors

Nicola Gambino[2] (University of Cambridge, UK)

In [1, 2] Andr´e Joyal introduced the notions of *species of structures* and *analytic functor* to provide a general understanding of formal power series, an important tool in enumerative combinatorics. A crucial aspect of the theory of species is that it provides a calculus (with operations such as composition, addition, multiplication, and derivation) that extends the familiar calculus of formal power series [3].

Motivated by the desire to give a categorical account of the calculus of species, we generalize Joyal's notions of species of structures and analytic functor, and show how these generalised notions can be organized into a bicategory and a 2-category, respectively. Our main results show that these two-dimensional structures are equivalent and come equipped with a cartesian closed structure. These results allow us to give a homogeneous presentation of calculi of species and to construct new graph models of the untyped lambda-calculus.

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Supremum Enriched Taxons and Sheaves

W. Dale Garraway (Eastern Washington University, USA)

An enriched taxon is a $\mathcal{V}$-enriched semicateory whose composition is a coequalizer. When $\mathcal{V}$ is **SUP**, the monoidal category of complete lattices, a **SUP**-category is sometimes called a quantaloid, The semicategory of modules on a **SUP**-taxon, $\mathcal{Q}$, is equivalent to the semicategory of infimum preserving lax-semifunctors and lax-transformations $LAX_{\mathbf{INF}}(\mathcal{Q}^{co}.REL)$. It can now be shown that the idempotent splitting completion of the category of matrices of $\mathcal{Q}$ is then equivalent to the category of infimum preserving semifunctors and transformations, **INF**-$TAX(\mathcal{Q}^{co}, REL)$. Thus **INF**-$TAX(\mathcal{Q}^{co}, REL)$ is the completion of a supremum enriched taxons with respect to coproducts and the splitting of idempotents. Extending this in the appropriate way to involutive taxons this tells us that for $\mathcal{E}$ a Grathendieck topos **INF**-$TAX^*(\mathbf{REL}(\mathcal{E})^{co}, REL)$ is equivalent to $\mathbf{REL}(\mathcal{E})$. Thus we can interpret a sheaf as an infimum preserving semifunctor. We next explore the relationship between these categories and the appropriate notion of the category of elements and profunctors. In particular we will explore the difference between modules on $\mathcal{Q}$ and the category of $\mathcal{Q}$-taxons and the appropriate slice categories.

Torsion Theories and Homological Categories

Marino Gran[3] (Universitè du Littoral Côte d'Opale, France)

In any homological category there is an intrinsic notion of short exact sequence. Several classical results in the homological algebra of groups and rings hold, more generally, in any homological category: this is the case, for instance, for the Short Five Lemma, the 3×3-Lemma and the Snake Lemma [2].

In this joint work with Dominique Bourn we show that also the classical notion of a torsion theory finds a natural setting in these categories, and that many results known in the abelian context can be extended to the homological categories. Some topological examples of torsion theories are covered by this new axiomatic approach, since the category $Grp(Top)$ of topological groups is homological, as is, more generally, any category $Th(Top)$ of models of a semi-abelian algebraic theory Th in the category of topological spaces [1].

After presenting some basic properties of a torsion theory in a homological category, we shall show that a torsion theory exactly corresponds to a semi-left-exact reflection [3]. We shall then focus our attention on two main examples. The first one is given by the reflection $F\colon Grp(Top) \to Grp(Haus)$ from the topological groups $Grp(Top)$ to the Hausdorff groups $Grp(Haus)$; in this case, the torsion part of the torsion theory consists in the indiscrete topological groups. The second example is given by the reflection $F'\colon Grp(CHaus) \to Grp(Prof)$ from the compact Hausdorff groups $Grp(CHaus)$ to the profinite groups $Grp(Prof)$; here the torsion part consists in the connected compact Hausdorff groups. Both these situations can be generalized by replacing the theory Grp of groups with any other semi-abelian theory Th, as for instance the theory Rng of rings.

We shall then show that the torsion theories in a homological category always correspond to a special kind of closure operator. For the two topological examples mentioned above we shall give an explicit description of this closure operator.

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The Shape of a Category up to Directed Homotopy

Marco Grandis (University of Genova, Italy)

This work is a contribution to a recent field, Directed Algebraic Topology, which studies structures where paths and homotopies cannot generally be reversed, like 'directed spaces' in some sense - ordered topological spaces, inequilogical spaces, simplicial and cubical sets, etc.

The study of homotopy invariance is far richer and more complex than in the classical case, where homotopy equivalence between 'spaces' produces a plain equivalence of their fundamental groupoids, for which one can simply take - as a minimal model - the categorical skeleton. Our directed structures have a *fundamental category* $\uparrow\Pi_1(X)$, which must be studied up to appropriate notions of *directed* homotopy equivalence, wider than categorical equivalence.

We shall use two (dual) directed notions, which take care, respectively, of variation 'in the future' or 'from the past': *future equivalence* (a symmetric version of an adjunction, with two units) and its dual, a *past equivalence* (with two counits); and then study how to combine them. Minimal models of a category, up to these equivalences, are then proposed to better understand the 'shape' and properties of the category we are analysing, as well as of the process it represents. In the simplest case, these models are obtained as the join of the *least full reflexive* and the *least full coreflexive subcategory.*

Category Theory and Cognitive Neural Systems: A Mathematical Semantic Model

Michael J. Healy[4] (Universities of New Mexico and Washington, USA)

We apply category theory to provide a mathematical semantic model for neural networks. The model expresses the hierarchical structure of conceptual knowledge and shows how representations of this structure can form incrementally in a neural system through Hebbian synaptic weight change. The formation of new representations associated with sensory patterns through the *re-use of prior knowledge* is a key feature of the semantic model. Limits and colimits are categorical constructions which model re-use as the extraction of abstract concepts and the formation of more specific concepts based upon existing concept representations in a neural network. The constructions express the categorical notion that deriving new concept representations from existing ones requires more than mere combining: It requires a structural synthesis involving concept "blending". Another key feature of the model is *knowledge coherence*. A separate concept representation is formed in each region of a multi-region network, associated with a sensor, a multi-sensory association region, or a functional region such as for planning, decision-making or motor control. The regional representations must be unified through network interconnection pathways so that they act as a single hierarchy. Categorical structural mappings known as natural transformations show how the interconnections must be organized to express this coherence property. Pursuing the implications of these and other notions of the categorical semantic model yields design principles to aid cognitive neuroscientists in understanding the brain and to aid researchers in building cognitive systems.

This material was presented in June at the Cognitive Systems Workshop, Santa Fe, New Mexico, USA.

[4] With Thomas P. Caudell (University of New Mexico, USA).

$\mathcal{K}$-Purity and Orthogonality

Michel Hébert (American University in Cairo, Egypt)

A morphism with domain A in a locally λ-presentable category $\mathcal{C}$ is called λ-*presentable* if it is a λ-presentable object of the comma category $(A \downarrow \mathcal{C})$. We prove that the λ-presentable morphisms are precisely the pushouts of morphisms between λ-presentable objects of $\mathcal{C}$ (a problem left open in [H]). For $\mathcal{K}$ a subcategory of $\mathcal{C}$, Ad´amek and Sousa ([A-S]) call a morphism f in $\mathcal{C}$ $\mathcal{K}_\lambda$-*pure* if for every commutative square $fk = gh$ with h a $\mathcal{K}$-epi between λ-presentable objects, there exists d such that $dh = k$. Then they characterize the λ-orthogonality classes in $\mathcal{C}$ (i.e., those defined by orthogonality with respect to a set of morphisms between λ-presentable objects), as those $\mathcal{K}$ which are closed under λ-directed colimits, $\mathcal{K}_\lambda$-pure subobjects and products (or, equivalently, limits). We call f *extra* $\mathcal{K}_\lambda$-pure if for every factorization $f = gh$, if h is a λ-presentable $\mathcal{K}$-epi, then it is a retraction. For $\mathcal{K}$ empty, extra $\mathcal{K}_\lambda$-purity is equivalent to the usual λ-purity (see [H]). Using our result above, we prove that extra $\mathcal{K}_\lambda$-purity is equivalent to $\mathcal{K}_\lambda$-purity if $\mathcal{K}$ is closed under λ-directed colimits and products. Then we characterize the classes closed (in $\mathcal{C}$) under $\mathcal{K}_\lambda$-pure subobjects and products as precisely the λ_m-orthogonality classes (i.e., those defined by orthogonality with respect to a class, in general proper, of λ-presentable morphisms). We also show that those are reflexive in $\mathcal{C}$. We finally identify the orthogonality hull of any subcategory of $\mathcal{C}$, as its closure under a related operator.

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Fibrations Between Fibrations and Logic of Parametric Polymorphism

Claudio Hermida (Queen's University, Canada)

We consider fibrations in the 2-category $\mathcal{F}ib$ of fibrations as a setup to model a predicate logic on a polymorphic type theory. Using the lifting/factorisation of adjoints in $\mathcal{F}ib$ from [1], we give a structural decomposition of fibrations therein, which we apply to describe the categorical structure necessary to model the above logic. Similarly, we obtain a lifting of simple products for composite fibrations, from which we deduce that the above logic yields a fibration $(p, q) : t \to b$ between $\lambda 2$-fibrations; the structure in t expresses the so-called *logical relations* for polymorphic lambda calculus. We complete the set-up adding structural equality in order to express *relational parametricity*. We show how such fibrations in $\mathcal{F}ib$ arise by externalisation of internal ones, and present some relevant constructions.

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Factorization Systems and (Dis)Connectedness

David Holgate (University of Stellenbosch, South Africa)

This paper presents a new adjunction for the study of (pre)factorization systems and (dis)connected morphisms. Our principal motivation is to obtain such classical factorization systems as (*Monotone*, *Light*) and (*Separable*, *Purely Inseparable*) via machinery analogous to that used to study (dis)connected objects.

Such objects in a category $\mathcal{C}$ have been studied via the adjunction $r \dashv l : \mathcal{P}(\mathcal{C}) \underset{l}{\overset{r}{\rightleftarrows}} \mathcal{P}(\mathcal{C})^{op}$ where $\mathcal{P}(\mathcal{C})$ is the collection of subobject classes of $\mathcal{C}$ ordered by inclusion, and for $\mathcal{A} \subseteq \mathcal{C}$, $r(\mathcal{A}) := \{X \in \mathcal{C}$: every $f : A \to X$ is constant for $A \in \mathcal{A}\}$, and $l(\mathcal{A}) := \{X \in \mathcal{C}$: every $f : X \to A$ is constant for $A \in \mathcal{A}\}$.

Taking $f : A \to B$ to be constant in a finitely complete category $\mathcal{C}$ iff $f \times f$ factors through $\delta_B : B \to B \times B$ leads us to introduce the following relation for morphisms $f : A \to B$ and $g : C \to D$ in $\mathcal{C}$:

$f||g$ iff for any commutative square $v \circ f = g \circ u$, $u \times_v u$ factors through $\delta_g : C \to C \times_D C$.

For $\mathcal{A} \subseteq \mathrm{Mor}\mathcal{C}$, put $R(\mathcal{A}) := \{f \in \mathrm{Mor}\mathcal{C} \ : \ g||f \ \forall \ g \in \mathcal{A}\}$, and $L(\mathcal{A}) := \{f \in \mathrm{Mor}\mathcal{C} \ : \ f||g \ \forall \ g \in \mathcal{A}\}$. The resulting adjunction $R \dashv L : \mathcal{P}(\mathrm{Mor}\mathcal{C}) \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{P}(\mathrm{Mor}\mathcal{C})^{op}$ is the centre of our study.

We introduce the above, investigate the fixed points of this adjunction and the (pre)factorization systems that arise as $(L(R(\mathcal{A})) \cap \mathrm{RegEpi}\mathcal{C}, R(\mathcal{A}))$.

Some Applications of Categorical Algebra in Universal Algebraic Geometry

Yefim Katsov (Hanover College, USA)

Originating in the works of B. Plotkin, G. Baumslag and others (see, for example, B. Plotkin, *Seven lectures on the universal algebraic geometry*. Preprint, Institute of Mathematics, Hebrew University, Jerusalem (2002), arXiv:math. GM/0204245; and G. Baumslag, A. Mysnikov, and V. N. Remeslennikov, *Algebraic geometry over groups*, J. Algebra **219** (1999), 16-79), quite recently a new, fascinating area of algebra — algebraic geometry over varieties of universal algebras — has been established. As its name suggests, this area holds many surprising similarities to classical algebraic geometry. It turns out that a description of automorphisms of a category Θ^0 of finitely generated free algebras of a variety Θ of universal algebras is quite important in developing algebraic geometry over the variety Θ. In this talk, after introducing basics of universal algebraic geometry, we focus on automorphisms of the category Θ^0 of an IBN-variety Θ, where a variety Θ is said to have *IBN* (*"Invariant Basis Number"*), or to be an *IBN-variety*, if any two isomorphic finitely generated free algebras of Θ have the same rank. In particular, a ring R is an IBN-ring iff the variety of (left, or right) R-modules over R is an IBN-variety. Just to illustrate, among other interesting and important results concerning different varieties of unars, monoids, Lie algebras, *etc.* and effectively using methods of categorical algebra that are discussed in the talk, we single out a description of automorphisms of categories of finitely generated free modules that resolves Plotkin's Problem 12 in the aforementioned preprint for IBN-rings.

Theorem. *All automorphisms of a category of finitely generated free modules over an IBN-ring R are "semi-inner"* (which will be rigorously defined in the talk).

Corollary. *Let R be an artinian (or noetherian, or commutative, or PI-) ring. Then all automorphisms of a category of finitely generated free modules over R are semi-inner.*

Finally, we conclude the talk by stating some interesting open problems and directions for further investigations.

The Information Flow Framework (IFF)

Robert E. Kent (Ontologos, USA)

The IEEE P1600.1 Standard Upper Ontology (SUO) project (`http://suo.ieee.org/`), aims to specify an upper ontology[5] that will provide a structure and a set of general concepts upon which object-level domain ontologies can be constructed. These object-level domain ontologies will utilize the SUO for applications such as data interoperability, information search and retrieval, automated inferencing, and natural language processing. The Information Flow Framework (IFF) is being developed to represent the structural aspect of the SUO. The categorical approach of the IFF provides a principled framework for the modular design of object-level ontologies via a structural metatheory. The IFF takes a building blocks approach towards the development of object-level ontological structure, using insights and ideas from the theory of information flow [1] and the theory of formal concept analysis [2]. The IFF provides an important practical application for category theory. It is an experiment in foundations, which follows a bottom-up approach to the logical description. The IFF represents metalogic, and as such operates at the structural level of ontologies. In the IFF, there is a precise boundary between the metalevel and the object level. The modular architecture of the IFF consists of metalevels, namespaces and meta-ontologies. There are four metalevels, **Ur**, **Top**, **Upper** and **Lower**, with the last three corresponding to the set-theoretic distinctions between generic, large and small collections, respectively. Each metalevel services the levels below by providing a metalanguage used to declare and axiomatize those levels. Corresponding to the four metalevels are the four metalanguages: IFF-UR, IFF-TOP, IFF-UPPER and IFF-LOWER, with each metalanguage including those above it. Within each metalevel, the terminology is partitioned into namespaces, and various namespaces are collected together into meaningful composites called meta-ontologies. Currently, the IFF contains meta-ontologies representing category theory, information flow, formal concept analysis, first order logic, the semantic integration of ontologies, institutions, multitudes, simple common logic, etc. All the various meta-ontologies in the IFF are anchored to the IFF core hierarchy (IFF stack) $\mathbf{Set} \subseteq \mathbf{Cls} \subseteq \mathbf{Col} \subseteq \mathbf{Ur}$[6], where all inclusions preserve composition and identities (are functorial), the first two preserve finite limits, and the first preserves the Cartesian closed structure. These preservation properties require use of the three fundamental relations of subcollection, (function) restriction and (relation) abridgment. The IFF structure is driven largely by the IFF *categorical design principle* (a goal), which states that the axiomatization of any non-core lower metalevel namespace should be categorical (only use category-theoretic operators; but not use quantifiers and logical connectives). This talk will discuss the IFF, its metalevel architecture and its metalanguages.

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Categorical Distribution Theory; Heat Equation

Anders Kock[7] (University of Aarhus, Denmark)

Distributions are an important tool in the theory of partial differential equations. We describe how the wave- and heat equations, and the distribution theory relevant for these, fit into the context of synthetic differential geometry. For the

[5] An ontology is a formal, explicit specification of a shared conceptualization. It is an abstract model of some phenomena in the world, explicitly represented as concepts, relationships and constraints, that is machine-readable and incorporates the consensual knowledge of some community.

[6] axiomatized in the lower core (IFF-LCO), upper core (IFF-UCO), top core (IFF-TCO) and ur (IFF-UR) meta-ontologies, respectively

[7] This is joint work with G. E. Reyes.

wave equation, this can be done purely synthetically, whereas the heat equation requires some "classical" functional analysis, notably the Frölicher-Kriegl theory of Convenient Vector Spaces. In particular, we describe and utilize an embedding of the category of convenient vector spaces into certain topos models of SDG.

Categorical Transition Systems and Comprehension

Jürgen Koslowski (Technische Universität Braunschweig, Germany)

An elementary observation concerning traditional labeled transition systems suggests a notion of "graph comprehension" between graph morphisms into a given graph G and graph morphisms from G to the bicategory spn. Collecting a number of known results by Benabou, Hermida, Pavlovi´c and Street concerning ordinary categorical comprehension, for each small category B the basic adjunction between the functor category [B,set] and the slice category Cat/B of commutative triangles over B can be factored into (at least) four canonical factors. Two intermediate categories in this factorization arise from bicategories with lax functors as objects, by restricting the attention to 1-cells that are left adjoint, ie, maps. This enables us to give a precise meaning to the notion of graph comprehension. Moreover, dropping the restriction to maps forces us to deal with non-trivial 2-cells, which were absent from the classical theory of comprehension. In particular, the computer-science motivated notion of simulation may now be formulated in this context. However, a priori sensible choices of 2-cells in the various bicategories are not necessarily compatible with the biequivalences induced by graph comprehension. This explains some of the discrepancies in certain categorical formulations of simulations.

Composing PROPs

Stephen Lack (University of Western Sydney, Australia)

A PROP is a way of encoding structure borne by an object of a symmetric monoidal category. We describe a notion of *distributive law* for PROPs, based on Beck's distributive laws for monads. A distributive law between PROPs allows them to be composed, and an algebra for the composite PROP consists of a single object with an algebra structure for each of the original PROPs, subject to compatibility conditions encoded by the distributive law. One example is the PROP for bialgebras, which is a composite of the PROP for coalgebras and that for algebras. Another is the PROP for commutative separable algebras.

These ideas can also be used to give a treatment of factorization systems on categories.

This talk is based on a paper of the same name which can be obtained from the following URL.

http://www.maths.usyd.edu.au/u/stevel/papers/prop.html

Substance and Form of Cohesive Space

F. William Lawvere (University at Buffalo, USA)

Smooth, continuous, and combinatorial conceptions of cohesive space produce categories X equipped with an endofunctor π_0 with the following properties: X has disjoint sums, products, and exponentiation with the usual adjointnesses and π_0 is an idempotent monad which preserves sums and products. The functor π_0 is the most basic "counting" of a space, and the subcategory fixed under it consists of spaces which are (relatively) non-cohesive (absolutely so if X is a topos and π_0 (truth-values) = 1 as is frequently the case). Defining new homs by $[X, Y] = \pi_0(Y^X)$ we obtain the Hurewicz category $H(X)$ with a canonical homotopy-type functor. Intuitively the objects of $H(X)$ are possible forms of spaces, neglecting the substance they are made of, and the form of a particular space is the extreme inequality of parts which remains when all deforming has been done. Grassmann considered inequality as the key feature of extensivity, so I also refer to $H(X)$ as a category of extensive quality. Intensive quality, on the other hand, is concerned with the kind of substance of which a space is made, neglecting the form in the large; this is detected as the isomorphism type of the tiny portion of a space almost equal to a given point. Making Cantor's bold assumption that the inclusion of noncohesive spaces X_0 into the "Mengen" X has also a right adjoint γ_0, I propose to identify the subcategory $X_\varepsilon = \{X|\gamma_0(X) \to \pi_0(X)$ is an isomorphism$\}$ with the intensive qualities (and the connected ones among these with substances). This coincidence of the Cantor core with the Poincar/'e partition means intuitively that each connected component of X has a unique point—however, not all points need be equally bare; in important examples, X_0 may retain operations of Galois groups and X_ε may contain spectra of semi-Gorenstein algebras. Additional features are often obtainable: (extensive) $H(X)$, like X_ε, should also be a "category of qualities" in that the functor π_0 becomes both right and left adjoint to the same noncohesive inclusion—this feature results if either π_0 is continuous in the sense that it preserves also $()^S$ for S noncohesive, or X is finitely combinatorial in the sense that all homsets are finite; (intensive) the inclusion of X_ε in X should have a right adjoint or Thom core γ_ε detecting the infinitesimal substance of each space—this connected geometric morphism $X \to X_\varepsilon$ results under certain conditions, one of which is (the Nullstellensatz) that the natural morphism $\gamma_0(X) \to \pi_0(X)$ is epimorphic for all X in X. "Material" is more richly structured than substance because it responds to any environment in a definite way; thus one aim of these considerations is to investigate the above constructions in a category X of second-order differential equations.

Nerves of Algebras

Tom Leinster (University of Glasgow, UK)

The standard nerve construction shows how a category can be regarded as a simplicial set with certain properties. So, if $\mathbf{T}$ is the theory of categories then the category of $\mathbf{T}$-algebras embeds fully into a presheaf category. It turns out that this is true not only for the theory $\mathbf{T}$ of categories, but for all theories $\mathbf{T}$ of a particular kind—namely, familially representable monads on presheaf categories. I shall explain what these are and how the embedding works.

The importance of this is as follows. Almost all of the proposed definitions of n-category are of one of two types:

- 'an n-category is an algebra for a certain familially representable monad', or
- 'an n-category is a presheaf with certain properties'.

The result here shows that every definition of the former type is equivalent to a definition of the latter type.

The L^p Naturality Gap

F. E. J. Linton (Wesleyan University, USA)

The spaces $l^p(I)$ of I-indexed sequences with absolutely p^{th}-power summable entries ($1 \leq p \leq \infty$) are covariantly functorial in I for p = 1, contravariantly functorial in I for $p = \infty$, but nothing much for other p.

By the same token, the l^p-style "products" of I-tuples of Banach spaces serve as their coproducts for $p = 1$, their products for $p = \infty$, but nothing much for other p.

Similar observations pertain to the various l^p-style "cross-norms" on tensor products of Banach spaces.

Again, the various spaces $l^p(I)$ behave somewhat like the components of a graded $l^\infty(I)$-algebra, but in a not entirely satisfactory way.

The talk will fill in some of the details behind the remarks above, and, so far as they are available, behind their L^p analogues.

A Simplicial Approach to Factorization Systems and Radicals

László Márki[8] (Hungarian Academy of Sciences, Hungary)

We shed light on an unexpected connection between radicals and factorization systems. We consider the simplicial set of short exact sequences in a category with zero object, kernels, and cokernels. We show how this notion can be specialized to give, on the one hand, factorization systems in a category and, on the other hand, Kurosh–Amitsur radicals as presented in a recent paper by Janelidze and M´arki.

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Atomic Toposes of Free Algebras

Matías Menni[9] (Universidad Nacional de La Plata, Argentina)

The Schanuel topos is the category of sheaves for the atomic topology on $\mathbf{I}^{\mathrm{op}}$, the opposite of the category of finite sets and injections. It is well known that this topos is equivalent to the category of pullback-preserving presheaves in $\mathbf{Set}^{\mathbf{I}}$ [1, 3]. It is not so well-known however that the Schanuel topos is also the category of free algebras for a monad on the presheaf topos of Joyal species $\mathbf{Set}^{\mathbf{B}}$, where $\mathbf{B}$ is the groupoid of finite sets and bijections [2, 5]. Indeed, the monad is the one arising from the adjunction $\iota_! : \mathbf{Set}^{\mathbf{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathbf{I}} : \iota^*$ induced by the inclusion $\iota : \mathbf{B}^{\mathrm{op}} \to \mathbf{I}^{\mathrm{op}}$.

This work is motivated by, and generalises, both of the above observations; concentrating on a class of atomic sheaf toposes that arise in the same way as the Schanuel topos does. Naturally, one is led to wonder about monads on toposes whose Kleisli categories are in turn toposes. The problem of fully characterising this situation is open.

Recall that for a factorisation system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$, the inclusion $\iota : \mathcal{M} \to \mathcal{C}$ induces the monadic adjunction $\iota_! : \mathbf{Set}^{\mathcal{M}^{\mathrm{op}}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} : \iota^*$. The associated monad on $\mathbf{Set}^{\mathcal{M}^{\mathrm{op}}}$ is referred to as the *factorisation* monad.

We consider sites equipped with a factorisation system. A *free site* is a site for which its sheaf topos is canonically equivalent to the Kleisli category of the factorisation monad. An *$\mathcal{E}$-flat site* is a site for which every presheaf that maps pushouts along $\mathcal{E}$-maps to quasi-pullbacks is a sheaf, and vice versa. If $\mathcal{M}$ is the class of isos then the site is called *flat*. The definition of $\mathcal{E}$-flat sites is motivated by the following result.

For a category with a factorisation system, the Kleisli category of the factorisation monad embeds in the category of presheaves that map pushouts along $\mathcal{E}$-maps to quasi-pullbacks. Moreover, if the category has pushouts along $\mathcal{E}$-maps and every section is in $\mathcal{M}$ then the embedding is an equivalence if and only if the category has no proper ω-chains of $\mathcal{E}$-maps. As a corollary it is established that a flat atomic site with pushouts is free if and only if it has no proper ω-chains.

We give a series of conditions on categories that yield flat atomic sites. It follows, for instance, that for an essentially small coregular category with pullbacks and such that regular subobjects have effective unions, the opposite of the underlying subcategory of regular monos is a flat atomic site.

Examples of free and flat atomic sites are functor categories $(\mathbf{I}^{\mathrm{op}})^{\mathbb{C}}$, for $\mathbb{C}$ a category with a finite set of objects. Further examples are the opposite of free (symmetric) monoidal categories with initial unit on groupoids; the simplest concrete such category being the opposite of the category of finite linear orders and injective monotone functions [1, 4].

[8] The result has been obtained jointly with George Janelidze.

[9] With Marcelo Fiore (University of Cambridge, UK).

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Parametrized Iterativity

Stefan Milius (Technical University of Braunschweig, Germany)

In previous work we introduced for any finitary endofunctor H on a locally finitely presentable category, the concept of an iterative algebra. The aim was to generalize, extend, and simplify the description of free iterative theories given by Calvin Elgot and his coauthors. In our investigation we followed the footsteps of Evelyn Nelson who introduced iterative Σ-algebras (in Set) and simplified the description of the free iterative theory on Σ: it is the theory given by all free iterative Σ-algebras. A free iterative Σ-algebra on the set Y is the algebra of all *rational* Σ*-trees*, i. e., all those Σ-trees having up to isomorphism only finitely many different subtrees (a characterization provided by Susanna Ginali).

In the present work we generalize the concept of iterative algebras, by considering in lieu of an endofunctor of $\mathcal{A}$, a *base*, i. e., a finitary functor $X \Box A$ of two variables such that $X \Box -$ is a monad for each object X of $\mathcal{A}$, or in other words, a base is a finitary functor from $\mathcal{A}$ to the category $\mathsf{Mnd}_{fin}(\mathcal{A})$ of finitary monads on $\mathcal{A}$. An algebra on an object A is here a monadic algebra $A \Box A \to A$ of the monad $A \Box -$. Bases were introduced recently by Tarmo Uustalu. Example: the classical case of an endofunctor H gives a functor $(X, A) \mapsto HX + A$ with the obvious monad structure given by coproduct. Other, non-classical, examples are given by free monoids on X or by free monads if $\mathcal{A}$ is the category of finitary endofunctors on another locally finitely presentable category.

In this talk we will introduce iterative algebras for a base and we prove that on every object Y a free iterative algebra RY exists, and furthermore, RY can be constructed as a colimit of the diagram of all coalgebras $X \to X \Box Y$, X a finitely presentable object, and their homomorphisms in a similar way as in the classical setting of endofunctors. We also introduce the notion of a base module, i. e., a base such that the functor $S : X \mapsto X \Box X$ is a monad and the base itself is a right S-module. The category of base modules is monadic over the category of bases. Now for a given base $L : \mathcal{A} \to \mathsf{Mnd}_{fin}(\mathcal{A})$ the monad $R(-)$ of free iterative algebras of L yields by composing with L a base module $L \cdot R$, and this is the free iterative base module on L. This generalizes the classical result for endofunctors, yet despite the increased technical difficulty its proof remains conceptually much clearer and shorter than the original one given by Elgot for the special case of polynomial endofunctors on Set.

The Geometric Realization of a Simplicial Sheaf on a Space B

Susan Niefield (Union College, USA)

A topological space X over B gives rise to a locale $\mathcal{O}(X)$ over $\mathcal{O}(B)$, or equivalently, to an internal locale in the topos ShB of sheaves on B. The interpretation of concepts in the internal logic of ShB leads to an internal, and hence sheaf-theoretic, approach to the homotopy theory of X over B. A singular functor $S\colon Top/B \to Sh(B, SS)$ replaces the usual singular functor $Top \to SS$ in this context, where Top/B is the category of topological spaces over B and $Sh(B, SS)$ is the category of simplicial sheaves on B. This functor S takes a space X over B to the simplicial sheaf defined by $\mathbf{X}(U, n) = Top/B(U \times \Delta_n, X)$, where Δ_n is the standard n-simplex in Top and $U \times \Delta_n$ is a space over B via the first projection. Its left adjoint, the geometric realization functor $|\ |: Sh(B, SS) \to Top/B$, is defined as a composition of left adjoints, going through the category $Sh(B, Top)$ of Top-valued sheaves on B via the sheafification of the usual geometric realization $SS \to Top$.

Presheaves and Chu Spaces Via the Same 2-Categorical Construction

Vaughan Pratt (Stanford University, USA)

We define a 2-category of categories with a small generator and cogenerator, whose functors preserve carriers and cocarriers up to isomorphism, and show that both the presheaf categories $V^{C^{\mathrm{op}}}$ and the Chu categories $\mathsf{Chu}(V, k)$ arise as the A-final (up to a certain global automorphism of homsets) categories of its connected components. (Ordinarily $V = \mathsf{Set}$ but in any case V must have a final object.) Every such A-final category is linearly distributive: $\mathsf{Chu}(V, k)$ in the usual way, but (unexpectedly) the presheaf categories as well including V itself, via the one uniform construction of *par*.

Spans, Hammocks, and Fences

Dorette Pronk (Dalhousie University, Canada)

Given a category $\mathbf{C}$ with a subcategory W, satisfying the 2-out-of-3 property, we will discuss various localizations of $\mathbf{C}$ with respect to W and the relationships between them.

One can consider the category **Span**$(\mathbf{C}, W)$ of spans where the backward arrows are in W. A well-known example of this is the category of partial maps. It follows from the 2-out-of-3 property that the vertical arrows between spans have to be in W. So the 2-cells in **Span**$(\mathbf{C}, W)$ are hammocks of length 1 (in the terminology of Dwyer and Kan). This talk will further discuss the functors between **Span**$(\mathbf{C}, W)$ and Dywer and Kan's hammock localization $L^H(\mathbf{C}, W)$, as well as the relationship between these localizations and $\mathbf{C}[W^*]$, the 2-category obtained by freely adding right adjoints to the arrows in W, as defined by Dawson, Par´e and the author.

Groupoid Quantales

Pedro Resende (Instituto Superior Técnico, Portugal)

The purpose of this talk is to establish a correspondence between groupoids and quantales. I will show that from a localic groupoid with reasonable properties (e.g., a "semiopen" multiplication map — this includes topological groups and ´etale groupoids) a quantale is obtained, coinciding with the locale of arrows of G equipped with a multiplication determined by the multiplication of G. I will provide a complete characterization of the quantales that arise in this way — among other things they are also locales and will be called *quantal frames* —, in addition showing that in each case the groupoid can be reconstructed from its quantal frame up to isomorphism. Hence, such localic groupoids (and some of their maps) live in the (opposite of the) category *QF* of quantal frames.

As an application I will describe the construction of the "universal groupoid" of an inverse semigroup S, which is usually defined as being a groupoid whose arrows are germs of the partial homeomorphisms induced by S on a space of units determined by the idempotents of S. In terms of *QF* the construction of this groupoid becomes point-free. It is given by a left adjoint $\mathcal{G}$ from inverse semigroups to *QF*, with the space of germs of the classical construction being replaced by the locale $\mathcal{G}(S)$, and it suggests that the "universality" of the groupoid can be best understood as coming from an adjunction that relates inverse semigroups and quantales, rather than inverse semigroups and groupoids.

Baer Sums in Pointed, Strongly Protomodular and Barr-exact Categories, Level 1 and 2

Diana Rodelo (Universidade do Algarve, Portugal)

Given an internal abelian group A in a Barr-exact category $\mathcal{E}$, we describe a method of realizing the cohomology groups $H^1(\mathcal{E}, A)$ and $H^2(\mathcal{E}, A)$ in terms of abelian objects (i.e autonomous Maltsev operations) and obtain a nine-term exact sequence. The process developed at level two is based on that of level one, suggesting the presentation of higher order cohomology groups and a long exact sequence by iteration.

Having in mind the interpretation of cohomology groups in the sense of Eilenberg-Maclane, we consider the particular case of a slice category $\mathcal{E}/Q$, where $\mathcal{E}$ is a pointed, strongly protomodular and Barr-exact category. Using Baer extension nomenclature, we show how to define the Baer sum of equivalence classes of extensions and the Baer sum of the equivalence classes of two-fold extensions in this context.

Split Structures

Robert Rosebrugh[10] (Mount Allison University, Canada)

In the early 1990's we proved that the full subcategory of sup-lattices determined by the constructively completely distributive (CCD) lattices is equivalent to the idempotent splitting completion of the bicategory of sets and relations. The result was extremely useful in that it had many corollaries. Much earlier, Raney had proved some of our results, but without mention of categories, for completely distributive lattices. Almost from the outset, we felt that our theorem was but a small specialization of a very general result. However, until recently even a precise statement, let alone a proof, had eluded us. We have now found both in a context that we had completely overlooked.

Let D be a monad on a category $\mathcal{C}$ in which idempotents split. Write $\mathbf{k}ar(\mathcal{C}_D)$ for the idempotent splitting completion of the Kleisli category. Write $\mathbf{s}pl(\mathcal{C}^D)$ for the category whose objects are pairs $((L,s),t)$, where (L,s) is an object of $\mathcal{C}^D$, the Eilenberg-Moore category, and $t : (L,s) \to (DL, mL)$ is a *homomorphism* that splits $s : (DL, mL) \to (L,s)$, with $\mathbf{s}pl(\mathcal{C}^D)(((L,s),t),((L',s'),t')) = \mathcal{C}^D((L,s)(L',s'))$.

Our main result is that $\mathbf{k}ar(\mathcal{C}_D) \cong \mathbf{s}pl(C^D)$. We also show that this implies the CCD lattice characterization theorem and time permitting, consider a more general context.

Accessible Quotients and Homotopy

Jiří Rosický (Masaryk University, Czech Republic)

The homotopy category Ho($\mathcal{K}$) of a model category $\mathcal{K}$ appears as the category of fractions $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ inverting weak equivalences of $\mathcal{K}$. Up to equivalence, this is the composition $P : \mathcal{K} \xrightarrow{R} \mathcal{K}_{cf} \xrightarrow{Q} \mathcal{K}_{cf}/\sim$ where $\mathcal{K}_{cf}$ is the full subcategory of $\mathcal{K}$ consisting of objects which are both fibrant and cofibrant, R is a fibrant-and-cofibrant replacement functor and Q is the quotient with respect to a homotopy. The later means that Q is a coequifier of γ_1 and γ_2 where $\gamma_1, \gamma_2 : \mathrm{Id} \to C$ where $C : \mathcal{K}_{cf} \to \mathcal{K}_{cf}$ is a cylinder functor. We say that $\mathcal{K}$ is *strongly combinatorial* if it is locally presentable, cofibrantly generated and both cofibrations and trivial cofibrations are closed in $\mathcal{K}^{\to}$ under λ-filtered colimits for some regular cardinal λ. The most of homotopy categories, including the classical homotopy category (of simplicial sets) and the stable homotopy category of spectra, arise from strongly combinatorial model categories. In this case, $\mathcal{K}_{cf}$ is an accessible category and R and C are accessible functors. Hence Ho($\mathcal{K}$) is a colimit of accessible categories. One cannot expect that Ho($\mathcal{K}$) is accessible because, following an old result of P. Freyd, the classical homotopy category is not even concrete.

[10]This is joint work with R. J. Wood.

We will show that injectivity classes in $\mathrm{Ho}(\mathcal{K})$, where $\mathcal{K}$ is strongly combinatorial, behave like injectivity classes in locally presentable categories. For example, $\mathcal{L} \subseteq \mathcal{K}$ is a small-injectivity class iff it is closed under products, λ-pure subobjects and λ-filtered colimits for some regular cardinal λ where the last concept has to be suitably defined. Sometimes, $\mathrm{Ho}(\mathcal{K})$ is even *weakly λ-accessible* in the sense that the canonical functor $E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ is bijective on objects (up to isomorphism) and full on morphisms; here Ind_λ is a free completion under λ-filtered colimits and $\mathcal{K}_\lambda$ consists of λ-presentable objects in $\mathcal{K}$. This is well-known to happen in the stable homotopy category of spectra for $\lambda = \omega$ and is called the Brown property in the axiomatic stable homotopy theory. We will show that this happens in the classical homotopy category for all λ. If $\mathrm{Ho}(\mathcal{K})$ is weakly λ-accessible then the third condition in the characterization of small-injectivity classes can be replaced by the closedness under weak λ-filtered colimits (coming out from E_λ). Let us add that, under set-theoretical Vopěnka's principle, this third condition can be omitted in any strongly combinatorial model category $\mathcal{K}$ at all.

Parameterized Accessibility

Vincent Schmitt[11] (University of Leicester, UK)

This is some on-going work with Max Kelly. One can define notions of P-flatness and Q-accessibility where the parameters P resp. Q stand for families of indexes of limits (resp. colimits) in the sense of Borceux-Kelly. The following points hold. Fixing a family P of indexes and Q denoting the family of P-flat indexes:
- For a small category A, a presheaf on A is P-flat if and only if it is a Q-colimit of representables.
- The full subcategory of the category of presheaves on a small A generated by P-flat preshaves is the free Q-cocompletion of A. Conversely Q-accessible categories are exactly categories of P-flat presheaves on small A's.

This correspondence yields meaningful internal descriptions of free-Q-cocomplete objects for suitable families Q. In particular Lawvere's Cauchy-complete categories may be retrieved as full subcategories of P_0-flat functors for P_0 the family of ALL indexes. Remarkably this theory of accessibility lifts to the enriched setting. Other completions by means of P-flat presheaf categories are investigated. One can obtain this way completions of metric spaces by means of "asymmetric" Cauchy filters. One also retrieves the so-called "algebraic" completion of partial orders.

I acknowledge Francis Borceux that suggested me a major improvement after a presentation of this work at the workshop in Amiens in Nov. 2003. The following works are related to the subject: [1], [2], [3], [4], [5], [6], [7], [8], [9].

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On Partial Morphisms in Categories of Lax Algebras

Christoph Schubert (University of Bremen, Germany)

Lax algebras (in the sense of [1], [2]) facilitate the simultaneous study of ordered, topological, uniform, and approach-structures as well as labelled transition systems.

We will present some results on the structure of extremal partial morphism classifiers in categories of (reflexive, unitary) lax algebras and strict extremal partial morphism classifiers in categories of reflexive and transitive lax algebras, generalising the corresponding results for the category of topological spaces.

This work is part of the author's Ph. D. thesis supervised by Prof. H.-E. Porst.

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Topology, Categories, and Lambda-Calculus

Dana S. Scott (Carnegie Mellon University, USA)

Many categories have injective objects, but their properties depend on what families of subobjects are allowed. In the case of topological T_0-spaces, for example, two alternatives suggest themselves: (1) arbitrary subspaces, and (2) dense subspaces. Both notions are interesting. Thus, a space D is injective in sense (1) iff for any space Y and subspace X and any continuous function $f : X \to D$ there is a continuous extension $f' : Y \to D$ of f. Injective spaces are easily proved to be closed under arbitrary products and continuous retracts, which facts provide many examples once a few such spaces are known. Perhaps it is not so obvious, however, that injective spaces are also closed under the formation of function spaces, once the space of continuous functions is given the right topology; indeed the category of injective spaces and continuous functions is a cartesian closed category. The talk will review old and new results about injective spaces, applications of the results, and a recent use of injectives to define a cartesian closed extension (called EQU) of the category of all T_0-spaces. This topological point of view makes it obvious that (untyped) lambda-calculus models exist in many forms. But the (typed) lambda-calculus can be related nicely to topology (M. Escardo) and EQU has a useful homology and homotopy theory (M. Grandis).

Inner and Outer Adjoints in Polarized Categories

R. A. G. Seely (McGill University, Canada)

Motivated by an analysis of Abramsky-Jagadeesan games, we consider a categorical semantics for a polarized notion of two-player games, a semantics which has close connections with the logic of (finite cartesian) sums and products, as well as with the multiplicative structure of linear logic. In each case, the structure is polarized, in the sense that it is modelled by two categories, one for each of two polarities, with a module structure connecting them.

In this talk we shall emphasise one aspect of this work, a new notion of adjunction which we call "inner" and "outer" adjoints. This is an abstraction of the polarized sums and products which arise from the two-player games.

This is joint work with Robin Cockett.

On Morita Equivalence of Algebraic Theories

Lurdes Sousa[12] (School of Technology of Viseu, Portugal)

We give a characterization of Morita equivalence of many-sorted algebraic theories, i.e., Lawvere theories over a power of Set. Following Dukarm [2] two theories are called Morita equivalent if their categories of algebras are equivalent. We present a construction which, starting from a theory $\mathcal{T}$, presents a Morita equivalent theory $\mathcal{T}'$; and we prove that every theory Morita equivalent to $\mathcal{T}$ is, as a category, equivalent to some such a $\mathcal{T}'$. This generalizes the procedure proposed for one-sorted theories in [2]. In the classical case studied by Morita [4], the one-sorted theories of R-modules are precisely the theories of R'-modules for equivalent rings R and R'. And we show the relationship of our result to the results on equivalence of monoids obtained in [1]-[3]. Finally, we describe all many-sorted theories of the category Set.

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Two Dimensional Algebra:
Can it Help us with (Weak) n-dimensional Categories?

Art Stone[13] (Vancouver, Canada)

While the (strict) Eilenberg-Moore construction can take us from n-categories to other n-categories, it cannot take us from n-categories up to $(n+1)$-categories. One way around the problem is to use what we'd like to call *two-dimensional* adjunctions; they can take us up to *categories* of n-categories, with their expected $(n+1)$-category structure.

In the past these *two-dimensional* adjunctions have also been called (by John Gray) *quasi* adjunctions, (by us) *soft* adjunctions and (by Ross Street and others) *lax* adjunctions. But for the current purpose the old terminologies don't quite work, and neither do many standard notations.

These two-dimensional adjunctions might be thought of as having adjunction-structure in two directions, *horizontal* and *vertical*.

We'll recall their origins, definitions and central properties — how their associated *two-dimensional simplicial objects* are replete with natural monads and co-monads, which are connected by (four kinds of) natural distributive laws, and how the proofs are simplified when we turn to $2 \times 2 \times 2$ stackings of *cubic hexagons*.

[12] This is joint work with Jiří Adámek and Manuela Sobral.

[13] Joint work with John MacDonald (UBC, Canada).

When we look at this way of producing

categories of *strict* n-categories,

we get examples of two-dimensional adjunctions that are somewhat degenerate. But when we push on to

categories of *weak* n-categories,

degeneracies disappear.

Orders and Ideals over a Base Quantaloid

Isar Stubbe (Louvain-la-Neuve, Belgium)

The topos $\mathsf{Sh}(\Omega)$ of sheaves on a locale Ω can be described in terms of Ω-sets. The ordered objects in $\mathsf{Sh}(\Omega)$ admit a similar elementary description in terms of "Ω-orders" [1]. When rewritten in a suitable manner, the latter axioms still make sense for a base quantale or even quantaloid $\mathcal{Q}$: they say that a "$\mathcal{Q}$-order" is a Cauchy complete $\mathcal{Q}$-enriched category-without-units. $\mathcal{Q}$-orders are the objects of a quantaloid $\mathsf{Idl}(\mathcal{Q})$ whose morphisms are ideals (order-compatible relations); and $\mathsf{Ord}(\mathcal{Q}) = \mathsf{Map}(\mathsf{Idl}(\mathcal{Q}))$ is the locally ordered category of $\mathcal{Q}$-orders and order-preserving morphisms. In particular is $\mathsf{Ord}(\Omega)$ precisely the category of ordered objects in $\mathsf{Sh}(\Omega)$ as put forward in [1]. An equivalent description of $\mathcal{Q}$-orders avoids the use of enriched categories-without-units: denote $\mathcal{Q}_{\mathsf{cc}}$ for the split-idempotent completion of $\mathcal{Q}$, then $\mathsf{Idl}(\mathcal{Q}) = \mathsf{Dist}(\mathcal{Q}_{\mathsf{cc}})$, the latter being the quantaloid of distributors between $\mathcal{Q}_{\mathsf{cc}}$-enriched categories. Applied to a locale Ω, this shows the consistency of the elementary description of Ω-orders with the work of [2].

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Topology and Small Maps

Javad Tavakoli (Okanagan University College, Canada)

In this talk we show that for a given topology in an elementary topos $\mathcal{E}$, the class of almost epi maps and the class of almost mono maps satisfy the axioms for open maps. Indeed, in a topos $\mathcal{E}$ with (IC), the class of almost epi maps satisfies the axioms for small maps in the Joyal-Moerdijk sense. Moreover, we prove some important properties of a topology in $\mathcal{E}$. For instance, we show that Ω_j is closed in Ω. Finally we will provide a necessary and sufficient condition for a map being almost mono.

On the Duality between Open and Closed

Paul Taylor (University of Manchester, UK)

`www.cs.man.ac.uk/~pt/ASD`

The principal aim of Abstract Stone Duality was to reformulate General Topology in such a way that the connection with Recursion Theory would be precise. However, this has paid a conceptual bonus, in the form of a much closer lattice duality between

finite intersection	finite union
open subspaces	closed subspaces
open maps	proper maps
discrete spaces	Hausdorff spaces
overt spaces	compact spaces
existential quantifier	universal quantifier

From both points of view the problem with the traditional axiomatisation of spaces or locales lies with infinitary joins, in particular the *directed* ones. ASD solved this by treating the lattice of open sets as a *topological* algebra, axiomatising *continuous* functions directly. Turning the idea of the Scott topology on its head (largely) makes the directed joins disappear.

In the new calculus, membership $x \in U$ of an open set is replaced by λ-application ϕx, and containment $K \subset U$ of a compact subspace by $A\phi$, where A is second order. (It was already apparent in the literature on function-spaces c1970 that $U \subset K$ and not $x \in K$, was the defining feature of a compact subspace; continuous lattices developed this idea, and ASD goes a step further.)

I will concentrate on the numerous topological interpretations of the "basis decompositions"

$$\phi x \ = \ \exists n. A_n \phi \wedge \beta^n x \ = \ \forall k. C_k \phi \vee \delta^k x$$

where k ranges over a compact Hausdorff space and n an overt discrete one. For example, C_k may represent an overt closed subspace with a dense sequence of points, providing yet another example of the need to replace the word "open".

With this principle, the lattice duality in the axiomatisation is apparently broken only by the fixed point axiom.

A Model Structure for Homotopy of Internal Categories

Tim Van der Linden[14] (Vrije Universiteit, Belgium)

It is well known that the category Cat of all small categories and functors between them may be equipped with the following "folk" Quillen model category structure: the weak equivalences (we) are the equivalences of categories; the fibrations (fib) are the star-surjective functors; the cofibrations (cof) are the functors which are injective on objects. The aim of this talk is to describe an analogous structure on the category $\mathsf{Cat}(\mathcal{C})$ of categories and functors in a given finitely complete category $\mathcal{C}$. In case $\mathcal{C}$ is the category Set of (small) sets and functions, we regain the folk structure on $\mathsf{Cat} = \mathsf{Cat}(\mathsf{Set})$.

We define a cocylinder on $\mathsf{Cat}(\mathcal{C})$ such that any two internal functors are naturally equivalent if and only if they are homotopic with respect to this cocylinder. We set we the class of homotopy equivalences, fib the class of internal functors with the homotopy lifting property, and cof the class of functors with the homotopy extension property. It turns out that the trivial fibrations (resp. cofibrations) are exactly the split epimorphic (resp. monomorphic) weak equivalences. Any cofibration is split monomorphic on objects, and fibrations may be characterized by a notion of star-surjectivity.

Descent Theory, Distributive Laws and Yang-Baxter Equation

Enrico M. Vitale[15] (Université Catholique de Louvain, Belgium)

Let $f\colon R \to S$ be a morphism of commutative unital rings, and consider the induced functor $f!\colon R$-mod $\to S$-mod defined by $f!(N) = N \otimes_R S$. The descent problem for f consists in recognizing when an S-module is of the form $f!(N)$ for some R-module N. A classical theorem asserts that, if f is faithfully flat, then an S-module is of the form

[14] This is joint work with Tomas Everaert and Rudger Kieboom.
[15] This is a work with S. Kasangian and S. Lack.

$f!(N)$ if and only if it is equipped with a $\mathbb{T}^*$-coalgebra structure, where $\mathbb{T}^*$ is the comonad on S-mod induced by the adjunction

$$R\text{-mod} \underset{f^*}{\overset{f!}{\rightleftarrows}} S\text{-mod} \qquad f! \dashv f^*$$

For this reason, a $\mathbb{T}^*$-coalgebra structure on an S-module M can be called a descent datum for M. It is also well-known that there are several equivalent ways of describing what a descent datum for M is, and a natural problem is to rise to a categorical level these others descriptions of descent data.

We replace the previous adjunction $f! \dashv f^*$ by the more general situation

$$\mathbb{C} \underset{R^{\mathbb{T}}}{\overset{L^{\mathbb{T}}}{\rightleftarrows}} \mathrm{Alg}(\mathbb{T}) \qquad L^{\mathbb{T}} \dashv R^{\mathbb{T}}$$

where $\mathbb{T}$ is a monad over an arbitrary category $\mathbb{C}$. We prove that, if the monad $\mathbb{T}$ is equipped with an invertible distributive law $K\colon T^2 \Rightarrow T^2$ satisfying the Yang-Baxter equation, then to give a descent datum on a $\mathbb{T}$-algebra X is equivalent to giving a K-braiding on X, that is a morphism $TX \to TX$ satisfying some suitable conditions.

The main example, which is a direct generalization of the classical case of commutative rings, is given by the monad $T = -\otimes S\colon \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ is a braided monoidal category and S is a (not necessarily commutative) monoid in $\mathbb{C}$; the distributive law is provided by the braiding $S \otimes S \to S \otimes S$. Another example comes from the theory of bialgebras.

Symmetric Separable Algebras in Monoidal Categories and Cospan (Graph)

Robert F. C. Walters[16] (University of Insubria, Italy)

A symmetric separable algebra object in a strict monoidal category has symmetric monoid and comonoid structures (with symmetry satisfying Yang-Baxter) and linked by the Frobenius [3] and one other axiom (U) [2]. We prove the following:

1)The free strict monoidal category with a symmetric separable algebra object is the category of cospans of finite sets.

2) The free strict monoidal category with a symmetric separable algebra object carrying an action is the category of cospans of finite graphs restricted to discrete objects (finite sets).

The second result provides a foundation for applications to concurrency in theoretical computer science including work by the authors, Gadducci and Heckel, and others. The proofs of these results use a presentation of morphisms as normal forms. Our results extend the work of Grandis who proved that the free strict monoidal category with a symmetric monoid is the category of finite sets [4] and is related to the work of Abrams [1] on 2-TQFT's and Frobenius algebras.

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[16]This is joint work with Robert Rosebrugh and Nicoletta Sabadini

Logical Aspects of the Ideal Completion of a Heyting Pretopos

Michael A. Warren (Carnegie Mellon University, USA)

The category theoretic study of set theory was initiated by Joyal and Moerdijk [3] and has recently been extended by Awodey et al. [1]. In this talk I show that the methods of Awodey et al. [1] may be modified to treat not only impredicative theories such as those considered in [3] and [1], but also predicative theories. Specifically, I show that every Heyting pretopos $\mathcal{E}$ may be embedded in a subcategory $\mathbf{Idl}(\mathcal{E})$ of sheaves on $\mathcal{E}$, called the *ideal completion* of $\mathcal{E}$, that is a model of a suitable predicative set theory. Moreover, if $\mathcal{E}$ is also locally cartesian closed, then the exponentiation axiom is also satisfied (cf. [2]). If time permits I will discuss additional recent results in the category theoretic study of predicative set theories.

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A Factorization of Regularity

R. J. Wood[17] (Dalhousie University, Canada)

Earlier, the authors showed that categories with regular factorizations (in the sense of Kelly) and with regular epimorphisms closed under composition form $\mathbf{cat_{ker}}^{\mathcal{R}}$ the 2-category of algebras for a KZ-doctrine $\mathcal{R}$ on the 2-category $\mathbf{cat_{ker}}$, of all categories and functors which preserve kernel inclusions. Here we show that the usual 2-category $\mathbf{lex}$, of categories with finite limits and left exact functors, is $\mathbf{cat_{ker}}^{\mathcal{L}}$ for a coKZ-doctrine $\mathcal{L}$. Combined with our earlier results this shows that the 2-category $\mathbf{reg}$, of regular categories and regular functors, is the 2-category of algebras for a distributive law $\mathcal{L}\mathcal{R} \longrightarrow \mathcal{R}\mathcal{L}$ over $\mathbf{cat_{ker}}$.

Internal Monotone-Light Factorization for Categories via Preorders

João J. Xarez (University of Aveiro, Portugal)

The classical monotone-light factorization of S. Eilenberg and G. T. Whyburn is a special case of the categorical one, as studied by A. Carboni, G. Janelidze, G. M. Kelly, and R. Par´e [1], when one considers the reflection **CompHaus** → **Stone**, between compact Hausdorff and Stone spaces.

We had showed in [3] that monotone-light factorization also exists for the reflection **Cat** → **Preord**, between categories and preordered sets.

We now show that, for a finitely-complete category **C** with coequalizers of kernel pairs, there may exist a reflection **Cat(C)** → **Preord(C)**, between categories and preorders in **C**, and an associated monotone-light factorization. Hence, the example **Cat** → **Preord** has been internalized, becoming the special case **C** = **Set**.

[17] This is joint work with Claudia Centazzo.

In fact, if every (to be defined) product-regular epi in **C** is also stably-regular, then there exist reflections **(R)Grphs(C)** → **(R)Rel(C)**, from (reflexive) graphs into (reflexive) relations in **C**, and **Cat(C)** → **Preord(C)**, from categories into preorders in **C**.

Furthermore, such a sucient condition ensures as well that these reflections do have stable units (in the sense of C. Cassidy, M. H´ebert, and G. M. Kelly [2]), and so that they are examples of categorical Galois theory. This stable units property is also shown to be equivalent to the existence of a monotone-light factorization, provided there are sufficiently many effective descent morphisms with domain in the respective full subcategory (as it happens with **CompHaus** → **Stone**, **Cat** → **Preord** and **(R)Grphs(C)** → **(R)Rel(C)**.

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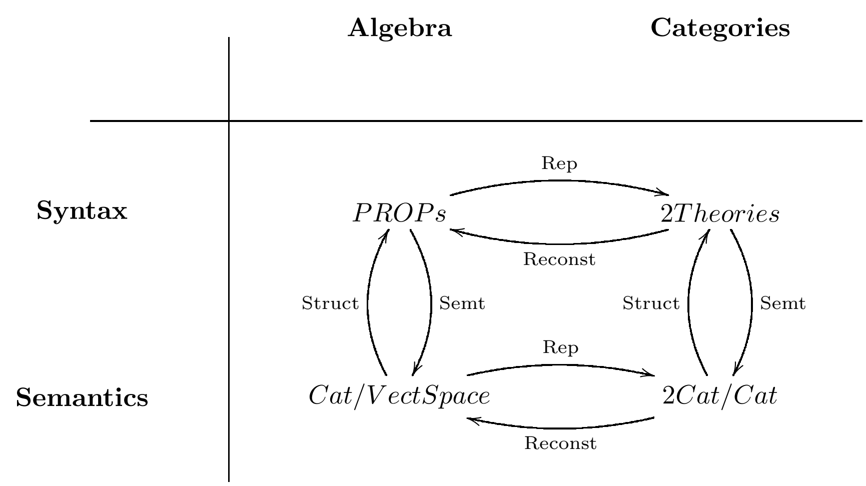
The Syntax and Semantics of Tannaka Duality

Noson S. Yanofsky (Brooklyn College, City University of New York, USA)

Certain types of algebras are mirrored by certain types of categories. For example the modules for a Drinfeld algebras has the structure of a monoidal category. The modules for a Quasi-triangular Drinfeld algebra form a braided monoidal category. Algebras with involutions are mirrored by categories with closed structures. We shall give a functorial way of going from a description of a type of algebra to a description of a type of category. There is also a functorial way to reverse this process. Algebras with extra structures are described by PROPs. Categories with extra structure are described by 2Theories (which are 2-categorical generalizations of algebraic theories.) Our aim is to present a dictionary between quantum group theory and category theory.

We shall "flesh out" the following "big" picture.



The downward maps take descriptions of structures to their (2-)categories of models. The upward maps take categories and their forgetful functors to the structures that describes them. The left-to-right maps take algebraic structures to their categories of representations. The right-to-left maps are reconstructions from categories to their algebraic structures. We shall look at each of the four corners of the above diagram in detail. By looking at subcategories, quotient categories and variations of each of the four corners we shall get different relationships with the other corners. There will be simple functors, quasi-adjunctions, adjunctions, quasi-equivalences and equivalences. By doing so, we will get (weak) universal properties of Tannaka duality.

Algebraic Systems as Monoidal Functor Categories

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An algebraic system consists of sets with operations satisfying some relations. This is broad enough to include the usual systems such as groups, modules, Hopf algebras, group representations, etc. Any algebraic structure can be regarded as a strict small monoidal category generated by a graph with some relations between arrows and an algebraic system as a category is equivalent to the monoidal functor category from such strict small monoidal category to any underlying monoidal category such as (**Set**, $\times$, $*$) and $(\mathbf{hTop}_*, \wedge, *)$.

Using this identification, Kan extensions in the 2-category of monoidal categories can yield information about adjunctions of functors between different algebraic systems, such as the existence of left adjoints of some forgetful functors. I shall also discuss interchanging algebraic structures in a fixed underlying monoidal category.