Chapter 1

Introductory Remarks on PDEs

1.1 First Order Equations

Let us begin at the beginning. Although they can be arbitrarily complicated, solutions to differential equations are based on a number of simple, basic principles. The very first of these is the single variable equation

$$u'(x) \equiv 0. \tag{1.1}$$

Of course the solution to this is just

$$u(x) \equiv C \tag{1.2}$$

where is a constant (i.e., independent of x).

This may appear to be very simple (and it is), but many more complicated equations are solved by reducing precisely to this case. For example, to solve

$$u'(x) \equiv 1,$$

we rewrite as

$$\frac{d}{dx}(u(x) - x) \equiv 0$$

and hence by (1.1)

$$u(x) - x \equiv C$$

or, in other words,

$$u(x) \equiv x + C.$$

This is just the most simple of examples – what's important is to understand the *philosophy* of reducing seemingly complicated equations to simpler ones!

Now suppose that $u = u(x, y)$ is a bivariate function and we wish to solve the most basic of *partial* differential equations:

$$\frac{\partial u}{\partial x}(x, y) \equiv 0. \tag{1.3}$$

What is the solution? Holding y fixed, (1.3) is just (1.1) in disguise, and so the solution is given by (1.2), i.e.,

$$u(x, y) \equiv C$$

where C is independent of x. The difference is that C independent of x means something slightly different: in this context, functions depend on x *and* y, and so if one is independent of x it remains a function of y! Hence, the solution of (1.3) is

$$u(x, y) = C(y) \tag{1.4}$$

where $C(y)$ is an arbitrary function of y.

Similarly, the solution of the trivariate equation

$$\frac{\partial u}{\partial y}(x, y, z) = 0$$

is given by

$$u(x, y, z) = C(x, z)$$

for some arbitrary function $C(x, z)$ (of x and z), and so on.

Now, what extra conditions determine what this $C(y)$ of (1.4) is exactly? Substituting e.g. $x = 0$ into (1.4) gives

$$C(y) = u(0, y) =: u_0(y)$$

so that $C(y)$ is given by the "initial function" $u_0(y)$. (Recall that in the univariate case, $u'(x) = 0$, the solution is determined by the initial *value* $u_0 := u(0) \in \mathbb{R}$.) Alternatively, we may specify an initial function on other lines (or even curves). For example, suppose we specify that $u(x, y) = x^2$ on the line $y = 2x$. Then, again substituting into (1.4) gives (noting that $x = y/2$)

$$C(y) = u(y/2, y) = (y/2)^2.$$

However, there is one important exception. We *cannot* specify an arbitrary initial function on lines of the form $y = k$. Why not? Let's see what happens if we do try to specify $u(x, y) = f(x)$ on the line $y = k$. Then, substituting into (1.4) yields

$$C(k) = f(x),$$

but this is clearly not possible for constant k! These exceptional lines (curves) are called the *characteristics* of the equation. We will return to them shortly.

Now for the next level of complication. Here is a simple physical problem. Suppose you had a fluid, say oil, flowing down a cylindrical pipe aligned along the x-axis, at a constant speed c. For simplicity, we assume that the density of the oil, u, depends only on the location x and time t, so that $u = u(x,t)$. What can be said about this density function $u(x,t)$? Well, since $u(x,t)$ is a density, we can use it to compute the mass of the oil in any section of the pipe, say for $a \le x \le b$. We do this by first finding the differential of the mass M, (or in other words, the mass of the infinitesimal slice between x and $x + dx$). Specifically,

$$\begin{aligned} dM &= \text{density} \times \text{cross-sectional area} \times \text{thickness} \\ &= u(x,t) \times A \times dx \\ &= Au(x,y)dx \end{aligned}$$

where A is the (constant) cross-sectional area of the pipe. To find the total mass for $a \le x \le b$, we "add":

$$M(t;a,b) = \int_{x=a}^{x=b} dM = \int_{x=a}^{x=b} Au(x,t)dx = A\int_{x=a}^{x=b} u(x,t)dx. \tag{1.5}$$

This is the mass at time t. At a slightly later time, $t + h$, the oil, since it is flowing at a constant speed c, has moved to the right by $c((t+h)-t) = ch$, i.e., the oil from $a \le x \le b$ has moved to $a+ch \le x \le b+ch$. Since the oil is neither created or destroyed (hopefully!) we must have

$$M(t+h; a+ch, b+ch) = M(t;a,b),$$

i.e.,

$$A\int_{x=a+ch}^{x=b+ch} u(x,t+h)dx = A\int_{x=a}^{x=b} u(x,t)dx.$$

Cancelling the A results in

$$\int_{x=a+ch}^{x=b+ch} u(x,t+h)dx = \int_{x=a}^{x=b} u(x,t)dx. \tag{1.6}$$

This is a prefectly valid *integral* equation – it is just a bit awkward to work with, and so we simplify. Since (1.6) holds for all b, we may differentiate with respect to b to obtain (by the Fundamental Theorem of Calculus; assuming that $u(x,t)$ is sufficiently regular),

$$u(b+ch, t+h) = u(b,t), \tag{1.7}$$

a *difference* equation. But, since (1.7) must hold for all h, we again differentiate, this time with respect to h, to obtain (by the Chain Rule),

$$\frac{\partial u}{\partial x}(b+ch,t+h)\frac{\partial}{\partial h}(b+ch) + \frac{\partial u}{\partial t}(b+ch,t+h)\frac{\partial}{\partial h}(t+h) = \frac{\partial}{\partial h}u(b,t) = 0,$$

i.e.,

$$c\frac{\partial u}{\partial x}(b+ch,t+h) + \frac{\partial u}{\partial t}(b+ch,t+h) = 0.$$

Now put $h = 0$ to obtain

$$c\frac{\partial u}{\partial x}(b,t) + \frac{\partial u}{\partial t}(b,t) = 0. \tag{1.8}$$

One last thing. Since (1.8) holds for all b we may write, for emphasis,

$$c\frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) = 0,$$

or, dropping the functional dependencies,

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0. \tag{1.9}$$

Equation (1.9) is called the Transport Equation, and we will now proceed to solve it. The strategy is just to recognize it as a disguised form of (1.3) (and hence of (1.1)!). The key observation is that $c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$ is really the directional derivative of u in the direction $\langle c, 1\rangle$, i.e.,

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = D_{\langle c,1\rangle}u$$

and hence (1.9) is the same as

$$D_{\langle c,1\rangle}u \equiv 0. \tag{1.10}$$

Note that in terms of directional derivatives, the model equation (1.3), $\frac{\partial u}{\partial x} = 0$, is just $D_{\langle 1,0\rangle}u = 0$. Since the equations are essentially the same, the solutions are the same. Equation (1.10) means that u should not depend on the direction $\langle c, 1\rangle$. But every function depends on the directions $\langle c, 1\rangle$ and its perpendicular direction $\langle c, 1\rangle^{\perp} = \langle 1, -c\rangle$ and so u has to depend only on $\langle c, 1\rangle^{\perp}$. Specifically,

$$\begin{aligned} u(x,t) &= C(\langle x,t\rangle \cdot \langle c,1\rangle^{\perp}) \\ &= C(\langle x,t\rangle \cdot \langle 1,-c\rangle) \\ &= C(x-ct), \end{aligned}$$

for some arbitrary univariate function C. Actually, to emphasize that C is a function we usually write

$$u(x-ct) = f(x-ct) \tag{1.11}$$

where f is an arbitrary (differentiable) univariate function.

If we set $t = 0$ in (1.11), then we see that

$$u(x,0) = f(x)$$

or, in other words, $f(x) = u_0(x) := u(x,0)$, so that

$$u(x,t) = u_0(x-ct). \tag{1.12}$$

There is another way of thinking about solving such equations which can be applied to non-constant coefficient equations. Specifically

$$D_{\langle c,1\rangle} u \equiv 0$$

can be interpretted to mean that u doesn't change in the direction $\langle c, 1\rangle$. If we evolve a point (x_0, t_0) back and forth along this direction then we generate a line through (x_0, t_0) with direction (tangent) vector $\langle c, 1\rangle$. The normal vector is then $\langle 1, -c\rangle$ and the equation of the line is

$$x - ct = (x_0 - ct_0) := d.$$

Hence $u(x, t)$ is constant along each of these lines $x - ct = d$ and the value of $u(x, t)$ depends only on which of these lines the point (x, t) happens to lie. How do we distinguish the lines one from the other? By the value of d of course! So then

$$u(x, t) = \text{“}f(d)\text{”}.$$

But $d = x - ct$, and so

$$u(x, t) = f(x - ct),$$

as before.

The advantage of this is that we can apply this reasoning to more general equations. Consider, for example, the equation

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

We begin again by recognizing this as a directional derivative:

$$D_{\langle 1,y\rangle} u = 0.$$

However, in this case the direction $\langle 1, y\rangle$ is not constant and so if we evolve a point continually in this direction we will generate a curve, not just a straight line. Which curve? It's tangent vector is $\langle 1, y\rangle$ and hence the slope of the tangent line is

$$\frac{dy}{dx} = \frac{y}{1}.$$

The solution to this first order ODE is $y = Ce^x$ for some constant C. Now, what does $D_{\langle 1,y\rangle} u = 0$ mean with regard to this curve? Since $\langle 1, y\rangle$ is tangent to the curve (by construction), this just says that $u(x, y)$ is constant on each of the curves $y = Ce^x$. Hence the value of $u(x, y)$ depends only on which curve the point (x, y) happens to lie. How do we distinguish the various curves, one from the other? By the C, of course. Hence

$$u(x, y) = \text{“}f(C)\text{”}.$$

But $C = ye^{-x}$ and hence the general solution is given by

$$u(x, y) = f(ye^{-x})$$

for some function f.

The curves ($x - ct = d$ for the Transport Equation and $y = Ce^{-x}$ for $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$) have an obvious importance for the solution of their respective equations. They have a name: Characteristic Curves. Moreover, they have some important, less obvious properties.

The first of these comes from considering initial value problems. Suppose that for our equation $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$, we want to specify the initial condition $u(0, y) = y^2$ (or any given function of y for that matter). Then

$$y^2 = u(0, y) = f(ye^{-0}) = f(y)$$

so that $f(y) = y^2$ and

$$u(x, y) = f(ye^{-x}) = (ye^{-x})^2 = y^2e^{-2x}.$$

So no problems. What, if instead we wished to specify that $u(x, 1) = e^x$. Then,

$$e^x = u(x, 1) = f(1e^{-x})$$

so that $f(e^{-x}) = e^x$. If we set $s = e^{-x}$, then $e^x = 1/e^{-x} = 1/s$ and so $f(s) = 1/s$. Hence

$$u(x, y) = f(ye^{-x}) = \frac{1}{ye^{-x}} = \frac{e^x}{y}.$$

Again, few problems. However, what about the condition $u(x, 0) = x^2$? Then, we would want

$$x^2 = u(x, 0) = f(0e^{-x}) = f(0).$$

But this is not possible, since on the left we have a function and on the right a constant! What went wrong? The problem is that the line $y = 0$ is a characteristic ($y = Ce^x$ with $C = 0$) and we know that on these characteristics, $u(x, y)$ must be *constant*. Hence we cannot specify just anything we want there! The characteristics are the only curves with this property, and we will come back to it later.

Returning to the Transport Equation, the solution (1.12) has some important features. First of all, for fixed $t = t_0, say$, the graph of

$$x \longmapsto u(x, t_0) = u_0(x - ct_0)$$

is just the graph of $u_0(x)$, transported or translated by ct_0 to the right. Notice that from $t = 0$ to $t = t_0$ it moved a distance ct_0 in time t_0 and hence at speed $ct_0/t_0 = c$. Thus the evolution of $u(x, t)$, as time progresses, is the graph of $u_0(x)$ (starting at time $t = 0$) moving to the right at constant speed c. If for example, we set

$$u_0(x) = h(x) := \begin{cases} 0 & \text{if } x < -1 \\ x + 1 & \text{if } -1 \le x < 0 \\ 1 - x & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x \end{cases}$$

the "hat" function, then the evolution of $u(x,t)$ is this "wave" moving at constant speed c to the right. (Note that, although it acts like a wave travelling through an undisturbed medium, the Transport Equation does *not* describe the more complicated physical behaviour of waves, such as reflection and scattering.)

Secondly, there is a somewhat peculiar quality of the solution $u(x,t) = u_0(x-ct)$ of the Transport equation in that the solution makes sense for functions which are *not* differentiable, i.e., even for functions that cannot be substituted back into the original differential equation! In particular, if $u_0(x) = h(x)$, as above, then $u_0(x)$ is not differentiable at $x = -1, 0, +1$. So what do mean then by saying that $u(x,t) = u_0(x-ct)$ is a solution of the Transport Equation? This is a subtle question that is important in the careful study of partial differential equations. We will make just a brief interlude to describe what this is all about. The key idea is to "judge" a function not by what it *is*, but by what it does; what its effect is on other functions.

So what is the "effect" of $u(x,t)$ on another function, $\phi(x,t)$, say? This might be interpreted in many different ways, but one way, natural in the context of Calculus (integrals and derivatives) is to say that the "effect" of $u(x,t)$ on $\phi(x,t)$ is defined by

$$E_u(\phi) := \iint_{\mathbb{R}^2} \phi(x,t)u(x,t)dxdt. \tag{1.13}$$

Now, this requires some restrictions in order for it to be well defined. First of all, we will only consider the "effects" of u on very "good" functions; good in the sense that they present no problems for the operations of Calculus. Specifically, we restrict $\phi(x,t)$ to the class of infinitely differentiable functions (so that there are no problems with taking any number of derivatives) that are of *compact support*, i.e., there is a number R (depending on ϕ) such that $|(x,t)| > R \Longrightarrow \phi(x,t) = 0$ (so that there are no problems integrating over all of $\mathbb{R}^2$). In symbols, $\phi \in C_c^\infty(\mathbb{R}^2)$, which just means exactly what we have just said about ϕ.

Secondly, for $E_u(\phi)$ to make sense, even for such well-behaved ϕ, we will require that $|u(x,t)|$ have a finite integral over every ball in $\mathbb{R}^2$ of finite radius (in symbols $u \in L^1_{\text{loc}}(\mathbb{R}^2)$).

With these restrictions, $E_u(\phi)$ is well-defined so that the "effect" of u on each ϕ can be determined. Moreover, we note that if we knew the effects of u on *all* $\phi \in C_c^\infty(\mathbb{R}^2)$ we could use this information to reconstitute $u(x,t)$ by means of a so-called approximate identity (intuitively, the idea is to consider a sequence of ϕ's becoming more and more Dirac-delta like at the point (x,t)). The details of this are beyond the scope of these notes, but the important fact is that given a $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ we can compute its effect on any $\phi \in C_c^\infty(\mathbb{R}^2)$ and conversely, given the effects of u on all such ϕ's we can recover what u was, so that no true information is lost. Because of this, we often make no distinction between a function and its effect on $C_c^\infty(\mathbb{R}^2)$.

Now, what about derivatives? Given a $u(x,t)$, what is the "effect" of say

$\dfrac{\partial u}{\partial x}$? By definition,

$$E_{\frac{\partial u}{\partial x}}(\phi) = \iint_{\mathbb{R}^2} \phi(x,t)\frac{\partial u}{\partial x}(x,t)dxdt. \tag{1.14}$$

But if u were itself smooth, then we could integrate by parts, to obtain

$$E_{\frac{\partial u}{\partial x}}(\phi) = -\iint_{\mathbb{R}^2} \frac{\partial \phi}{\partial x}(x,t)u(x,t)dxdt \tag{1.15}$$

(the boundary effects are zero since, by assumption, there is an $R > 0$ such that $\phi(x,t) = 0$ for $|(x,t)| > R$). Note that (1.14) is defined only if $\dfrac{\partial u}{\partial x}$ exists, but that (1.15) is defined for *any* $u \in L^1_{\text{loc}}(\mathbb{R}^2)$. Hence, we define the "effect" of the derivative to be given by (1.15). We repeat, for emphasis, that in this sense, the "effect" of the derivative is defined even when the classical derivative does not exist!

This is an important point and so we pause to work out an explicit example. It is slightly easier to retreat to one variable functions $u(x)$, where

$$E_{u'}(\phi) := -\int_{\mathbb{R}} \phi'(x)u(x)dx. \tag{1.16}$$

Consider

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

which is obviously not differentiable at the origin. However,

$$\begin{aligned} E_{u'}(\phi) &= -\int_{-\infty}^{\infty} \phi'(x)u(x)dx \\ &= -\int_{0}^{\infty} \phi'(x) \times 1\, dx \\ &= -\phi(x)\,|_{x=0}^{x=\infty} \\ &= -\{\phi(\infty) - \phi(0)\} \\ &= \phi(0) \quad \text{(since } \phi \text{ has compact support)} \end{aligned}$$

which is certainly well-defined for every $\phi \in C_c^{\infty}(\mathbb{R})$.

In other words, $E_{u'}$ as defined by (1.16), generalizes (the effect of) the derivative to any $u \in L^1_{\text{loc}}(\mathbb{R})$.

Now back to seeing why any function $u(x,t) = u_0(x - ct)$ can be considered to be a generalized (or weak) solution of the Transport Equation. In particular we will show that, in the generalzied sense,

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0.$$

But recall taht this is equivalent to the directional derivative

$$D_{\langle c,1\rangle}u = 0.$$

Now the "effect" of $D_{\langle c,1\rangle}u$, in the generalized sense of (1.15), is just

$$\begin{aligned} E_{D_{\langle c,1\rangle}u}(\phi) &= -\iint_{\mathbb{R}^2} (D_{\langle c,1\rangle}\phi)(x,t)u(x,t)dxdt \\ &= -\iint_{\mathbb{R}^2} (D_{\langle c,1\rangle}\phi)(x,t)u_0(x-ct)dxdt. \end{aligned}$$

Now change variables in the integral by setting $a = \langle c,1\rangle \cdot \langle x,t\rangle = cx + t$ and $b = \langle c,1\rangle^{\perp} \cdot \langle x,t\rangle = x - ct$, so that $da = cdx + dt$, $db = dx - cdt$ and

$$dadb = \text{abs}\begin{vmatrix} c & 1 \\ 1 & -c \end{vmatrix} dxdt = (c^2+1)dxdt.$$

Hence,

$$\begin{aligned} E_{D_{\langle c,1\rangle}u}(\phi) &= -\frac{1}{c^2+1}\int_{b=-\infty}^{b=+\infty}\int_{a=-\infty}^{a=+\infty} \frac{\partial\phi}{\partial a}(a,b)u_0(b)dadb \\ &= -\frac{1}{c^2+1}\int_{b=-\infty}^{b=+\infty} u_0(b)\left(\int_{a=-\infty}^{a=+\infty}\frac{\partial\phi}{\partial a}(a,b)da\right)db \\ &= -\frac{1}{c^2+1}\int_{b=-\infty}^{b=+\infty} u_0(b)\left\{\phi(a,b)\mid_{a=-\infty}^{a=+\infty}\right\}db \\ &= -\frac{1}{c^2+1}\int_{b=-\infty}^{b=+\infty} u_0(b)\times 0\,db \quad (\text{since } \phi\in C_c^{\infty}(\mathbb{R}^2)) \\ &= 0. \end{aligned}$$

Hence the generalized effect of $D_{\langle c,1\rangle}u$ is zero or, in other words, in the generalized (or weak) sense $D_{\langle c,1\rangle}u = 0$.

We remark here that such generalized derivatives are also known as distributional derivatives. We will return to this later.

Before continuing, at this point we can also describe a second important property of characteristics. We see now that solutions may have "singularities", e.g. jumps in the first derivative as for the hat function. For the Transport equation, $u(x,t) = u_0(x-ct)$ and so, in the (x,t)-plane, any jumps occur only across lines of the form $x - ct = d$, i.e. across characteristics! This is not an accident. In fact, suppose that u is a solution of some first order linear PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0.$$

Consider a curve γ in the xy-plane. We assume that $u(x,y)$ is continuous across γ, but we ask if it is possible that $u(x,y)$ have different normal vectors on either side of γ? If γ is *not* a characteristic, then the solution $u(x,y)$ (and its derivatives) is uniquely determined by its values on γ. But this means that its derivatives normal to γ are also determined and hence *no*, we cannot make an arbitrary jump. Consequently, for there to be a singularity, γ must be a characteristic curve. In "physical terms", singularities are propagated along characteristics!

1.2 The One Dimensional Wave equation

Let us now move on the the one (spatial) dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t). \tag{1.17}$$

This is a second order equation and so we first consider the simplest of such equations. For example, what about

$$\frac{\partial^2 u}{\partial x^2} = 0?$$

This is quite simple. We just write

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$$

and then rename $\frac{\partial u}{\partial x} = v$, say. Then $\frac{\partial v}{\partial x} = 0$ so that $v(x,t) = f(t)$ for some function f of t. Hence, integrating, we have

$$u(x,t) = xf(t) + g(t).$$

The equation $\frac{\partial^2 u}{\partial t^2} = 0$ can be solved in a similar manner. There is also a third possibility, $\frac{\partial^2 u}{\partial x \partial t} = 0$. It can be solved in exactly the same way, and since it is actually very close (as we shall see) to the wave equation, we will work it out in detail:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial t} &= 0 \\
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) &= 0 \\
\frac{\partial v}{\partial x} &= 0 \quad (\text{setting } v = \frac{\partial u}{\partial t}) \\
v &= f(t) \quad (\text{for some function } f) \\
\frac{\partial u}{\partial t} &= f(t) \\
u(x,t) &= F(t) + G(x)
\end{aligned}$$

where $F(t)$ is an anti-derivative of $f(t)$ and $G(x)$ is some function of x. (Note that to determine the two functions $F(t)$ and $G(x)$ we will need *two* initial conditions.)

Now back to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t).$$

First rewrite as

$$c^2\frac{\partial^2 u}{\partial x^2}(x,t)-\frac{\partial^2 u}{\partial t^2}(x,t)=0$$

and then notice that this is exactly the same as

$$\left(c\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)u(x,t)=0.$$

Indeed, we calculate

$$\begin{aligned}
&\left(c\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)u(x,t)\\
=\;&\left(c\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\left(c\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}\right)\\
=\;&c\left(c\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial t}\right)+c\frac{\partial^2 u}{\partial t\partial x}-\frac{\partial^2 u}{\partial t^2}\\
=\;&c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2},
\end{aligned}$$

using the fact that (generally speaking) $\frac{\partial^2 u}{\partial x\partial t}=\frac{\partial^2 u}{\partial t\partial x}$. Again we recognize, $c\frac{\partial u}{\partial x}\pm\frac{\partial u}{\partial t}$ as directional derivatives so that we may rewrite the wave equation as

$$D_{\langle c,1\rangle}D_{\langle c,-1\rangle}u(x,t)=0. \tag{1.18}$$

This is really the same type of equation as $\frac{\partial^2 u}{\partial x\partial t}=0$ except that in this (seemingly simpler) case the two directions are $\langle 1,0\rangle$ and $\langle 0,1\rangle$ instead of $\langle c,1\rangle$ and $\langle c,-1\rangle$. Hence the solution to (1.18) follws from the same calculations as before, yielding

$$\begin{aligned}
u(x,t)&=F(\langle c,1\rangle^{\perp}\cdot\langle x,t\rangle)+G(\langle c,-1\rangle^{\perp}\cdot\langle x,t\rangle)\\
&=F(\langle 1,-c\rangle\cdot\langle x,t\rangle)+G(\langle 1,c\rangle\cdot\langle x,t\rangle)\\
&=F(x-ct)+G(x+ct) \qquad (1.19)
\end{aligned}$$

for two arbitrary (univariate) functions F and G. Of course, if F and G are not conventionally twice differentiable then this is a generalized (or weak) solution.

Equation (1.19) may be interpreted as the superimpostion of two waves, one moving to the right ($F(x-ct)$) and one to the left ($G(x+ct)$), both at speed c.

To see some of the physical implications of the wave equation, let us consider a simple scattering problem. Suppose the medium in which the wave is travelling has a discontinuity, as is often the case in geophysical applications. Specifically, suppose that the medium is such that for $x<0$ the wave speed is c_1 and for $x>0$ the speed is c_2. Consider some wave $u(x,t)=w(x-c_1t)$ coming in from the left, towards the origin (we guarantee this by requiring that $w(x)=0$ for $x>0$). We want to see what happens to the wave when it hits the discontinuity at $x=0$. To gain some physical intuition it is often useful to consider extreme cases. If $c_2 = c_1$ then there really is no discontinuity and we would expect the wave just to travel on through without any disturbance. However, if $c_2 = 0$ then the wave gets stuck in the mud, so to speak, and effectively hits a barrier. In this case we would expect the wave to reflect back and return (inverted) from whence it came. In other, non-extreme cases, we might reasonably expect then that part of the wave reflects back and part continues on. Let's figure out *exactly* what does happen.

Now, since there are two regimes, $x < 0$ and $x > 0$, the solution of the wave equation must be of the form

$$u(x,t) = \begin{cases} F_1(x - c_1 t) + G_1(x + c_1 t) & x < 0 \\ F_2(x - c_2 t) + G_2(x + c_2 t) & x > 0 \end{cases}. \tag{1.20}$$

We are also asking that $u(x,t)$ satisfy the initial conditions

$$\begin{aligned} u(x,0) &= w(x - c_1 t)\mid_{t=0} = w(x), \\ \frac{\partial u}{\partial t}(x,0) &= \frac{\partial}{\partial t}\{w(x - c_1 t)\}\mid_{t=0} = -c_1 w'(x). \end{aligned} \tag{1.21}$$

It follows from this that

$$\begin{aligned} F_1(y) &= w(y), \quad y < 0 \\ G_1(y) &= 0, \quad y < 0 \\ F_2(y) &= 0, \quad y > 0 \\ G_2(y) &= 0, \quad y > 0. \end{aligned} \tag{1.22}$$

But for $x > 0$ and $t \geq 0$, $x + c_2 t > 0$ and so $G_2(x + c_2 t) \equiv 0$. Similarly, for $x < 0$ and $t \geq 0$, $x - c_1 t < 0$ and hence $F_1(x - c_1 t) = w(x - c_1 t)$. Thus we have

$$u(x,t) = \begin{cases} w(x - c_1 t) + G_1(x + c_1 t) & x < 0 \\ F_2(x - c_2 t) & x > 0 \end{cases}$$

(with the middle two conditions of (1.22) holding as well).

Now, the wave itself, as a physical object, should be such that $u(x,t)$ is continuous at $x = 0$ and also that $\frac{\partial u}{\partial x}(x,t)$ is continuous at $x = 0$. For the continuity of $u(x,t)$:

$$u(0^-,t) = w(0 - c_1 t) + G_1(0 + c_1 t) = w(-c_1 t) + G_1(c_1 t)$$

while

$$u(0^+,t) = F_2(0 - c_2 t) = F_2(-c_2 t)$$

so that

$$w(-c_1 t) + G_1(c_1 t) = F_2(-c_2 t), \quad t \geq 0. \tag{1.23}$$

For the continuity of $\frac{\partial u}{\partial x}(x,t)$:

$$\frac{\partial u}{\partial x}(0^-,t) = w'(0 - c_1 t) + G_1'(0 + c_1 t) = w'(-c_1 t) + G_1'(c_1 t)$$

while

$$\frac{\partial u}{\partial x}(0^+,t) = F_2'(0-c_2t) = F_2'(-c_2t)$$

so that

$$w'(-c_1t) + G_1'(c_1t) = F_2'(-c_2t). \tag{1.24}$$

Differentiating (1.23) with respect to t gives

$$-c_1w'(-c_1t) + c_1G_1'(c_1t) = -c_2F_2'(-c_2t). \tag{1.25}$$

Then, c_1(1.24)−(1.25) yields

$$2c_1w'(-c_1t) = (c_1+c_2)F_2'(-c_2t)$$

and hence

$$F_2'(-c_2t) = \frac{2c_1}{c_1+c_2}w'(-c_1t)$$

and, upon integrating,

$$\frac{F_2(-c_2t)}{-c_2} = \frac{2c_1}{c_1+c_2}\frac{w(-c_1t)}{-c_1} + \alpha,$$

i.e.,

$$F_2(-c_2t) = \frac{2c_2}{c_1+c_2}w(-c_1t) + \alpha.$$

By (1.22), the constant $\alpha = 0$ and we have, upon setting $y = -c_2t < 0$,

$$F_2(y) = \frac{2c_2}{c_1+c_2}w(\frac{c_1}{c_2}y), \quad y < 0.$$

Similarly, by combining c_1(1.24)+(1.25), we may calculate that

$$G_1(y) = \frac{c_2-c_1}{c_1+c_2}w(-y), \quad y > 0.$$

Hence

$$u(x,t) = \begin{cases} w(x-c_1t) + \dfrac{c_2-c_1}{c_1+c_2}w(-(x+c_1t)) & x<0 \\ \\ \dfrac{2c_2}{c_1+c_2}w(\dfrac{c_1}{c_2}(x-c_2t)) & x>0. \end{cases} \tag{1.26}$$

We see that $u(x,t)$ is composed of three terms: the first term in (1.26), $w(x-c_1t)$, is the original (incoming) wave moving to the right towards the origin at speed c_1; the second term $\frac{c_2-c_1}{c_1+c_2}w(-(x+c_1t))$ is the reflected part of the wave moving to the left, away from the origin at speed c_1, but with "amplitude" $\frac{c_2-c_1}{c_1+c_2}$; the third term $\frac{2c_2}{c_1+c_2}w(\frac{c_1}{c_2}(x-c_2t))$ is the transmitted wave, moving to the right at speed c_2, but with "amplitude" $\dfrac{2c_2}{c_1+c_2}$. The quantities $\dfrac{c_2-c_1}{c_1+c_2}$ and $\dfrac{2c_2}{c_1+c_2}$ are known as the reflection and transmission coefficients, respectively.

Note that in the extreme cases considered initially we would have

$$u(x,t) = \begin{cases} w(x-ct) & x<0 \\ w(x-ct) & x>0 \end{cases}$$

if $c_1 = c_2 = c$, and

$$u(x,t) = \begin{cases} w(x-c_1t) - w(-(x+c_1t)) & x<0 \\ 0 & x>0 \end{cases}$$

if $c_2 = 0$, which is in agreement with our earlier observations.

Now, to return to the general solution of the wave equation. As noted previously, two initial conditions are required to specify the two arbitrary functions F and G. Most often these are taken to be

$$\begin{aligned} u(x,0) &= \phi(x), \quad \text{the initial position} \\ \frac{\partial u}{\partial t}(x,0) &= \psi(x), \quad \text{the initial velocity.} \end{aligned} \tag{1.27}$$

In this case there is a particularly convenient way of writing the soultion, due to d'Alembert:

$$u(x,y) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

It is easy to verify, after the fact, that this is indeed the solution of the wave equation with the initial conditions (1.27). It is also not particularly difficult to derive it by applying the initial conditions to the solution $u(x,t) = F(x-ct) + G(x+ct)$ and solving for F and G. However, it is also possible to derive by means of the Fourier transform. Since this transform is of more general use (for example in signal processing) we will first take an interlude to discuss some of its basic properties and then show how it can be used to solve the wave (and other!) equations.

Suppose first that $f(x)$ is a function, defined on all of $\mathbb{R}$ with the property that

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

Then the (continuous) Fourier transform of $f(x)$ is defined to be

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} e^{-iwx} f(x)dx; \quad \omega \in \mathbb{R}.$$

For each $\omega \in \mathbb{R}$, $\widehat{f}(\omega)$ is a finite value since

$$\begin{aligned} |\widehat{f}(\omega)| &\leq \int_{-\infty}^{\infty} |e^{-i\omega x}|\,|f(x)|\,dx \\ &= \int_{-\infty}^{\infty} |f(x)|dx < \infty, \quad \text{by assumption.} \end{aligned}$$

If $f(x)$ happened to be zero outside the interval $[-\pi, \pi]$, then

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx = \int_{-\pi}^{\pi} e^{-i\omega x} f(x)dx$$

and so

$$\frac{1}{2\pi}\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx$$

is the nth (complex) Fourier series coeffcient of $f(x)$. More generally then, $\widehat{f}(\omega)$ may be thought of as giving the "amplitude" of the ω-frequency part of $f(x)$, and $\widehat{f}(\omega)$ is often said to be "in the frequency domain". These considerations are important and extremely useful, especially in signal processing, but here we will just make use of some of the basic algebraic properties of the Fourier transform. The first, since we are considering differential equations, is what the Fourier transform does to derivatives. Suppose then that $f(x)$ is such that both $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ and $\int_{-\infty}^{\infty} |f'(x)|dx < \infty$, as well as that $\lim_{x\to\pm\infty} f(x) = 0$. Then we calculate, integrating by parts, that

$$\begin{aligned} \widehat{f'}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x} f'(x)dx \\ &= e^{-i\omega x} f(x)\ |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-i\omega x}(-i\omega) f(x)dx \\ &= 0 + (i\omega)\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \\ &= (i\omega)\widehat{f}(\omega). \end{aligned} \tag{1.28}$$

Moreover, if in addition $\int_{-\infty}^{\infty} |xf(x)|dx < \infty$,

$$\begin{aligned} \frac{d}{d\omega}\widehat{f}(\omega) &= \frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \\ &= \int_{-\infty}^{\infty} (\frac{d}{d\omega} e^{-i\omega x}) f(x)dx \\ &= \int_{-infty}^{\infty} e^{-i\omega x}(-ixf(x))dx \\ &= \widehat{-ixf(x)}(\omega). \end{aligned}$$

The remarkable fact is that (for sufficiently regular functions), the Fourier transform transforms differentiation into multiplication, and hence Calculus problems into (simpler) Algebra problems. This makes it eminently worthwhile to examine some of its other properties.

Now, we have already had to write down the condition $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ several times, for several different functions. It is convenient at this point to introduce some time saving notation. We set

$$L^1(\mathbb{R}) := \{f \mid \int_{-\infty}^{\infty} |f(x)|dx < \infty\}.$$

(The 1 in the superscript refers to the fact that $|f(x)|$ is to the power one; later we will use e.g. the space $L^2(\mathbb{R}) := \{f \mid \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$.)

- If $\widehat{f}(\omega) \in L^1(\mathbb{R})$, we may write

  $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{+i\omega x} d\omega,$$

  i.e., we recover $f(x)$ from its Fourier transform. This Fourier inverse transform allows us to go back and forth from the spatial domain ($f(x)$) to the frequency domain ($\widehat{f}(\omega)$).

- Given two functions $f(x), g(x) \in L^1(\mathbb{R})$, their *convolution* product $(f * g)(x)$ is defined to be

  $$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

  (and can be shown to also be in $L^1(\mathbb{R})$).

  It can be shown that

  $$\widehat{(f * g)}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega).$$

  Hence convolutions arise naturally when inverting the product of two Fourier transforms.

- If $a \in \mathbb{R}$, the the translation oeprator T_a is defined by

  $$(T_a f)(x) := f(x-a),$$

  i.e., $T_a f$ is just f translated to the right by the amount a. Then we may calculate

  $$\begin{aligned} \widehat{T_a f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x} T_a f(x) dx \\ &= \int_{-\infty}^{\infty} e^{-i\omega x} f(x-a) dx. \end{aligned}$$

Now change variables by letting $y = x - a$ so that $x = y + a$ and $dx = dy$, to obtain

$$\begin{aligned}\widehat{T_a f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega(y+a)} f(y)dy \\ &= e^{-i\omega a}\int_{-\infty}^{\infty} e^{-i\omega y} f(y)dy \\ &= e^{-i\omega a}\widehat{f}(\omega). \end{aligned} \tag{1.29}$$

The Fourier transform, although initially defined only for $L^1(\mathbb{R})$ functions can be extended to other functions. In particular, it can be extended to $L^2(\mathbb{R})$ functions, and we even have

$$f \in L^2(\mathbb{R}) \iff \widehat{f} \in L^2(\mathbb{R})$$

(the corresponding statment for $L^1(\mathbb{R})$ is not true!). Importantly, it can also be extended to generalized functions, or distributions.

We have already seen the use of generalized derivatives which came from considering functions from the point of view of their "effect" on $C_c^\infty(\mathbb{R})$ functions. Now every function $u \in L^1_{\text{loc}}(\mathbb{R})$ has an effect:

$$\phi \longmapsto E_u(\phi) := \int_{\mathbb{R}} \phi(x)u(x)dx,$$

but not everything that has an effect is a function! The classic example is the so-called Dirac-delta, where the effect is

$$E_\delta(\phi): \quad \phi \longmapsto \phi(0).$$

This is a perfectly good effect but it cannot be represented as the effect of a classical function. Hence we may consider δ as a generalized function. There are many others. In order to handle these we have to be slightly more organized and precise about exactly what we mean.

Definition. A distribution, or generalized function, f is a mapping $C_c^\infty(\mathbb{R}) \to \mathbb{R}$, denoted by

$$\phi \longmapsto \langle f, \phi\rangle$$

with two regularity properties:

(a) Linearity $\langle f, a_1\phi_1 + a_2\phi_2\rangle = a_1\langle f, \phi_1\rangle + a_2\langle f, \phi_2\rangle$

(b) Continuity if $\phi_n \to \phi$ (in the sense of $C_c^\infty(\mathbb{R})$), then $\langle f, \phi_n\rangle \to \langle f, \phi\rangle$.

But what does $\phi_n \to \phi$ in the sense of $C_c^\infty(\mathbb{R})$ mean? This just means that

1. there is a *common* interval $[-R, R]$ such that $\phi_n(x) = 0$ *outside* $[-R, R]$ for all n, and

2. $\phi_n^{(j)} \to \phi^{(j)}$ uniformly on $[-R, R]$ for all orders of derivatives j.

(This definition guarantees that if $\phi_n \to \phi$, in the sense of $C_c^\infty(\mathbb{R})$, then ϕ is also in $C_c^\infty(\mathbb{R})$.)

Some simple examples are:

- if $f \in L^1_{\text{loc}}(\mathbb{R})$ then $\langle f, \phi \rangle := \int_{\mathbb{R}} \phi(x) f(x) dx$
- if $f = \delta$ then $\langle \delta, \phi \rangle := \phi(0)$
- $\langle f', \phi \rangle := \phi'(0)$.

every distribution has a generalized (distributional) derivative given by

$$\langle f', \phi \rangle := -\langle f, \phi' \rangle.$$

For example,

$$\begin{aligned} \langle \delta', \phi \rangle &:= -\langle \delta, \phi' \rangle \\ &= -\phi'(0). \end{aligned}$$

Another example: consider, as before,

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Then,

$$\begin{aligned} \langle f', \phi \rangle &:= -\langle f, \phi' \rangle \\ &= -\int_{\infty}^{\infty} f(x) \phi'(x) dx \\ &= -\int_0^{\infty} \phi'(x) dx \\ &= -\{\phi(x) \mid_{x=0}^{x=\infty}\} \\ &= -\{\phi(\infty) - \phi(0)\} \\ &= \phi(0) \quad (\phi(\infty) = 0 \text{ since } \phi \in C_c^\infty(\mathbb{R})) \end{aligned}$$

so that $f' = \delta$.

So now, what is the Fourier transform of a distribution? We need first another basic property of Fourier transforms:

$$\int_{\mathbb{R}} \widehat{f}(x) g(x) dx = \int_{\mathbb{R}} f(x) \widehat{g}(x) dx$$

(for sufficiently regular f and g).

Now to define $\widehat{f}(\omega)$, we have to define its effect on $\phi \in C_c^\infty(\mathbb{R})$; $\langle \widehat{f}, \phi \rangle$. But by the above, $\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle$ and so we make this the definition. Specifically, if f is a distribution, then $\widehat{f}$ is the distribution defined by

$$\langle \widehat{f}, \phi \rangle := \langle f, \widehat{\phi} \rangle.$$

For example, if $f = \delta$,

$$\begin{aligned}\langle \widehat{\delta}, \phi \rangle &:= \langle \delta, \widehat{\phi} \rangle \\ &= \widehat{\phi}(0) \\ &= \int_{\mathbb{R}} e^{-i0x} \phi(x) dx \\ &= \int_{\mathbb{R}} 1 \times \phi(x) dx \\ &= \langle 1, \phi \rangle.\end{aligned}$$

Hence $\widehat{\delta} = 1$.

Alternatively, we may compute,

$$\begin{aligned}\widehat{\delta} &= \int_{\mathbb{R}} e^{-i\omega x} \delta(x) dx \\ &= \langle \delta, e^{-i\omega x} \rangle \\ &= e^{-i\omega 0} \\ &= 1.\end{aligned}$$

Although $e^{-i\omega x} \notin C_c^\infty(\mathbb{R})$, and hence this latter calculation is not really legal, it does give the correct answer.

How do we use this for PDE's? Let's first go back to the Transport Equation: $\frac{\partial u}{\partial t}(x,t) = -c\frac{\partial u}{\partial x}(x,t)$. Take the Fourier transform of both sides to obtain

$$\int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial u}{\partial t}(x,t) dx = -c \int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial u}{\partial x}(x,t) dx.$$

On the left, we may take the derivative with respect to t outside the integral (the "sum" of the derivatives equals the derivative of the "sum"; at least most of the time). On the right, we have just the Fourier transform of a derivative (holding t fixed), and so by (1.28),

$$\frac{d}{dt} \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t) dx = -c(i\omega) \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t) dx,$$

i.e.,

$$\widehat{u}(\omega, t) = -c(i\omega)\widehat{u}(\omega, t). \tag{1.30}$$

Hence, by taking the Fourier transform we have converted a PDE in the space domain to a simpler ODE in the frequency domain. We now solve this ODE, by standard elementary methods;

$$\widehat{u}(\omega, t) = Fe^{-ic\omega t}$$

for some F, constant with respect to t, i.e.,

$$\widehat{u}(\omega, t) = F(\omega)e^{-ic\omega t}.$$

If we write $F(\omega) = \widehat{f}(\omega)$ for some function $f(x)$, then we have

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-ic\omega t}$$

and so, by the translation formula, (1.29),

$$\widehat{u}(\omega, t) = \widehat{T_{ct}f}(\omega).$$

We now invert to obtain

$$u(x, t) = (T_{ct}f)(x) = f(x - ct),$$

which we recognize as the general solution of the Transport Equation.

Now let us apply the same technology to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Again, take the Fourier transform of both sides to obtain

$$\int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial^2 u}{\partial t^2}(x, t)dx = c^2 \int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial^2 u}{\partial x^2}(x, t)dx.$$

On the left, we again take the time derivative outside the integral. On the right, we have the Fourier transform of a second derivative (holding t fixed) to which we apply the formula (1.28) twice, to obtain

$$\frac{d^2}{dt^2} \int_{-\infty}^{\infty} e^{-i\omega x} u(x, t)dx = c^2(i\omega)^2 \int_{-\infty}^{\infty} e^{-i\omega x} u(x, t)dx,$$

i.e.,

$$\frac{d^2}{dt^2}\widehat{u}(\omega, t) = -c^2\omega^2\widehat{u}(\omega, t), \tag{1.31}$$

a second order linear ODE in t (for fixed ω). It's characteristic equation is $\lambda^2 + c^2\omega^2 = 0$, so that $\lambda = \pm ic\omega$ and its general solution is given by

$$\widehat{u}(\omega, t) = F_1 e^{ic\omega t} + F_2 e^{-ic\omega t} \tag{1.32}$$

where F_1, F_2 are constant with respect to t, i.e., $F_1 = F_1(\omega)$, and $F_2 = F_2(\omega)$.

Now impose the initial conditions

$$u(x, 0) = \phi(x); \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x)$$

so that

$$\widehat{u}(\omega, 0) = \widehat{\phi}(\omega) \text{ and } \frac{\partial}{\partial t}\widehat{u}(\omega, 0) = \widehat{\psi}(\omega)$$

(assuming that these exist). Applying these to (1.32) yields

$$F_1(\omega) + F_2(\omega) = \widehat{\phi}(\omega),$$

$$(ic\omega)F_1(\omega) - (ic\omega)F_2(\omega) = \widehat{\psi}(\omega).$$

This system can be solved for $F_1(\omega)$ and $F_2(\omega)$ to obtain

$$\begin{aligned} F_1(\omega) &= \frac{1}{2}\widehat{\phi}(\omega) + \frac{1}{2ic\omega}\widehat{\psi}(\omega) \\ F_2(\omega) &= \frac{1}{2}\widehat{\phi}(\omega) - \frac{1}{2ic\omega}\widehat{\psi}(\omega). \end{aligned}$$

Then substituting back into (1.32) gives

$$\widehat{u}(\omega, t) = \widehat{\phi}\left\{\frac{e^{ic\omega t} + e^{-ic\omega t}}{2}\right\} + \widehat{\psi}(\omega)\left\{\frac{e^{ic\omega t} - e^{-ic\omega t}}{2ic\omega}\right\}. \tag{1.33}$$

Now, just as before,

$$\widehat{\phi}\frac{e^{ic\omega t}}{2} = \frac{1}{2}(\widehat{T_{-ct}\phi})(\omega)$$

and

$$\widehat{\phi}\frac{e^{-ic\omega t}}{2} = \frac{1}{2}(\widehat{T_{ct}\phi})(\omega),$$

so we know where these terms come from. What about the last term in (1.33)? Now,

$$\widehat{\psi}(\omega)\left\{\frac{e^{ic\omega t} - e^{-ic\omega t}}{2ic\omega}\right\} = \widehat{\psi}\frac{\sin(c\omega t)}{c\omega}$$

is the product of two functions. Hence if we could recognize $\sin(c\omega t)/(c\omega)$ as the Fourier transform of some function $f(x)$, we would have $\widehat{\psi}\widehat{f}$ which we know is the Fourier transform of $\psi * f$, and we would be done. So what is this f? Finding f is actually quite easy. We just take the Fourier inverse transform (holding t fixed) of $\dfrac{\sin(\omega ct)}{c\omega}$ to obtain

$$f(x) = \begin{cases} 1/(2c) & -ct \le x \le ct \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that this is the case. Thus

$$\widehat{\psi}(\omega)\frac{\sin(c\omega t)}{c\omega} = (\widehat{\psi * f})(\omega).$$

Finally, we invert (1.33) and see that

$$\begin{aligned} u(x,t) &= \frac{1}{2}T_{-ct}\phi(x) + \frac{1}{2}T_{ct}\phi(x) + (\psi * f)(x) \\ &= \frac{\phi(x+ct) + \phi(x-ct)}{2} + (\psi * f)(x). \end{aligned}$$

This is already suspiciously like the d'Alembert solution, and indeed it is, since we may calculate

$$\begin{aligned}
(\psi * f)(x) &= \int_{-\infty}^{\infty} \psi(x-y) f(y) dy \\
&= \frac{1}{2c} \int_{y=-ct}^{y=ct} \psi(x-y) dy \\
&= \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \quad \text{(with } s = x - y\text{)}.
\end{aligned}$$

Please note however that using the Fourier transform implicitly puts restrictions on the functions ϕ and ψ. The advantage of the Fourier transform is that it can be used to solve many (linear) PDE's including the wave equation in higher dimensions! Let's see how this works.

1.3 The Wave Equation in Space

In $\mathbb{R}^3$ the wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1.34}$$

where $u = u(x_1, x_2, x_3, t)$ and $\nabla^2 u$ is the so-called Laplacian of u in the space variables, i.e.,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}.$$

Before proceeding to the general solution, let us consider the important special case of a radial wave, i.e., when u depends only on $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$ and time t. It is natural in this case to switch to spherical coordinates:

$$\begin{aligned}
x_3 &= r\cos(\phi) \\
x_2 &= r\sin(\phi)\sin(\theta) \\
x_1 &= r\sin(\phi)\cos(\theta).
\end{aligned}$$

Using the Chain Rule (and a little bit of mathematical elbow grease) it can be shown that the Laplacian is expressed in spherical coordinates by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \phi^2} + \cot(\phi)\frac{\partial u}{\partial \phi} + \frac{1}{\sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2}\right).$$

Since we suppose that u has no angular dependence, this simplifies to

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}.$$

Thus we wish to solve the "spherical wave equation"

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

for $u = u(r, t)$.

Set $v := ru$, and calculate

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{r}\frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial u}{\partial r} &= -\frac{1}{r^2}v + \frac{1}{r}\frac{\partial v}{\partial r}, \\ \frac{\partial^2 u}{\partial r^2} &= \frac{2}{r^3}v - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial^2 v}{\partial r^2} \end{aligned}$$

and so, substituting these back into the spherical wave equation yields

$$\begin{aligned} \frac{1}{r}\frac{\partial^2 v}{\partial t^2} &= c^2\left(\frac{2}{r^3}v - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\left\{-\frac{1}{r^2}v + \frac{1}{r}\frac{\partial v}{\partial r}\right\}\right) \\ &= c^2\left(\frac{1}{r}\frac{\partial^2 v}{\partial r^2}\right) \end{aligned}$$

so that v is a solution of the one-dimensional wave equation

$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial r^2}!$$

It follows that

$$u(r, t) = \frac{1}{r}\{F(r + ct) + G(r - ct)\}$$

for some functions F and G. In particular we see that the amplitude of a spherical wave decays as $1/r$.

Now for the general solution. We again take the Fourier transform of both sides of (1.34). However, this time we must use its multivariate version:

$$\widehat{f}(\omega) = \iiint_{\mathbb{R}^3} e^{-i\omega\cdot x} f(x)dV(x)$$

where $\omega = (\omega_1, \omega_2\omega_3)$ and $x = (x_1, x_2, x_3)$ are vectors (and $\omega \cdot x$ their dot product). many of the properties of the univariate transform carry over in the obvious way. For example,

$$\widehat{\frac{\partial f}{\partial x_j}}(\omega) = (i\omega_j)\widehat{f}(\omega).$$

In particular then,

$$\begin{aligned} \widehat{\nabla^2 f}(\omega) &= \{(i\omega_1)^2 + (i\omega_2)^2 + (i\omega_3)^2\}\widehat{f}(\omega) \\ &= -(\omega_1^2 + \omega_2^2 + \omega_3^2)\widehat{f}(\omega) \\ &= -|\omega|^2\widehat{f}(\omega). \end{aligned}$$

Applying this to (1.34), we have

$$\frac{d^2}{dt^2}\widehat{u}(\omega, t) = -c^2|\omega|^2\widehat{u}(\omega, t) \tag{1.35}$$

which is again, for fixed ω, a second order linear ODE in t, and completely analogous to the univaraite case (1.31). We solve, just as before, to obtain

$$\widehat{u}(\omega,t) = F_1(\omega)e^{+ic|\omega|t} + F_2(\omega)e^{-ic|\omega|t}. \tag{1.36}$$

Next, apply the initial conditions

$$\begin{aligned} u(x,0) &= \phi(x), \quad x \in \mathbb{R}^3 \\ \frac{\partial u}{\partial t}(x,0) &= \psi(x), \quad x \in \mathbb{R}^3 \end{aligned}$$

so that

$$\begin{aligned} \widehat{u}(\omega,0) &= \widehat{\phi}(\omega), \quad \omega \in \mathbb{R}^3 \\ \frac{d\widehat{u}}{dt}(\omega,0) &= \widehat{\psi}(\omega), \quad \omega \in \mathbb{R}^3 \end{aligned}$$

and solve to obtain

$$\begin{aligned} F_1(\omega) &= \frac{1}{2}\widehat{\phi}(\omega) + \frac{1}{2ic|\omega|}\widehat{\psi}(\omega) \\ F_2(\omega) &= \frac{1}{2}\widehat{\phi}(\omega) - \frac{1}{2ic|\omega|}\widehat{\psi}(\omega). \end{aligned}$$

We substitute these back into (1.36) to get a formula for $\widehat{u}(\omega,t)$:

$$\widehat{u}(\omega,t) = \widehat{\phi}(\omega)\left\{\frac{e^{ic|\omega|t} + e^{-ic|\omega|t}}{2}\right\} + \widehat{\psi}(\omega)\left\{\frac{e^{ic|\omega|t} - e^{-ic|\omega|t}}{2ic|\omega|}\right\}. \tag{1.37}$$

The fact that there are $|\omega|$ terms in the exponentials means that we are not dealing with simple translations anymore, but the problem remains the same: we need to identify the factors $\frac{e^{ic|\omega|t} + e^{-ic|\omega|t}}{2}$ and $\frac{e^{ic|\omega|t} - e^{-ic|\omega|t}}{2ic|\omega|}$ as the Fourier transforms of certain functions. Let us start with the second factor. In fact, we claim that

$$\frac{e^{ic|\omega|t} - e^{-ic|\omega|t}}{2ic|\omega|}$$

is the Fourier transform of the "spherical mean" *distribution*

$$\phi \longmapsto \frac{1}{4\pi c^2 t}\iint_{S(0,ct)} \phi \, d\sigma$$

where $S(a,\rho)$ is the *sphere*, centred at $a \in \mathbb{R}^3$, of radius ρ, so that

$$S(0,ct) = \{x \in \mathbb{R}^3 \,:\, |x| = ct\}.$$

Here, $d\sigma$ refers to the surface area on the sphere.

To verify this claim we just compute the Fourier transform of this distribution as

$$\frac{1}{4\pi c^2 t}\iint_{S(0,ct)} e^{-i\omega\cdot x}d\sigma(x).$$

Now, the integral is rotationally invariant in x and so we may conveniently choose coordinates so that the north pole of the sphere is along the vector ω. Then

$$\frac{\omega\cdot x}{|\omega|\,|x|} = \cos(\text{angle between } \omega \text{ and } x)$$

so that

$$\begin{aligned} \omega\cdot x &= |\omega|\,|x|\cos(\text{angle between the north pole and } x) \\ &= |\omega|\,|x|\cos(\phi) \end{aligned}$$

in the usual spherical coordinate system.

Hence

$$\begin{aligned} \frac{1}{4\pi c^2 t}\iint_{S(0,ct)} e^{-i\omega\cdot x}d\sigma(x) &= \frac{1}{4\pi c^2 t}\int_0^{2\pi}\int_0^{\pi} e^{-i|\omega|\,|x|\cos(\phi)}(ct)^2\sin(\phi)d\phi\, d\theta \\ &= \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-i|\omega|\,|x|\cos(\phi)}\sin(\phi)d\phi\, d\theta \\ &= \frac{t}{4\pi}\int_0^{2\pi}\left(\frac{e^{-i|\omega|\,|x|\cos(\phi)}}{i|\omega|ct}\Bigg|_{\phi=0}^{\phi=\pi}\right)d\theta \\ &= \frac{t}{4\pi}2\pi\frac{e^{i|\omega|ct}-e^{-i|\omega|ct}}{|\omega|ct} \\ &= \frac{e^{i|\omega|ct}-e^{-i|\omega|ct}}{2|\omega|c} \end{aligned}$$

as claimed.

For the second factor, notice that

$$\frac{e^{ic|\omega|t}+e^{-ic|\omega|t}}{2} = \frac{d}{dt}\left(\frac{e^{ic|\omega|t}-e^{-ic|\omega|t}}{2ic|\omega|}\right)$$

and hence $\frac{e^{ic|\omega|t}+e^{-ic|\omega|t}}{2}$ is the Fourier transform of the distribution

$$\phi \longmapsto \frac{d}{dt}\left(\frac{1}{4\pi c^2 t}\iint_{S(0,ct)}\phi d\sigma\right).$$

It follows, taking the Fourier inverse transform of (1.37), that

$$u(x,t) = \frac{1}{4\pi c^2 t}\iint_{S(x,ct)}\psi d\sigma + \frac{d}{dt}\left\{\frac{1}{4\pi c^2 t}\iint_{S(x,ct)}\phi d\sigma\right\}. \tag{1.38}$$

This formula is known as Kirchoff's formula. Notice the similarity with the one dimensional d'Alembert solution!

An interesting immediate consequence is Huygen's Principle, i.e., that $u(x,t)$ depends on the values of ϕ and ψ on the sphere $S(x,ct)$ and *not* on the inside! Conversely, given a point in space $x_0 \in \mathbb{R}^3$, we can ask where do the values of $\phi(x_0)$ and $\psi(x_0)$ have an effect on the wave $u(x,t)$. The answer is easy; x_0 must lie on the sphere $S(x,ct)$, i.e., $|x-x_0| = ct$. Hence the "domain of influence" of the point x_0 is $\{(x,t) : |x-x_0| = ct\}$. In space-time, (x,t), this is geometrically a cone, sometimes called a *light cone*.

If we we think of the wave as originating from a point source at x_0 so that ϕ and ψ are concentrated at (or near to) x_0, then an observer located at the spatial point x_1 will feel the wavefront at time

$$t = \frac{|x-x_0|}{c},$$

but then it will passs by and s/he will not feel any further disturbance. The two-dimensional case is completely different in this regard. In fact, given the solution to the three-dimensional problem, it is easy to solve the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(x_1,x_2,t) = c^2\nabla^2 u(x_1,x_2,t). \tag{1.39}$$

In fact, if u is a solution of (1.39), then $\dfrac{\partial^2 u}{\partial x_3^2} = 0$ and so u is necessarily a solution of the three-dimensional wave equation. Conversely, if a solution of the three-dimensional wave equation happens not to depend on x_3, then it is a solution of the two-dimensional problem. Hence, the solutions of (1.39) are given by Kirchoff's formula as follows:

$$u(x_1,x_2,t) = \frac{1}{4\pi c^2 t}\iint_{S(x,ct)} \psi(s_1,s_2)d\sigma(s) + \frac{d}{dt}\left[\frac{1}{4\pi c^2 t}\iint_{S(x,ct)} \phi(s_1,s_2)d\sigma(s)\right]$$

where $x = (x_1,x_2,0)$. The surface integrals

$$\iint_{S(x,ct)} \psi(s_1,s_2)d\sigma(s) \text{ and } \iint_{S(x,ct)} \phi(s_1,s_2)d\sigma(s)$$

can be simplified. For example,

$$\begin{aligned}
\iint_{S(0,ct)} \phi(s_1,s_2)d\sigma &= 2\iint_{s_3=\sqrt{(ct)^2-s_1^2-s_2^2}} \phi(s_1,s_2)d\sigma(s) \\
&= 2\iint_{s_1^2+s_2^2\le c^2t^2} \phi(s_1,s_2)\sqrt{1+(\frac{\partial s_3}{\partial s_1})^2+(\frac{\partial s_3}{\partial s_2})^2}ds_1ds_2 \\
&= 2\iint_{s_1^2+s_2^2\le c^2t^2} \phi(s_1,s_2)\frac{ct}{\sqrt{c^2t^2-s_1^2-s_2^2}}ds_1ds_2.
\end{aligned}$$

Hence,

$$\begin{aligned} u(x_1,x_2,t) &= \frac{1}{2\pi c}\iint_{D(x,ct)} \frac{\psi(s_1,s_2)}{\sqrt{c^2t^2-(s_1-x_1)^2+(s_2-x_2)^2}}ds_1ds_2 \\ &\quad + \frac{d}{dt}\left\{\frac{1}{2\pi c}\iint_{D(x,ct)} \frac{\phi(s_1,s_2)}{\sqrt{c^2t^2-(s_1-x_1)^2+(s_2-x_2)^2}}ds_1ds_2\right\} \end{aligned}$$

where $D(x, ct)$ is the disk, centred at x of raidus ct, i.e.,

$$D(x,ct) = \{(s_1,s_2) \,:\, (s_1-x_1)^2+(s_2-x_2)^2 \le c^2t^2\}.$$

We see that the value of u at (x_1, x_2, t) depends on not just the surface, but the entire ball (disk)

$$\{(s_1,s_2) \,:\, (s_1-x_1)^2+(s_2-x_2)^2 \le c^2t^2\}.$$

An observer will experience a disturbance like a frog on a lilly pad experiences ripples in a pond.

Returning to the three-dimensional case (1.38), notice that there are two parts to the solution; one for ψ and one for ϕ. In fact, if we consider the operator

$$\mathcal{A}_t\psi(s) := \frac{1}{4\pi c^2 t}\iint_{S(x,ct)} \psi d\sigma,$$

then

$$u(x,t) = \mathcal{A}_t\psi(x) + \frac{d}{dt}\{\mathcal{A}_t\phi(x)\}$$

and so clearly $\mathcal{A}_t$ is the most important part of the solution. It has an interesting physical interpretation. Recall that it arose as the convolution with the ("Spherical mean") distribution whose Fourier transform was

$$\frac{e^{ic|\omega|t}-e^{-ic|\omega|t}}{2ic|\omega|}.$$

But, by comparing with (1.37) this is just $\widehat{u}(\omega,t)$ for $\widehat{\phi}(\omega) \equiv 0$ and $\widehat{\psi}(\omega) \equiv 1$. In other words, the "spherical mean" distribution

$$\phi \longmapsto \frac{1}{4\pi c^2 t}\iint_{S(0,ct)} \phi d\sigma$$

(which may also be thought of as $\dfrac{1}{4\pi c^2 t}\delta(r-ct)$) is the solution of the wave equation with initial conditions $\phi(x) \equiv 0$ and $\psi(x) = \delta(x)$, a "point source". Hence $\dfrac{1}{4\pi c^2 t}\delta(r-ct)$ is the so-called "source function" of the three dimensional wave equation and, correspondingly, $\mathcal{A}_t$ the "source operator". Incidentally, the solution of the non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = f(x,t),$$

i.e, with "source" $f(x,t)$, can be nicely expressed in terms of this sorurce operator. Specifically,

$$u(x,t) = \mathcal{A}_t\psi(s) + \frac{d}{dt}\{\mathcal{A}_t\phi(x)\} + \int_0^t \mathcal{A}_{t-s}f(x,s)ds. \tag{1.40}$$

We leave the details to the reader.

1.3.1 Energy

A wave not subject to any "friction" should maintain its total "energy". This is just common sense, but there is a precise mathematical notion that confirms this intuition. Starting with a solution of the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0,$$

we can multiply by $\dfrac{\partial u}{\partial t}$ to obtain

$$\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} - c^2\nabla^2 u\frac{\partial u}{\partial t} = 0.$$

Now, we recognize that

$$\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2$$

and also that we can rewrite $\nabla^2 u\dfrac{\partial u}{\partial t}$, as a derivative with respect to t plus a derivative with respect to the space variables. Specifically,

$$\nabla^2 u\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}\right)^2 + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left\{\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right\}.$$

Hence

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + c^2\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}\right)^2\right\} - c^2\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left\{\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right\} = 0. \tag{1.41}$$

But, by the Fundamental Theorem,

$$\iiint_{\mathbb{R}^3}\frac{\partial}{\partial x_j}\left\{\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right\}dV(x) = 0$$

if $\dfrac{\partial u}{\partial t}\dfrac{\partial u}{\partial x_j}$ is 0 at $\pm\infty$. Hence, if this indeed the case,

$$\iiint_{\mathbb{R}^3}\frac{1}{2}\frac{\partial}{\partial t}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + c^2\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}\right)^2\right\}dV(x) = 0$$

and so

$$\frac{d}{dt}\left\{\iiint_{\mathbb{R}^3}\left(\frac{\partial u}{\partial t}\right)^2+c^2\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}\right)^2 dV(x)\right\}=0.$$

In other words,

$$\iiint_{\mathbb{R}^3}\left(\frac{\partial u}{\partial t}\right)^2+c^2\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}\right)^2 dV(x) \tag{1.42}$$

must be constant, i.e., must be conserved. The quantity (1.42) is known as the *energy* of the system. Of course, since we have made restrictions on u, this energy is not always finite for arbitrary solutions, but in the physical world it would very unusual to encounter a wave of "infinite energy"!

1.3.2 Back to Characteristics

What are the characteristics of the wave equation? Let us first consider the one dimensional equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}.$$

Consider then a curve γ in (x,t) space-time given parametrically as

$$\gamma: \quad x=f(s),\ \ t=g(s).$$

We want to determine if γ is a characteristic or not. Thus we specify some initial data on γ, $u\mid_\gamma=\phi(s)$ and $\left.\frac{\partial u}{\partial n}\right|_\gamma=\psi(s)$ (here $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction normal to γ), and ask whether or not this data determines the solution (at least in a neighbourhood of γ). The first question is whether or not this low derivative information determines even the three second partial derivatives of u, $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x\partial t}$ on γ. We need to calculate a bit. First,

$$u\mid_\gamma=\phi(s) \implies u(f(s),g(s))=\phi(s)$$

so that, differentiating,

$$\frac{\partial u}{\partial x}f'(s)+\frac{\partial u}{\partial t}g'(s)=\phi'(s).$$

Also, the tangent vector to γ is $\frac{d}{ds}\langle f(s),g(s)\rangle=\langle f'(s),g'(s)\rangle$ and so the normal vector is $\langle -g'(s),f'(s)\rangle$. Hence,

$$\left.\frac{\partial u}{\partial n}\right|_\gamma=\frac{\partial u}{\partial x}(-g'(s))+\frac{\partial u}{\partial t}f'(s)=\psi(s).$$

These two equations can be solved (since the determinant is $(f'(s))^2+(g'(s))^2 > 0$) for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in terms of ϕ', ψ, f' and g'. For simplicity, le us denote

$$\left.\frac{\partial u}{\partial x}\right|_\gamma = \alpha(s) \text{ and } \left.\frac{\partial u}{\partial t}\right|_\gamma = \beta(s).$$

Now, we may differentiate again to obtain (on γ)

$$\frac{d}{ds}\frac{\partial u}{\partial x}(f(s), g(s)) = \frac{\partial^2 u}{\partial x^2}f'(s) + \frac{\partial^2 u}{\partial x \partial t}g'(s) = \alpha'(s)$$

and

$$\frac{d}{ds}\frac{\partial u}{\partial t}(f(s), g(s)) = \frac{\partial^2 u}{\partial t \partial x}f'(s) + \frac{\partial^2 u}{\partial t^2}g'(s) = \beta'(s).$$

hence, together with the original PDE we have the 3×3 system for the three partial derivatives:

$$\begin{aligned} 1\frac{\partial^2 u}{\partial t^2} + 0\frac{\partial^2 u}{\partial x \partial t} - c^2\frac{\partial^2 u}{\partial x^2} &= 0 \\ 0\frac{\partial^2 u}{\partial t^2} + g'(s)\frac{\partial^2 u}{\partial x \partial t} + f'(s)\frac{\partial^2 u}{\partial x^2} &= \alpha'(s) \\ g'(s)\frac{\partial^2 u}{\partial t^2} + f'(s)\frac{\partial^2 u}{\partial x \partial t} + 0\frac{\partial^2 u}{\partial x^2} &= \beta'(s). \end{aligned}$$

This system uniquely determines the second partials iff its determinant

$$\begin{vmatrix} 1 & 0 & -c^2 \\ 0 & g'(s) & f'(s) \\ g'(s) & f'(s) & 0 \end{vmatrix} = c^2(g'(s))^2 - (f'(s))^2 \neq 0.$$

In other words, the characteristics are given by

$$c^2(g'(s))^2 = (f'(s))^2,$$

i.e.,

$$f'(s) = \pm c g'(s).$$

Integrating, we have

$$f(s) = \pm c g(s) + k.$$

But then, since $s = f(s)$ and $t = g(s)$,

$$x \pm ct = k,$$

as expected.

The higher dimensional case is somewhat more complicated, and hence we won't give all the details, just enough to understand how the equation for the characteristics arises.

To keep things as simple as possible, let's consider a surface S given in functional form by $t = f(x)$, where $x \in \mathbb{R}^3$. We then impose initial conditions on S (i.e., when $t = f(x)$):

$$u(x, f(x)) = \phi(x) \tag{1.43}$$

and

$$\frac{\partial u}{\partial n}(x, f(x)) = \psi(x)$$

where, again, $\frac{\partial}{\partial n}$ denotes the normal derivative with respect to S. Indeed, since the normal vector to S is just $\vec{n} = \langle \nabla f(x), -1 \rangle$, we have

$$\nabla_x u \cdot \nabla f(x) - \frac{\partial u}{\partial t} = \psi(x). \tag{1.44}$$

We may differentiate (1.43) with respect to x and obtain

$$\nabla_x u(x, f(x)) + \frac{\partial u}{\partial t}(x, f(x)) \nabla f(x) = \nabla \phi(x). \tag{1.45}$$

The linear system (1.44) and (1.45) can be solved (the determinant is non-zero, just as in the one-dimensional case) for $\nabla_x u$ (a vector!) and $\frac{\partial u}{\partial t}$ to obtain

$$\frac{\partial u}{\partial t}(x, f(x)) = \alpha(x) \tag{1.46}$$

$$\frac{\partial u}{\partial x_j}(x, f(x)) = \beta_j(x), \quad 1 \leq j \leq n \tag{1.47}$$

for some functions $\alpha(x)$ and $\beta_j(x)$. Differentiating (1.46) and the jth equation of (1.47) with respect to x_j, we have

$$\frac{\partial^2 u}{\partial t \partial x_j}(x, f(x)) + \frac{\partial^2 u}{\partial t^2}(x, f(x)) \frac{\partial f}{\partial x_j}(x) = \frac{\partial \alpha}{\partial x_j}(x) \tag{1.48}$$

and

$$\frac{\partial^2 u}{\partial x_j^2}(x, f(x)) + \frac{\partial^2 u}{\partial x_j \partial t}(x, f(x)) \frac{\partial f}{\partial x_j}(x) = \frac{\partial \beta_j}{\partial x_j}(x) \tag{1.49}$$

for $j = 1, \cdots, n$.

Then, (1.49) $- \frac{\partial f}{\partial x_j}(x)$(1.48) yields

$$\frac{\partial^2 u}{\partial x_j^2}(x, f(x)) - \left(\frac{\partial f}{\partial x_j} \right)^2 \frac{\partial^2 u}{\partial t^2}(x, f(x)) = \frac{\partial \beta_j}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial \alpha}{\partial x_j}.$$

Adding for $j = 1, \cdots, n$ then results in

$$\nabla^2 u(x, f(x)) - \left\{ \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j} \right)^2 \right\} \frac{\partial^2 u}{\partial t^2}(x, f(x)) = \sum_{j=1}^{3} \left\{ \frac{\partial \beta_j}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial \alpha}{\partial x_j} \right\}.$$

But

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

and so

$$\nabla^2 u(x, f(x)) \left\{ 1 - c^2 \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j} \right)^2 \right\} = \sum_{j=1}^{3} \left\{ \frac{\partial \beta_j}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial \alpha}{\partial x_j} \right\}.$$

Clearly, there is a degeneracy if

$$c^2 \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j} \right)^2 = 1$$

and this is indeed the equation that gives the characteristics of the wave equation. One should check that this is consistent with our previous one-dimensional result. Incidentally, this equation is known as the *eikonal* equation and is important in ray theory.

1.4 Vibrating Strings and Plates

Suppose that we have a taut string or wire between $x = 0$ and $x = \ell$, tied down at both end points. Plucking the string will cause vibrations so that the displacement $u(x, t)$ will satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \ell$$

with inital conditions

$$\begin{aligned} u(x, 0) &= \phi(x) \quad 0 < x < \ell \\ \frac{\partial u}{\partial t}(x, 0) &= \psi(x) \quad 0 < x < \ell. \end{aligned}$$

But now we have to account for the fact that the string is tied down at $x = 0$ and $x = \ell$. We express this through *boundary* conditions

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0 \\ u(\ell, t) &= 0, \quad t \geq 0. \end{aligned}$$

How do we solve this equation? Now, for each fixed t, $u(x, t)$ is a function of x and *if* $u(x, t) \in L^2(0, \ell)$, it has a Fourier sine series expansion

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{\ell} x).$$

Since for each different t, the displacement is a different function of x, the coefficients a_n depend on t, i.e., $a_n = a_n(t)$. Hence, more precisely,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(\frac{n\pi}{\ell} x). \tag{1.50}$$

Since $\sin(\frac{n\pi}{\ell}x)$ is zero for $x = 0$ and $x = \ell$, the boundary conditions are automatically satisfied for such $u(x,t)$ and so we need to find the $a_n(t)$ so that $u(x,t)$ satifies the wave equation and also the initial conditions.

Putting $t = 0$, we see that

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin(\frac{n\pi}{\ell} x)$$

so that

$$a_n(0) = \alpha_n,$$

the Fourier sine coefficient of ϕ (i.e. $\sum_{n=1}^{\infty} \alpha_n \sin(\frac{n\pi}{\ell}x) = \phi(x)$, in the sense of $L^2(0,\ell)$). Similarly,

$$\psi(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} a_n'(0) \sin(\frac{n\pi}{\ell} x)$$

so that $a_n'(0) = \beta_n$, the Fourier coefficients of $\psi(x)$.

Now, what about the wave equation, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, itself? Substituting the series expansion (1.50) into the equation, we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \sum_{n=1}^{\infty} a_n(t) \sin(\frac{n\pi}{\ell} x) &= c^2 \frac{\partial^2}{\partial x^2} \sum n = 1^{\infty} a_n(t) \sin(\frac{n\pi}{\ell} x); \\ \sum_{n=1}^{\infty} a_n''(t) \sin(\frac{n\pi}{\ell} x) &= c^2 \sum_{n=1}^{\infty} a_n(t) \left\{ -\left(\frac{n\pi}{\ell}\right)^2 \sin(\frac{n\pi}{\ell} x) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} a_n''(t) &= -c^2 \left(\frac{n\pi}{\ell}\right)^2 a_n(t), \quad n = 1, 2, \cdots \\ a_n(0) &= \alpha_n \\ a_n'(0) &= \beta_n. \end{aligned}$$

This is an infinite system of second order linear ODEs, but (by some miracle!) it is decoupled, i.e. the equations for each $a_n(t)$ do not interact with each other. Hence the system is easily solved. In fact,

$$a_n(t) = \alpha_n \cos(\frac{n\pi c}{\ell} t) + \frac{\beta_n}{\left(\frac{n\pi c}{\ell}\right)} \sin(\frac{n\pi c}{\ell} t),$$

so that

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \alpha_n \cos(\frac{n\pi c}{\ell} t) + \frac{\beta_n}{\left(\frac{n\pi c}{\ell}\right)} \sin(\frac{n\pi c}{\ell} t) \right\} \sin(\frac{n\pi}{\ell} x).$$

Now, why exactly did this system decouple? The reason is that

$$\frac{d^2}{dx^2} \sin(\frac{n\pi}{\ell} x) = -\left(\frac{n\pi}{\ell}\right)^2 \sin(\frac{n\pi}{\ell}), \tag{1.51}$$

i.e, the second spatial derivative of a basis function is a multiple of *itself* only and involves no other basis functions! Abstractly, this is an example of an eigenvalue problem. The second derivative is a (differential) operator. Let's call it $\mathcal{A}$. Then (1.51) just says that

$$\mathcal{A} \sin(\frac{n\pi}{\ell} x) = \lambda_n \sin(\frac{n\pi}{\ell} x)$$

for

$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2,$$

the nth *eigenvalue* of $\mathcal{A}$. Moreover, the *eigenfunctions* $\sin(\frac{n\pi}{\ell} x)$ satisfy the Boundary Conditions of our equation. Of special importance is the fact that the collection of eigenfuctions $\left\{\sin(\frac{n\pi}{\ell} x)\right\}_{n=1}^{\infty}$ forms a (Hilbert space) basis for $L^2(0,\ell)$, i.e., every function $f \in L^2(0,\ell)$ has an expansion

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{\ell} x).$$

This is not an accident – there is a general theory of when this fortuitous circumstance occurs.

Notice that the eigenfunctions $\sin(\frac{n\pi}{\ell} x)$ are orthogonal with respect to the inner product

$$\langle f, g \rangle := \int_0^{\ell} f(x) g(x) dx.$$

This is again not an accident. The operator $\mathcal{A}$ is what is called self-adjoint, and in this respect an analogue of a symmetric matrix. You might recall that for a symmetric matrix, the eigenvectors belonging to different eigenvalues are always orthogonal. For almost exactly the same reason, the eigenfunctions of a self-adjoint operator that belong to different eigenvalues, must always be orthogonal.

Notice also that the eigenvalues $\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2$ are discrete (indexed by n) and negative. For example, we might wonder whether there are any positive eigenvalues.

But if $\mathcal{A}u = \lambda u$, $u(0) = e(\ell) = 0$, then

$$\int_0^{\ell} (\mathcal{A}u) u dx = \lambda \int_0^{\ell} u^2(x) dx,$$

i.e.,

$$\int_0^\ell \frac{d^2u}{dx^2} u dx = \lambda \int_0^\ell u^2(x) dx.$$

Then, integrating by parts,

$$\left.\frac{du}{dx}u\right|_0^\ell - \int_0^\ell \frac{du}{dx}\frac{du}{dx} dx = \lambda \int_0^\ell u^2(x) dx.$$

But, $u(0) = u(\ell) = 0$ by assumption, and so we have

$$-\int_0^\ell \left(\frac{du}{dx}\right)^2 dx = \lambda \int_0^\ell u^2(x) dx,$$

i.e.,

$$\lambda = -\int_0^\ell \left(\frac{du}{dx}\right)^2 dx \Big/ \int_0^\ell u^2(x) dx$$

which is clearly negative.

Now let us apply the same ideas to a higher dimensional case; specifically to a vibrating (square) plate. For simplicity, let's take the plate to be situated at $0 \le x \le \pi$, $0 \le y \le \pi$ and consider the problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad 0 < x, y < \pi$$

with initial conditions

$$\begin{aligned} u(x, y, 0) &= \phi(x, y) \quad 0 < x, y < \pi \\ \frac{\partial u}{\partial t}(x, y, 0) &= \psi(x, y) \quad 0 < x, y < \pi \end{aligned}$$

and boundary conditions

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0$$

for $0 < x, y < \pi$ and $t \ge 0$.

The boundary conditions express the fact we are considering a plate or membrane attached along its edges to its "frame".

Here the Laplacian ∇^2 plays the role that the second derivative did in the vibrating string case. Our eigenvalue problem becomes

$$\nabla^2 u = \lambda u$$

where u must satisify the boundary conditions above. Since $\{\sin(nx)\}$ are the eigenfunctions for $[0, \pi]$ and the square $[0, \pi] \times [0, \pi]$ is the product of $[0, \pi]$ with itself, it seems reasonable to try and see if the (tensor) product of $\{\sin(nx)\}$

with $\{\sin(ny)\}$ gives the eigenfunctions of the Laplacian on the square. Let's see what happens:

$$\begin{aligned}
\nabla^2(\sin(nx)\sin(my)) &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sin(nx)\sin(my)) \\
&= -n^2\sin(nx)\sin(my) - m^2\sin(nx)\sin(my) \\
&= -(n^2+m^2)\sin(nx)\sin(my).
\end{aligned}$$

Hence $u_{nm}(x,y) := \sin(nx)\sin(my)$ is indeed an eigenfunction of ∇^2 with eigenvalue $\lambda_{nm} - (n^2+m^2)$. Note also that these $u_{nm}(x,y)$ do sastify the boundary conditions, as required. Actually, since the set $\{u_{nm}(x,y)\}$ is dense in $L^2([0,\pi]^2)$, there are no others.

Hence, to solve our vibrating plate problem, we look for an expansion of the form

$$u(x,y,t) = \sum_{n,m=1}^{\infty} a_{nm}(t)u_{nm}(x,y).$$

Such a u automatically staisfies the boundary conditions (since the eigenfunctions do), and so we need only worry about the initial conditions and the differential equation itself.

For the initial conditions, just set $t=0$ to see that

$$\phi(x,y) = u(x,y,0) = \sum_{n,m=1}^{\infty} a_{nm}(0)u_{nm}(x,y)$$

so that $a_{nm}(0) = \alpha_{nm}$, the nm-"Fourier" coefficient of ϕ with respect to the basis $\{u_{nm}\}$. Also,

$$\psi(x,y) = \frac{\partial u}{\partial t}(x,y,0) = \sum_{n,m=1}^{\infty} a'_{nm}(0)u_{nm}(x,y)$$

so that $a'_{nm}(0) = \beta_{nm}$, the nm-"Fourier" coefficient of ψ with respect to the basis $\{u_{nm}\}$.

Now for the differential equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\nabla^2 u \\
\frac{\partial^2}{\partial t^2}\sum_{n,m=1}^{\infty} a_{nm}(t)u_{nm}(x,y) &= c^2\nabla^2\sum_{n,m=1}^{\infty} a_{nm}(t)u_{nm}(x,y) \\
\sum_{n,m=1}^{\infty} a''_{nm}(t)u_{nm}(x,y) &= c^2\sum_{n,m=1}^{\infty} a_{nm}(t)\nabla^2 u_{nm}(x,y) \\
&= c^2\sum_{n,m=1}^{\infty} a_{nm}(t)\lambda_{nm}u_{nm}(x,y).
\end{aligned}$$

Hence,

$$\begin{aligned} a''_{nm}(t) &= c^2\lambda_{nm}a_{nm}(t) = -c^2(n^2+m^2)a_{nm}(t), \\ a_{nm}(0) &= \alpha_{nm}, \\ a'_{nm}(0) &= \beta_{nm} \end{aligned}$$

which is again an infinite system of decoupled equations. They are easily solved:

$$a_{nm}(t) = \alpha_{nm}\cos(c\sqrt{n^2+m^2}t) + \frac{\beta_{nm}}{c\sqrt{n^2+m^2}}\sin(c\sqrt{n^2+m^2}t)$$

so that

$$u(x,y,t) = \sum_{n,m=1}^{\infty}\left\{\alpha_{nm}\cos(c\sqrt{n^2+m^2}t) + \frac{\beta_{nm}}{c\sqrt{n^2+m^2}}\sin(c\sqrt{n^2+m^2}t)\right\}u_{nm}(x,y).$$