Generalized triangle inequalities with applications to algebraic groups

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This is a survey of my joint work with Bernhard Leeb and John Millson.

Everybody knows how to construct triangles with the prescribed side-lengths α_1, α_2, α_3 in the Euclidean plane: The necessary and sufficient conditions for this are the usual triangle inequalities $\alpha_i \le \alpha_j + \alpha_k$. In this talk I will explain how to solve (in a unified fashion) the analogous problem for other geometries X: nonpositively curved symmetric spaces (and their infinitesimal analogues) and Euclidean buildings. The notion of "side-length" in this generality becomes more subtle: *Side-lengths* are elements of the appropriate Weyl chamber Δ. One of the suprising results is that the "generalized triangle inequalities" for X determine a polyhedral cone $D(X) \subset \Delta^3$, which depends on X and on the type of geometry only weakly: $D(X)$ is completely determined by the finite Coxeter group corresponding to X. The linear inequalities describing $D(X)$ are determined by the "Schubert calculus" (computing the integer cohomology ring) for the associated generalized generalized Grassmanians. Our techniques for proving these results about $D(X)$ are mostly geometric (with a bit of dynamics): By relating triangles with weighted configurations "at infinity".

I then talk about application of these results to several problems in the algebraic group theory: Decomposing tensor products of representations of complex reductive Lie groups, computing structure constants of Hecke rings and generalizations of the Weyl's and Thompson's problems on eigenvalues (resp. singular values) of sums (resp. products) of $n \times n$ matrices.