Recursive seismic imaging *

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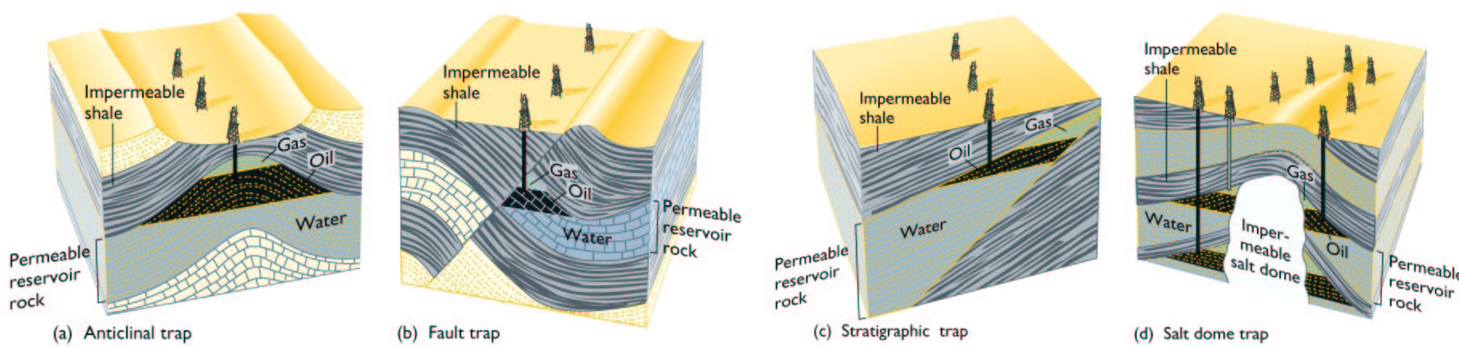
*Seismic wave simulation and seismic imaging: a PIMS summer school 2003

Class Outline

- Seismic reflectivity
  - An intermediary between rocks and waves
- The seismic shot record
  - The basic recording
- Imaging condition
  - Write a reflectivity estimate to the output space
- Wavefield extrapolators
  - One-way for simplicity
- Anisotropy
  - Parameterized in p space - natural for one-way methods
- Assignment
  - Develop and implement an imaging system that will run in a reasonable amount of time

Introduction

- Imaging goals:
  1. Resolve the spatial relationships between rock layers in a subsurface that can be heterogeneous and anisotropic
  2. Identify lithology and fluid content of layers

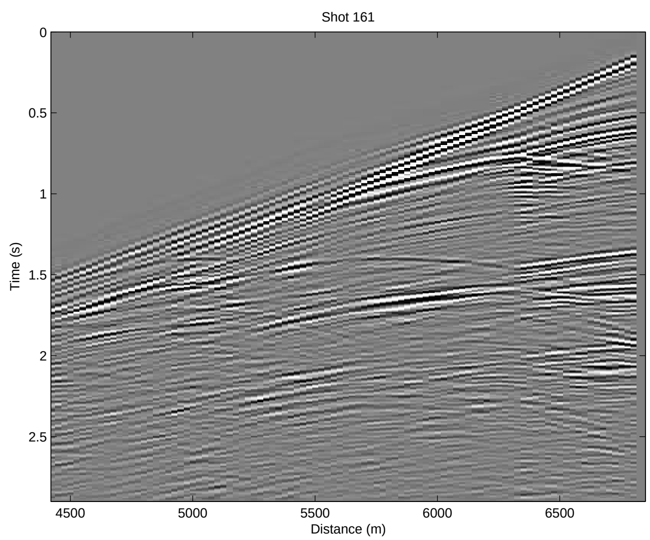


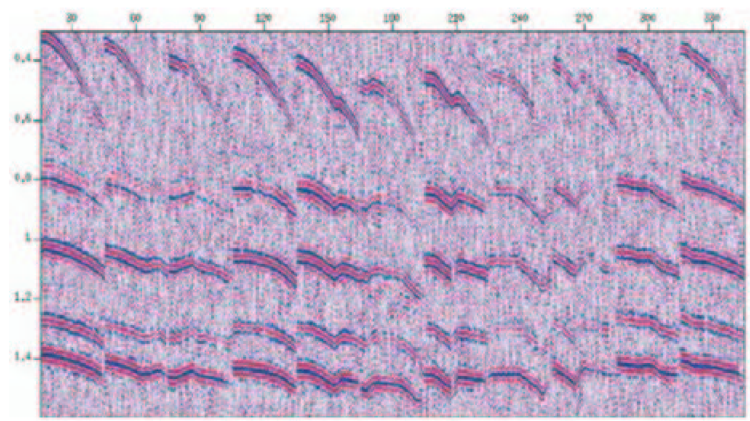
Data and physics

- Data consists of seismic and prior geologic knowledge
- Physics controls how we assemble an image and estimate lithology using the data

Seismic data

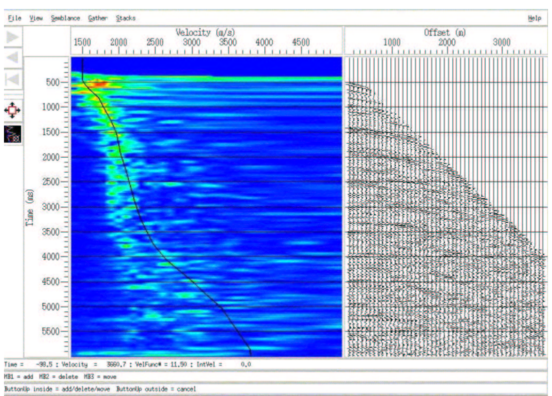
- Bang the ground and record the shaking
  - Shaking comes directly from the source and from echos in the subsurface

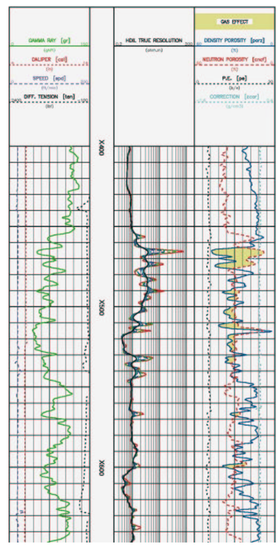


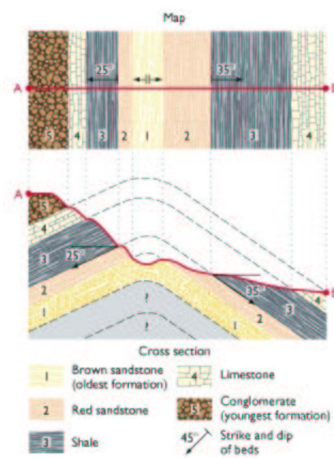


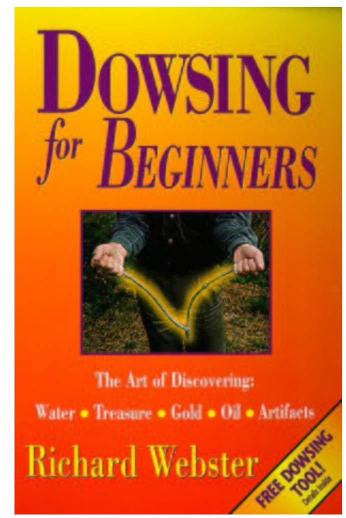
Geologic data

- Obtain a macro model of the subsurface from moveout analysis, log data, geologic outcrop, previous experience, intuition ...



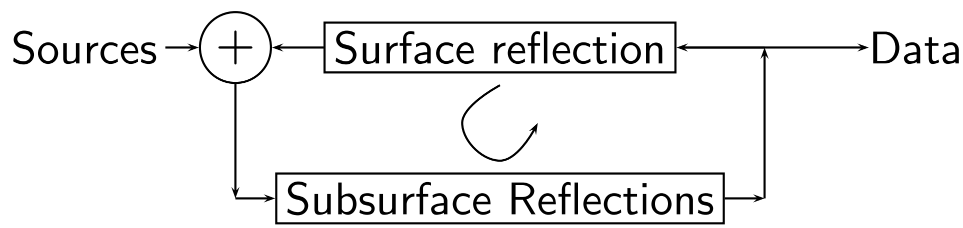






Physics

- Build a physical model of wave propagation specific to our data acquisition



- Derive a wave equation that relates waves and lithology
- Derive a representation of waves consistent with seismic data and the wave equation

Imaging

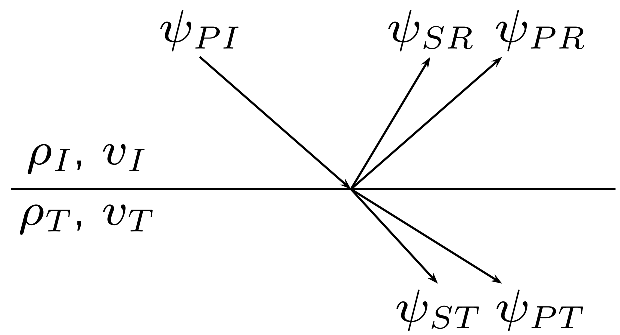
- To locate lithology of interest, manipulate the data according to the physics
- There are four main manipulations generally called *imaging*
  1. Pure inversion – model independent
  2. Constrained inversion – model dependent
  3. Mapping from time to depth using *Kirchhoff migration* followed by inversion – model dependent
  4. Extrapolation of surface data into the subsurface using *recursive migration* followed by inversion – model dependent
- The last of these is the focus of this class

Part 1: Seismic data and reflectivity

- Seismic reflections, along with seismic noise, are recorded as seismic data
- Reflections are generated where incident wavefields encounter changes in ρ and/or v
- Because the intent is to apply recursive imaging to seismic data, it is convenient to parameterize reflectivity with slowness instead of angle
- The recorded data in the form of a **shot gather** will be examined in detail to gain an appreciation for the challenge of locating and identifying different lithologic units

Reflectivity

- At the interface between differing lithologic units, the energy of an incident (I) seismic wave is parceled out between reflection (R) and transmission (T) of P-waves (ψ_P) and S-waves (ψ_S)



where ρ_i are the densities of the upper $(i = 1)$ and lower $(i = 2)$ lithologic layers, and v_i are the corresponding seismic velocities given by

$$v = \left[\langle p, p \rangle + q^2 \right]^{-\frac{1}{2}} \tag{1}$$

where, $p \in \Re^2$ and $q \in \Im^1$ are are slownesses in the horizontal and vertical directions respectively

- From the above figure, energy is assigned to the various wave modes according to the variation in ρ and q across the interface
- If we can relate the energy assigned to ψ_{PR} and ψ_{SR} to ρ and v analytically, we may hope to say something geological using the data that we record
- Similarly for ψ_{SI}

Equations and Boundary conditions

- To quantify ψ_{PR} at a point of reflection, we have at hand the following boundary conditions
  - Continuity of displacement
$$\psi_I + \psi_R = \psi_T \tag{2}$$
  - Continuity of traction
$$\tau_I + \tau_R = \tau_T \tag{3}$$
- Also, we have the following analytic relationships:
  - Planewave representation of seismic wavefields
$$\psi\left(x, z, t\right) = \frac{1}{\left(2\pi\right)^2} \int \varphi\left(p, \omega\right) e^{\pm i\omega\left[\langle p, x\rangle + q(p)z - t\right]} dp d\omega \tag{4}$$
  - Hooke's Law relating tractions τ to infinitesimal strains ε through the elastic coefficients C
$$\tau_{ij} = C_{ijkl}\,\varepsilon_{k,l} \tag{5}$$

where indecies i, j, k and l can take on values of 1, 2, and 3, and ε is related to displacement u through

$$\varepsilon_{kl} = \frac{1}{2}\left[u_{k,l} + u_{l,k}\right] \tag{6}$$

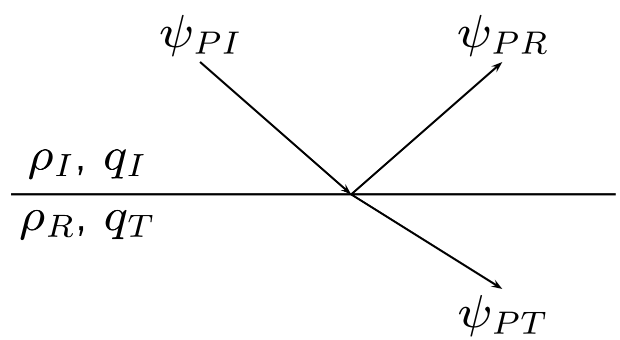
- For an elastic medium there are 36 dependent elastic coefficients – certain crystal symmetries can significantly reduce this number
- Continuity of displacement (equation 2) is satisfied at the interface ($z = 0$) by substituting a planewave (equation 4, constant p and ω) for the various ψ's with the result

$$\varphi_I + \varphi_R = \varphi_T \tag{7}$$

- Satisfaction of continuity of tractions (equation 3) is a little more difficult, so the following simplifying example is instructive

Fluid/fluid

- A boundary between two inviscid media provides a simple scenario that we can analyze to gain insight



  - This will lead us to an *imaging condition* and a very useful approach to prestack depth migration
- Only the trace of ε is non-zero, the Lamé parameter $\mu = 0$, and Hooke's Law is greatly simplified

$$\tau_{ii} = \lambda \varepsilon_{ii} = \lambda u_{i,i} \tag{8}$$

- ... and at the fluid/fluid interface, only tractions normal to the boundary are conserved

$$\tau_{33} = \lambda u_{3,3} \tag{9}$$

and

$$\lambda_I \left[u_{3,3;I} + u_{3,3;R} \right] = \lambda_T u_{3,3;T} \tag{10}$$

- Upon substitution of the planewave (equation 4) for u, conservation of tractions at the interface $(z = 0)$ is written

$$\begin{aligned} &\lambda_I \, q_I \left(p, \omega\right) \left[\varphi_I \left(p, \omega\right) - \varphi_R \left(p, \omega\right) \right] \\ &= \lambda_T \, q_T \left(p, \omega\right) \varphi_T \left(p, \omega\right) \end{aligned} \tag{11}$$

- Now, by defining

$$R\left(p, \omega\right) = \frac{\varphi_R \left(p, \omega\right)}{\varphi_I \left(p, \omega\right)} \tag{12}$$

and

$$T\left(p, \omega\right) = \frac{\varphi_T \left(p, \omega\right)}{\varphi_I \left(p, \omega\right)} \tag{13}$$

equations (7) and (11) give us 2 equations in 2 unknowns (R and T)

- Solving for R

$$R\left(p,\omega\right)=\frac{\lambda_I\, q_I\left(p,\omega\right)-\lambda_T\, q_T\left(p,\omega\right)}{\lambda_I\, q_I\left(p,\omega\right)+\lambda_T\, q_T\left(p,\omega\right)} \tag{14}$$

... and for T

$$T\left(p,\omega\right)=\frac{2\lambda_I\, q_I\left(p,\omega\right)}{\lambda_I\, q_I\left(p,\omega\right)+\lambda_T\, q_T\left(p,\omega\right)} \tag{15}$$

- R is the *reflection coefficient* and controls the percentage of downgoing ψ_I that is converted into upgoing ψ_R
- T is the *transmission coefficient* and controls the percentage of downgoing ψ_I that is converted into transmitted ψ_T
- Though R and T are derived here for the special case of a fluid/fluid interface, some general statements can be made:
  - Because R and T depend on $p=\frac{\sin\theta}{v}$, they are *angle dependent*
  - Planewaves are multiplied by R and T

$$\varphi_R\left(p,\omega\right)=R\left(p,\omega\right)\varphi_I\left(p,\omega\right) \tag{16}$$

$$\varphi_T\left(p,\omega\right)=T\left(p,\omega\right)\varphi_I\left(p,\omega\right) \tag{17}$$

– Monochromatic wavefields are convolved with R and T

$$\psi_R(x,\omega) = R(x,\omega) * \psi_I(x,\omega) \tag{18}$$

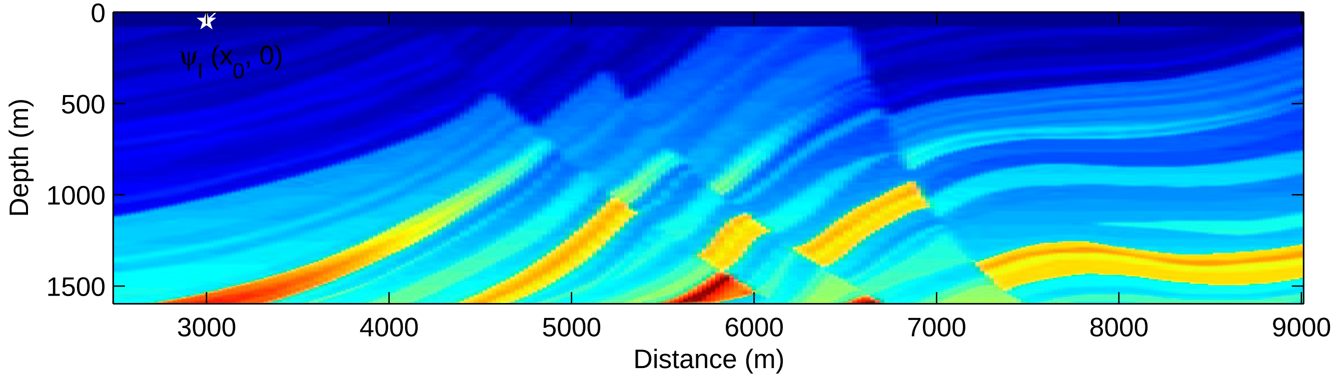
$$\psi_T(x,\omega) = T(x,\omega) * \psi_I(x,\omega) \tag{19}$$

The seismic shot record

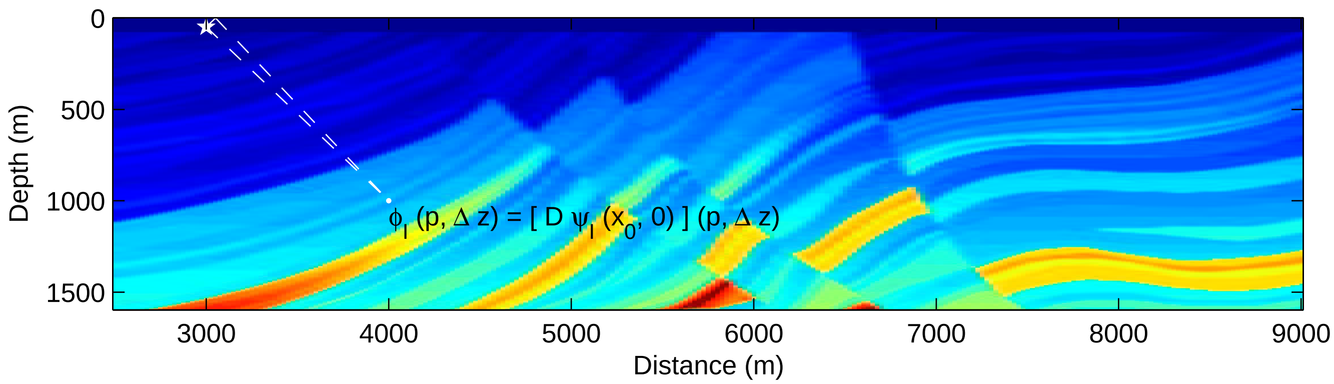
The seismic shot record is the fundamental measure in a seismic survey.

- Through derivation of R and T above, relationships are established between seismic waves and rock properties
  - Wave amplitudes are a measure of R and T at interfaces, and R and T are related analytically to ρ and v of the bounding media
- Recorded wavefields are a superposition of all wave modes originating at all points in the subsurface including:
  - Reflected P- and S-waves
  - Refracted P- and S-waves (q_P and q_S are complex valued)
  - Surface multiples, internal multiples
  - Surface waves (land) (q_P and q_S are complex valued)
- To restrict ourselves to the current practice of seismic imaging, we will concentrate on reflected waves and eliminate all waves corresponding to complex q

- For reflected arrivals $(q \geq 0)$, the following process occurs:



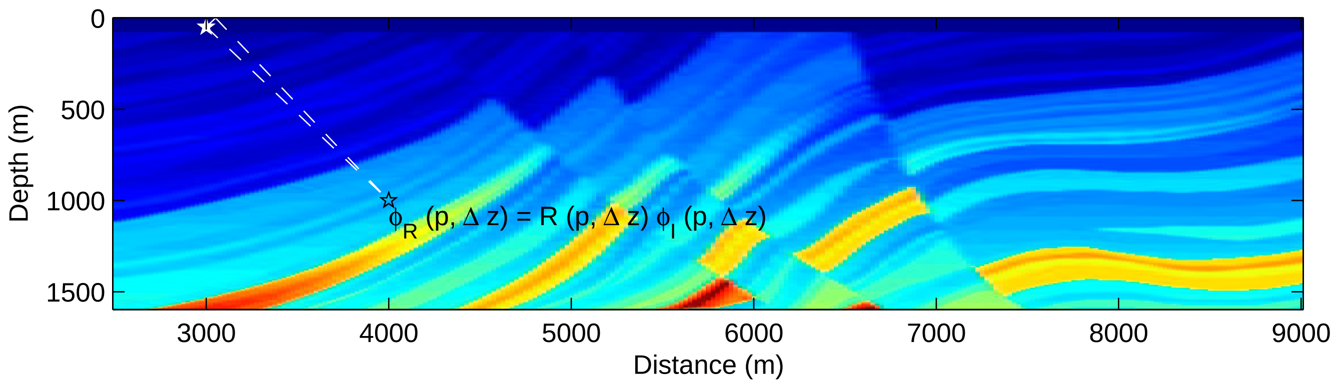
1. A seismic source ψ_I is excited



2. The source propagates $Down$ to $z = \Delta z$

$$\varphi_I\,(p, \Delta z) = \left[D\psi_I\,(x_0, 0)\right](p, \Delta z) \tag{20}$$

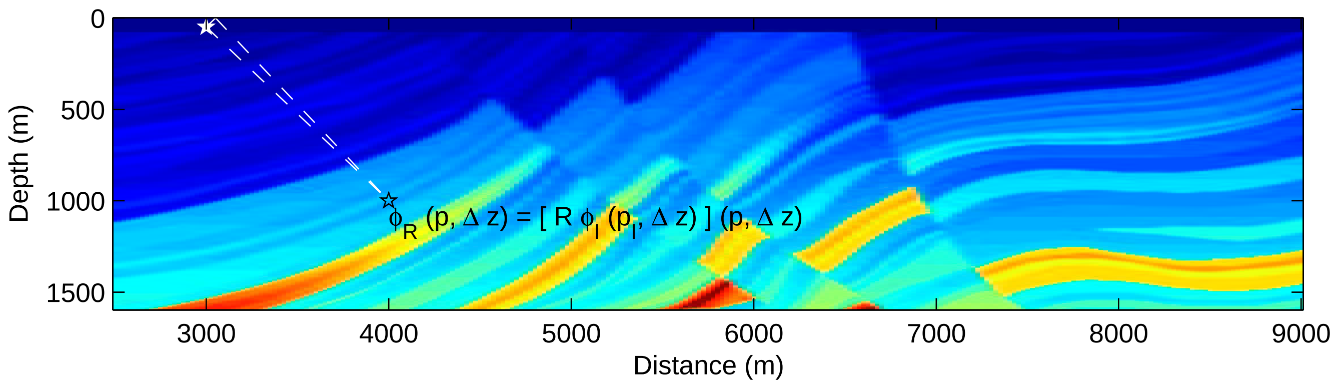
D is a wavefield extrapolator that incorporates the anisotropy and heterogeneity of the medium between Δz and 0, as well as the $x_0 \longrightarrow p$ transform



3. At Δz, part of ψ_I is converted to ψ_R

$$\varphi_R \left(p, \Delta z\right) = R \left(p, \Delta z\right) \varphi_I \left(p, \Delta z\right) \tag{21}$$

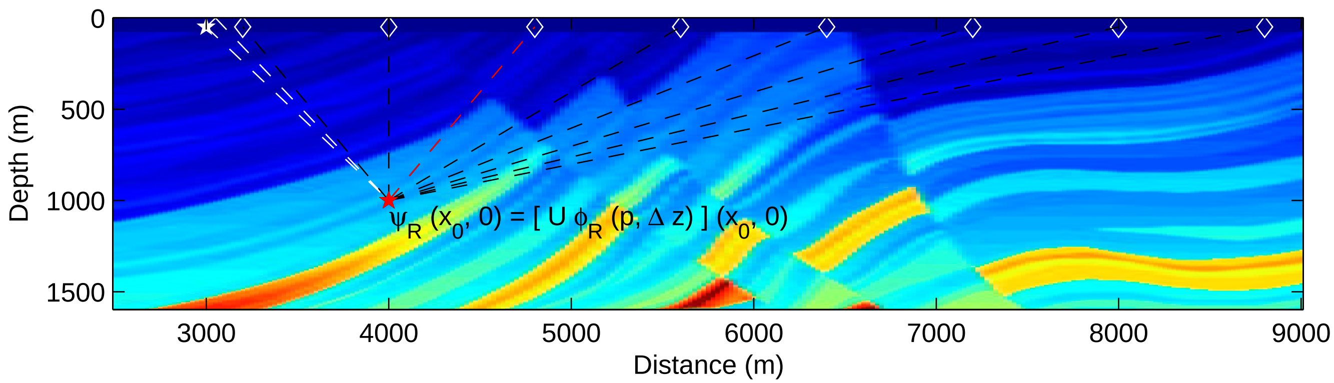
(multiplication when $p = p_I = p_R$)



4. If arbitrary geologic dip is encountered so that, all directions $p \neq p_I$ must be accounted for

$$\varphi_R\left(p, \Delta z\right) = \left[R\varphi_I\left(p_I, \Delta z\right)\right]\left(p, \Delta z\right) \tag{22}$$

(R is now some kind of operator)



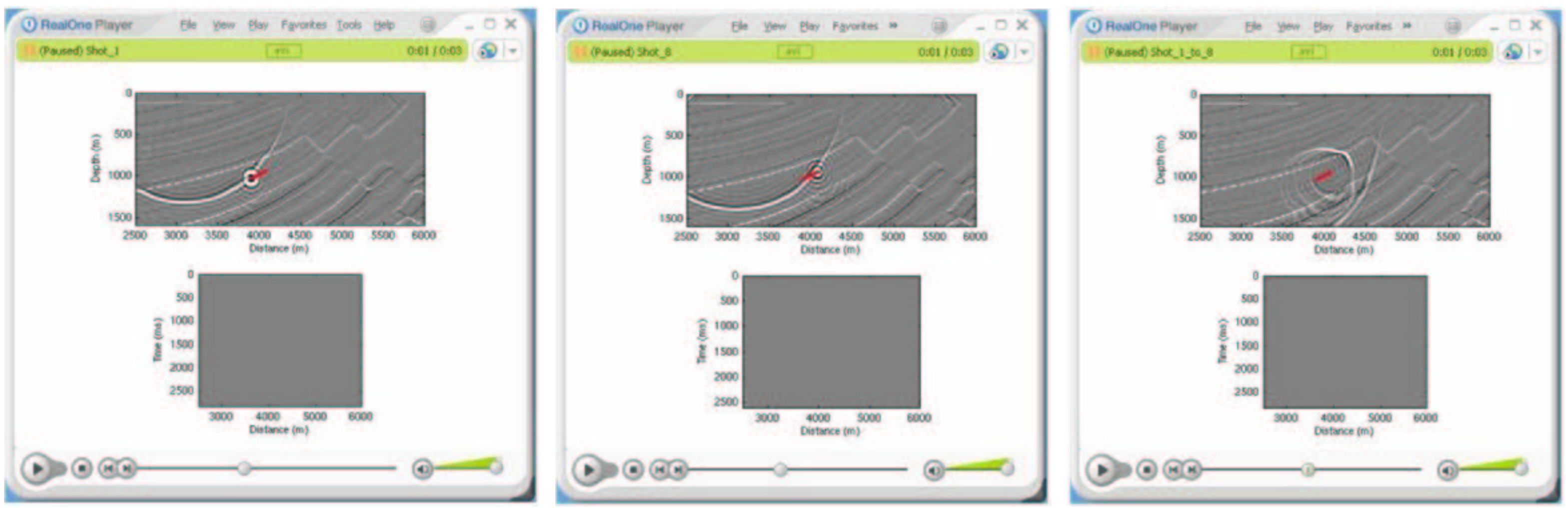
5. Reflected wavefield φ_R propagates $Upward$

$$\psi_R\,(x_0,0) = [U\varphi_R\,(p,\Delta z)]\,(x_0,0) \tag{23}$$

U is a wavefield extrapolator that incorporates the anisotropy and heterogeneity of the medium between Δz and 0, as well as the $p \longrightarrow x_0$ transform

Non-specular vs. specular reflections

- So far, discussion of R has been restricted to point reflectors
- If linear reflectors can be represented as a set of point reflectors, then ψ_R can be represented as a superposition of ψ_{Rj} from j point reflectors

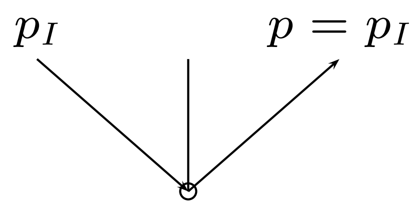


Part 2: Imaging conditions and wavefield extrapolators

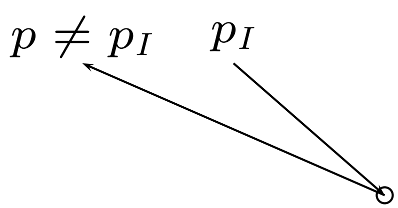
- In recursive seismic imaging, the **imaging condition** and the **wavefield extrapolators** are critical to ensuring the legitimacy of the estimate of R
- From the model of the **shot record**, the imaging condition will be seen to be the point at which the downgoing wave is converted into an upgoing wave
  - Imaging condition is a term common in geophysics and means simply "...estimate R and write it to the output space..."
- The wavefield extrapolators are simply the operators D and U used to carry ψ_I to a reflection point (imaging condition), and from there up to the surface

Dip dependent reflectivity

- In the p domain, multiplicative R was worked out for a horizontal reflector

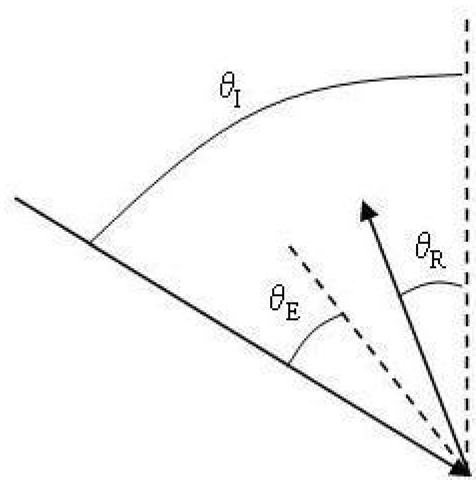


- For scatterers, (step 4 from the section above) an explicit relationship between p_I and p is required



Effective p

- For $p \neq p_I$ effective slowness p_e is computed based on the following picture



$$\begin{aligned} vp_E &= \sin\left(\frac{\theta_I + \theta_R}{2}\right) \\ &= \sqrt{\frac{1 - \cos\left(\theta_I + \theta_R\right)}{2}}, \text{ (half angle formula)} \\ &= \sqrt{\frac{1 - \cos\theta_I \cos\theta_R + \sin\theta_I \sin\theta_R}{2}}, \text{ (addition formula)} \end{aligned} \tag{24}$$

then, using $\sin\theta = vp$ and $\cos\theta = vq$ the effective slowness p_E is:

$$p_E = \Re\left\{\frac{1}{v\sqrt{2}}\sqrt{1 - v^2\left(p_I p - q_I q\right)}\right\} \tag{25}$$

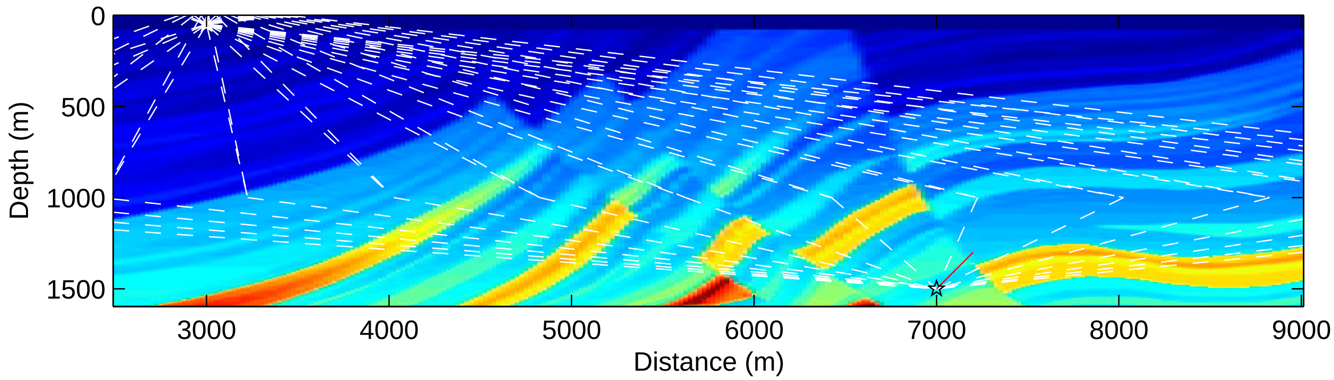
- Real valued p_E restricts ψ_I to downgoing wavefields
- Vertical slowness q is given in terms of p as

$$\begin{aligned} q &= \frac{1}{v}\sqrt{1-(vp)^2} \\ &= \Re\{q\} + i\,|\Im\{q\}| \end{aligned} \quad (26)$$

- Equation (26) ensuers that only reflections are considered

Apply dip-dependent R

- Consider reflection from the next level in our model



- Incident wavefield ψ_I now comes from all directions
- Reflected ψ_R in the p direction from a single point is now a superposition of scaled ψ_I's

φ_R for a single p

$$\begin{array}{c}\varphi_R \\ [\varphi_p]\end{array} = \begin{bmatrix} R_{p,-n} & \dots & R_{p,0} & \dots & R_{p,n} \end{bmatrix} \begin{array}{c}\varphi_I \\ \begin{bmatrix} \varphi_{-n} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_n \end{bmatrix}\end{array} \quad (27)$$

Subscripts for p are used for simplicity, and index n corresponds to the spatial Nyquist frequency $n\Delta p$

φ_R for all ps

$$\begin{bmatrix} \varphi_R \\ \varphi_{-n} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} R_{-n,-n} & \cdots & \cdots & \cdots & R_{-n,n} \\ \vdots & \ddots & & & \vdots \\ \vdots & & R_{0,0} & & \vdots \\ \vdots & & & \ddots & \vdots \\ R_{n,-n} & \cdots & \cdots & \cdots & R_{n,n} \end{bmatrix} \begin{bmatrix} \varphi_I \\ \varphi_{-n} \\ \vdots \\ \varphi_0 \\ \vdots \\ \varphi_n \end{bmatrix} \tag{28}$$

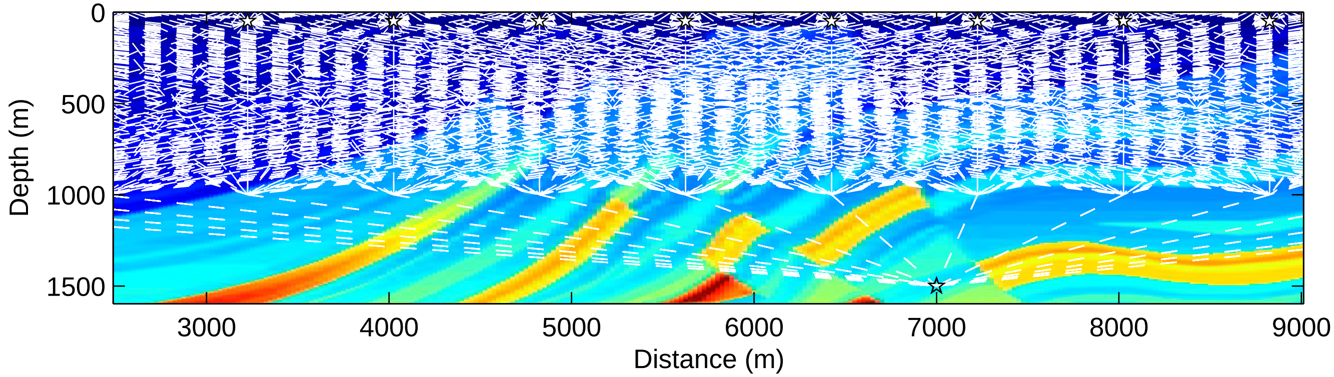
where subscript n corresponds to the spatial Nyquist frequency

- Equation (28) can be written compactly in vector notation

$$\vec{\varphi_R} = \vec{\vec{R}}\vec{\varphi_I} \tag{29}$$

All shot records

- R as given by equations (29) and (28) are for one monochromatic shot record
- Because a seismic survey involves many shot records with significant overlap, each reflection point is sensed multiple times



- For each reflection point, *all* sources and receivers can be included in the vector relationship

$$\left[\varphi_{\vec{R}_1}\ \varphi_{\vec{R}_2}\ \cdots\right] = \vec{\vec{R}}\left[\varphi_{\vec{I}_1}\ \varphi_{\vec{I}_2}\ \cdots\right] \tag{30}$$

or more compactly,

$$\vec{\vec{\varphi}}_R = \vec{\vec{R}}\vec{\vec{\varphi}}_I \tag{31}$$

Estimate R

- From equation (31) we have representation of all reflected wavefield in terms of all incident wavefields and angle dependent R
- So, for a reflection point, if all φ_R and φ_I are known, R can be computed by inversion

$$\vec{\vec{R}} = \left[\vec{\vec{\varphi}}_I^{\dagger} \vec{\vec{\varphi}}_I\right]^{-1} \vec{\vec{\varphi}}_I^{\dagger} \vec{\vec{\varphi}}_R \tag{32}$$

- Seismic imaging, also called seismic *migration*, by recursive methods refers to the process of estimating R for all points in the subsurface
- So, for point x, z in the subsurface:
  1. Compute spectrum $\varphi_I(p, z)$ for each wavefield $\psi_I(x_0, 0)$ using forward wavefield extrapolator D
  2. Compute $\varphi_R(p, z)$ for each $\psi_R(x, 0)$ using reverse wavefield extrapolator U^{-1}
  3. Compute R using equation (32)

4. Sum and normalize R over ω
5. Write to the output array

- In migration, steps 4 and 5 are the imaging condition
  - Each ω provides an estimate of R, and summing averages them

Wavefield extrapolators

- Until this point, discussion has concentrated on:
  - Modeling: compute reflected wavefield ψ_R from incident wavefield ψ_I and reflectivity R
  - Imaging: compute R from ψ_I and ψ_R
- For modeling and imaging, ψ_I and ψ_R must be computed from wavefields at the surface
  - For ψ_I we need a model of the source near the source location
  - For ψ_R we have the recorded wavefield
- Extrapolators U and D are required to move the wavefields around
- The choice of extrapolator depends on:
  - Knowledge of the variation of ρ and v between the surface and the target
    * If known poorly, and exact extrapolator may not tolerate error compared to an approximate operator

  - Computer power available
    * An approximate extrapolator may have better run times compared to an exact operator

- Generally, recursion based imaging methods use *one-way* operators derived from the wave equation
  - Two-way operators (finite differences or eignvalue decomposition) are not favored due to long run times and low tolerance for error, though they are very often used in modeling

One-way wavefield extrapolation

- From Taylor series, a monochromatic wavefield $\psi\left(z+\Delta z\right)$ can be obtained from a wavefield $\psi\left(z\right)$ recorded at the surface

$$\psi\left(z+\Delta z\right)=\sum_{j=0}^{\infty}\frac{\left(\Delta z\right)^{j}}{j!}\partial_{z}^{j}\psi\left(z\right) \tag{33}$$

- To find the required ∂_{z}^{j}, begin with the exact description of ∂_{z}^{2} from the Helmholtz equation

$$\partial_{z}^{2}\psi\left(z\right)=-\left[\left(\frac{\omega}{v}\right)^{2}+\partial_{x}^{2}\right]\psi\left(z\right) \tag{34}$$

- Replacing ψ on the right hand side with the inverse transform of it's spectrum

$$\psi(x,z) = \frac{1}{(2\pi)^2} \int \varphi(p,z)\, e^{-i\langle \omega p, x\rangle} dp \tag{35}$$

we have

$$\partial_z^2 \psi(x,z) = -\frac{1}{(2\pi)^2} \int \left[\left(\frac{\omega}{v} \right)^2 + \partial_x^2 \right] \varphi(p,z)\, e^{-i\langle \omega p, x\rangle} dp \tag{36}$$

- Note, by moving v inside the $p \longrightarrow x$ transform isotropic v is assumed
- In equation (36), ∂_x^2 acts only on the Radon kernal

$$\begin{aligned} &\partial_z^2 \psi(x,z) = \\ &-\frac{1}{(2\pi)^2} \int \varphi(p,z) \left[\left(\frac{\omega}{v} \right)^2 - \omega^2 \langle p, p\rangle \right] e^{-i\langle \omega p, x\rangle} dp \\ &= \frac{1}{(2\pi)^2} \int \varphi(p,z) \left[i\omega\, q(p) \right]^2 e^{-i\langle \omega p, x\rangle} dp \end{aligned} \tag{37}$$

where

$$\begin{aligned} q\left(p\right) &= \frac{1}{v}\sqrt{1 - v^2\left\langle p, p\right\rangle} \\ &= \Re\left\{q\left(p\right)\right\} + i\,\mathsf{sign}\left\{\Delta z\omega\right\}\left|\Im\left\{q\left(p\right)\right\}\right| \end{aligned} \tag{38}$$

- In extrapolation, q is a phase term $\phi = i\omega\Delta zq$, and the second line of equation (38) ensures $\phi\left(\Im\left\{q\right\} \neq 0\right)$ is real and negative
  - Also, q parameterized as above introduces dispersion when $v = v\left[1 + i\gamma\right]$ for $\gamma << 1$
- From equation (37), infer that

$$\partial_z^j\psi\left(x, z\right) \approx \frac{1}{\left(2\pi\right)^2}\int\left[\pm i\omega\, q\left(p\right)\right]^j\varphi\left(p, z\right)e^{-i\left\langle\omega p, x\right\rangle}dp \tag{39}$$

Accuracy of depth derivatives ∂_z^n

- The prescription for ∂_z^n given by equation (39) is exact for $n = 2$, but it is approximate for all $n \neq 2$
- To see this, compare two applications of ∂_z to ∂_z^2

$$\partial_z \partial_z \psi\left(x, z\right) = \frac{1}{\left(2\pi\right)^2} \int \varphi\left(p, z\right) c\left(x, p\right) e^{i\omega \langle p, x\rangle} dp \tag{40}$$

where, from the composition formula for two pseudo-differential operators

$$c\left(x, p\right) = \left[i\omega\, q\left(x, p\right)\right]^2 + \left(i\omega\right)^2 \sum_{j=1}^{\infty} \frac{i^j}{j!} \partial_p^j q\left(x, p\right) \partial_x^j q\left(x, p\right) \tag{41}$$

- The series terms in equation (41) represent error
  - Error is a function of the variation of q with x and p

- Here are some generalizations
  1. ∂_z is exact for invariant isotropic-media,
  2. ∂_z^j is exact for invariant isotropic-media,
  3. ∂_z is approximate for variable isotropic/anisotropic-media,
  4. ∂_z^j is approximate for variable isotropic/anisotropic-media
- Note, generalizations 2 and 4 can be proved independently

Replace ∂_z^n in the series for $\psi\left(z+\Delta z\right)$

- Returning to the series description for $\psi\left(z+\Delta z\right)$, substitute equation (39) for $\partial_z^n\psi$

$$\psi\left(x,z+\Delta z\right)=$$
$$\frac{1}{\left(2\pi\right)^2}\int\varphi\left(p,z\right)\left\{\sum_{j=0}^{\infty}\frac{1}{j!}\left[i\omega q\left(x,p\right)\Delta z\right]^j\right\}e^{-i\langle\omega p,x\rangle}dp \tag{42}$$

- Recognizing the exponential in the $\{\,\}$ braces above

$$\psi\left(x,z+\Delta z\right)=\frac{1}{\left(2\pi\right)^2}\int\varphi\left(p,z\right)e^{i\Delta z\omega q\left(x,p\right)}e^{-i\langle\omega p,x\rangle}dp \tag{43}$$

note that $+q$ has been selected to be consistent with the sign on Δz

- Equation (43) provides wavefield $\psi(z+\Delta z)$ from spectrum $\varphi(z)$ coincident with a $p \longrightarrow x$ transform – it is suitable for the U extrapolator of the model of the shot record
- A development similar to the above leads to a second extrapolator

$$\varphi(p, z+\Delta z) = \int \psi(x, z)\, e^{i\Delta z \omega q(x,p)} e^{i\langle \omega p, x\rangle} dx \tag{44}$$

- Equation (44) computes $\varphi(z+\Delta z)$ based on input $\psi(z)$ coincident with a $x \longrightarrow p$ transform – it is suitable for the D extrapolator of the model of the shot record

Up and down going waves

- For heterogeneous anisotropic media, equation (43) provides an approximate form for extrapolator D

$$[D\psi(x,z)](p,z+\Delta z) = \int \psi(x,z)\, e^{i\Delta z \omega q(x,p)} e^{-i\langle \omega p, x\rangle} dx \tag{45}$$

- Extrapolator U is given by equation (44)

$$[U\varphi(p,z+\Delta z)](x,z) = \frac{1}{(2\pi)^2} \int \varphi(p,z+\Delta z)\, e^{i\Delta z \omega q(x,p)} e^{-i\langle \omega p, x\rangle} dp \tag{46}$$

Reverse the up going waves

If spectra φ_I and φ_R can be estimated for all points in the subsurface, $R\left(p\right)$ can be computed for all points in the subsurface Spectrum $\varphi_I\left(z+\Delta z\right)$ can be computed using D and a model of the source $\varphi\left(z\right)$, but spectrum $\varphi_R\left(z+\Delta z\right)$ must be computed by reversing wavefield $\psi_R\left(z\right)$ and transforming $x\rightarrow p$

- φ_I is computed by extrapolating a model of the source using D

$$\varphi_I\left(p,z+\Delta z\right)=\left[D\psi\left(x,z\right)\right]\left(p,z+\Delta z\right) \quad (47)$$

- φ_R is computed by reversing the recorded wavefield using U^{-1}

$$\varphi_R\left(p,z+\Delta z\right)=\left[U^{-1}\psi_R\left(x,z\right)\right]\left(p,z+\Delta z\right) \quad (48)$$

where,

$$U^{-1}=\left[U^{\dagger}\,U\right]^{-1}U^{\dagger} \quad (49)$$

Summary of assumptions for wavefield extrapolator

- Along the way some assumptions were made
  - Isotropic v was assumed for exact ∂_z^2
  - Stationary v in x and z was assumed for approximate ∂_z^n
- Q: Can we relax the assumptions of stationarity and isotropy of v?
  - For $\Delta z \rightarrow 0$, v approaches stationarity in z
  - For practicality, an approximate R relation will be used for recursive imaging – perfect amplitudes cannot be demanded
  - For the medium, ρ and v are only approximately known (or why would we be imaging and inverting?) – perfect amplitudes and kinematics cannot be demanded
  - The medium may vary slowly and ∂_x terms in the error series may not be significant
  - Similarly anisotropy
  - Built upon the stationary assumption, reflections and mode-conversions not generated – fewer spurious amplitudes in media where ρ and v are not known
- A: Low-resolution R is usually better than no image

Thin layers and cracks

- Analytic descriptions for velocity in heterogeneous media are rare
- Common are analytic descriptions of anisotropy
- Some say that all rocks are fractured at all scales and, waves propagate anisotropically (propagate at different speeds) depending on their orientation to the fractures
- Others say that stacks of thin isotropic layers ($\Delta z <$ a seismic wavelength) constitute anisotropic media Most agree that, besides being heterogeneous, rocks are anisotropic
- In many sedimentary environments, part of the anisotropy of rocks is the result of compaction and lithification of sediments under gravitational loading
- Presumably, the tiny grains of the original sediments settle with their longest dimension normal to gravity
- As sediments accumulate, pore space is closed, fluids are driven out, and the dimensional sorting of grains is exaggerated
- Also, as seas advance and retreat, different sediments are deposited in layer upon layer of often very thin accumulations

- Sedimentary rocks, in particular shales, tend to be strongly anisotropic relative to an axis normal to deposition

Anisotropy

- In an elastic medium, *conservation of force* due to the passage of a seismic wave is given by

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j} \tau_{ij} \tag{50}$$

where subscript $i = 1, 2, \text{or } 3$ is direction, $j = 1, 2, \text{or } 3$ is the side of an infinitesimal cube of density ρ, u is displacement, $\ddot{u}$ is acceleration of particles within the cube, and τ are pressures due to the passage of the wave that *depend on direction*, and upon what face of the cube they are applied

- Using Hooke's Law equation (5), equation (50) becomes

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j} C_{ijkl} \frac{\partial u_k}{\partial x_l} \tag{51}$$

- Assuming homogeneous C, equation (51) becomes

$$\rho \ddot{u}_i = C_{ijkl} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_l} \tag{52}$$

- Using the plane wave description of equation (4), $\ddot{u}_i$ is

$$\ddot{u}_i = (i\omega)^2 \, u_i \tag{53}$$

and the spatial derivatives $u_{k,l}$ are

$$\frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_l} = (i\omega)^2 \, u_i \frac{\partial \hat{t}}{\partial x_j} \frac{\partial \hat{t}}{\partial x_l} \tag{54}$$

where

$$\hat{t} = p_1 x_1 + p_2 x_2 + p_3 x_3 \tag{55}$$

so that equation (52) can be written

$$\frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_l} = (i\omega)^2 \, u_k p_j p_l \tag{56}$$

- Based on equation (52), and using equations (56) and (53), and defining $u_i = \delta_{ik} u_k$, the solution for $q = p_3$ can be determined by solving

$$\text{Det}\left[\frac{C_{ijkl}}{\rho} p_j p_l - \delta_{ik}\right] = 0 \tag{57}$$

for p_3

- The three solutions for p_3 correspond to P-waves, and the fast and slow S-waves

PIMS summer school 2003, Thursday assignment

In seismic imaging using recursive migration, an angle dependent estimate of reflectivity is made for every point in the subsurface. This assignment is concerned with determining the computational cost of the major steps, making assumptions to reduce the cost, and creating a MATLAB program to image the shot gather produced Monday.

1. Part of the process of computing R is obtaining $\varphi_I(z+\Delta z)$ from downward propagation of a model of the source $\psi_I(z)$.
   (a) If $\varphi_I(z+\Delta z)$ is computed using equation (45) estimate the number of computations required in 2D.
   (b) Assume $q(x,p) = a(x) + b(p)$, modify equation (45) accordingly, and re-estimate the number of computations.
2. The next step toward computing R is obtaining $\varphi_R(z+\Delta z)$ from reverse propagation of the reflected wavefield $\psi_R(z)$.
   (a) If $\varphi_R(z+\Delta z)$ is computed using equation (46), what is the cost in 2D.

(b) Assume $q\left(x,p\right)=a\left(x\right)+b\left(p\right)$, modify equation (46) accordingly, and re-estimate cost.

3. Suggest mathematical descriptions for $a\left(x\right)$ and $b\left(p\right)$.
4. Having obtained φ_I and φ_R efficiently, R is ready to be computed using matrix equation (32).
   (a) Estimate the cost in 2D (assume m shots and m receivers/shot)
   (b) Assume $R\left(p\right)=R\left(p_I\right)$ and re-estimate the cost
   (c) Assume $R\left(p\right)=R$ and re-estimate the cost
5. Using the FFT provides significant computational savings when q is approximated as $a\left(x\right)+b\left(p\right)$. Are there any sacrifices using FFT in the computation of approximate U, D? What about exact/approximate R?

6. Earlier in the class, students generated a shot record using a finite difference operator. Using the following program outline, write a MATLAB program to recursively image your shot record for the P - P mode. MATLAB function `ps_space.m` will be provided to extrapolate wavefields. Extra credit is awarded for cleverness and cunning.

Pseudocode for recursive imaging

```
List of arrays:
I -- Model of source in (t, x)
R -- Shot record in (t, x)
M -- Velocity model in (z, x)
input(I, R, M)
fft(I) % FT t -> w
fft(R) % FT t -> w
for j = 1:nz % nz is the number of depths in M
   temp1 = forward(I, M)
   temp2 = reverse(R, M)
   REFLECTIVITY(j,:)  = image(temp1, temp2)
   I = temp1
   R = temp2
end
output(REFLECTIVITY)
```

PIMS summer school 2003, Thursday assignment: answer key

In seismic imaging using recursive migration, an angle dependent estimate of reflectivity is made for every point in the subsurface. This assignment is concerned with determining the computational cost of the major steps, making assumptions to reduce the cost, and creating a MATLAB program to image the shot gather produced Monday.

1. Part of the process of computing R is obtaining $\varphi_I(z+\Delta z)$ from downward propagation of a model of the source $\psi_I(z)$.
   (a) If $\varphi_I(z+\Delta z)$ is computed using equation (45) estimate the number of computations required in 2D.
   (b) Assume $q(x,p) = a(x) + b(p)$, modify equation (45) accordingly, and re-estimate the number of computations.
   (a) A: Because of the dependence of q on p, the FFT can't be used, and D costs as much as a slow Fourier transform $\sim m^2$ for m receivers. **Cost is** $\sim \mathbf{m^2}$
   (b) A: Using $q(x,p) = a(x) + b(p)$, D becomes

$$[D\psi(x,z)](p, z+\Delta z) = e^{i\Delta z \omega b(p)} \int \psi(x,z)\, e^{i\Delta z \omega a(x)} e^{i\omega p x} dx. \qquad (58)$$

   The FFT can be used, and **cost is** $\sim \mathbf{m} \log \mathbf{m}$.

2. The next step toward computing R is obtaining $\varphi_R(z + \Delta z)$ from reverse propagation of the reflected wavefield $\psi_R(z)$.

   (a) If $\varphi_R(z + \Delta z)$ is computed using equation (46), what is the cost in 2D.

   (b) Assume $q(x, p) = a(x) + b(p)$, modify equation (46) accordingly, and re-estimate cost.

   (a) A: Like D, the U will cost m^2. For U^{-1} given by equation (49), however, $U^\dagger U$ is computed at a cost $\sim m^3$, add to that the cost $\sim m^3$ to invert the result, and add another m^3 to multiply $u^\dagger$ by the inverse. **Cost, then, is $\sim \mathbf{m^3}$**.

   (b) A: From equation (46), operator $U^\dagger$ extrapolates in the reverse direction and applies the $x \longrightarrow p'$ transform

$$\left[U^\dagger \psi(x, z + \Delta z)\right](p', z) = \frac{1}{2\pi} \int \psi(x, z + \Delta z)\, e^{-i\Delta z \omega q\left(x, p'\right)} e^{i\omega p' x} dp'. \tag{59}$$

Using $q(x,p) = a(x) + b(p)$, equation (59) becomes

$$\left[U^{\dagger}\psi(x,z+\Delta z)\right](p',z) = e^{-i\Delta z\omega a(x)}\frac{1}{2\pi}\int \psi(x,z+\Delta z)\, e^{-i\Delta z\omega b\left(p'\right)} e^{i\omega p' x} dp', \tag{60}$$

and the FFT can be used at a cost $\sim m \log m$. Then, using

$$\psi(x,z+\Delta z) = \left[U\varphi(p,z)\right](x,z+\Delta z), \tag{61}$$

substitute equation (61) into equation (60) to get

$$\begin{aligned}\left[U^{\dagger}\psi(x,z+\Delta z)\right](p',z) &= e^{-i\Delta z\omega b\left(p'\right)}\frac{1}{2\pi}\int \varphi(p,z)\, e^{i\Delta z\omega b(p)}\delta\left(p'-p\right) dp, \\ &= \varphi\left(p',z\right).\end{aligned} \tag{62}$$

Operator $U^{\dagger}U$ is simply identity I. **Cost is** $\sim \mathbf{m}\log\mathbf{m}$.

3. Suggest mathematical descriptions for $a\,(x)$ and $b\,(p)$.

   (a) A: In 2D, slowness q that is a function of space-variable and anisotropic velocity v is defined by

$$q^2 = \frac{1}{v^2}\left[1 - (vp)^2\right], \tag{63}$$

   where the x, p dependence of v is implicit. Introduce space-invariable $\bar{q}$

$$\bar{q}^{\,2} = \frac{1}{\bar{v}^2}\left[1 - (\bar{v}p)^2\right], \tag{64}$$

where the p dependence of $\bar{v}$ is implicit. Subtract $\bar{q}$ from equation(63)

$$\begin{aligned} q^2 &= \bar{q}^{\,2} + q^2 - \bar{q}^{\,2} \\ &= \bar{q}^{\,2} + \frac{1}{v^2}\left[1 - (vp)^2\right] - \frac{1}{\bar{v}^{\,2}}\left[1 - (\bar{v}p)^2\right] . \\ q &= \bar{q}\left[1 + \frac{\bar{v}^{\,2} - v^2}{(\bar{q}v\bar{v})^2}\right]^{\frac{1}{2}} \\ &= \bar{q}\left[1 - \frac{1}{2}\frac{v^2 - \bar{v}^{\,2}}{(\bar{q}v\bar{v})^2} - \cdots - \mathcal{O}^{\infty}\right] , \end{aligned} \tag{65}$$

where $\sqrt{1-x} = \left[1 - \frac{1}{2}x - \cdots - \mathcal{O}^{\infty}\right]$ has been used. Slowness q can now be separated into terms A and b, where

$$A = -\frac{1}{2}\frac{v^2 - \bar{v}^{\,2}}{\bar{q}\,(v\bar{v})^2} - \cdots - \mathcal{O}^{\infty}, \tag{66}$$

and

$$b(p) = \bar{q}(p)\,. \tag{67}$$

In equation (66), to get rid of the p dependence, expand $\bar{q}$ about p and truncate to one term $\bar{q} \sim \frac{1}{\bar{v}}$. Then, expand v and $\bar{v}$ about $p = p_{const}$ and truncate them both to single terms $\hat{v}$ and $\tilde{v}$ respectively (compute the velocity of the *effective medium* – usually $p = 0$)

$$a = -\frac{1}{2}\frac{\hat{v}^{\,2} - \tilde{v}^{\,2}}{\hat{v}^{\,2}\tilde{v}} - \cdots - \mathcal{O}^{\infty}, \tag{68}$$

and, for computational practicality, truncate the series in equation (68) to $n < \infty$ terms

$$a(x) = \frac{1}{2}\frac{\tilde{v}^{\,2} - \hat{v}(x)^2}{\hat{v}(x)^2\,\tilde{v}} + \cdots + \mathcal{O}^{n}. \tag{69}$$

Explicitly in terms of p, b is

$$b(p) = \frac{1}{\bar{v}(p)}\sqrt{1 - (\bar{v}(p)\,p)^2}. \tag{70}$$

Summary of velocities used here:

(a) $\hat{v}$: Space variant, anisotropy of effective medium

(b) $\tilde{v}$: Space invariant, anisotropy of effective medium

(c) $\bar{v}$: Space invariant, anisotropic

4. Having obtained φ_I and φ_R efficiently, R is ready to be computed using matrix equation (32).
   (a) Estimate the cost in 2D (assume m shots and m receivers/shot)
   (b) Assume $R(p) = R(p_I)$ and re-estimate the cost
   (c) Assume $R(p) = R$ and re-estimate the cost
   (a) A: Equation (32) requires 3 matrix multiplies plus one matrix inversion. For m shots and m receivers, the cost of the matrix multiplies is $\sim m^3$. The cost of the inversion of a matrix is $\sim m^3$. **Cost is** $\sim \mathbf{m^3}$
   (b) A: Reflectivity for horizontal reflectors lies on the trace of R where $p = p_I$, and equation (32) simplifies

$$\begin{bmatrix} \varphi_{R,-n} \\ \vdots \\ \varphi_{R,0} \\ \vdots \\ \varphi_{R,n} \end{bmatrix} = \begin{bmatrix} R_{-n}\,\varphi_{I,-n} \\ \vdots \\ R_0\,\varphi_{I,0} \\ \vdots \\ R_n\,\varphi_{I,n} \end{bmatrix} \tag{71}$$

R is then computed for each p for each shot and averaged using

$$R(p) = \frac{1}{N}\sum_j \frac{\varphi_R(p)_j}{\varphi_I(p)_j} \tag{72}$$

where subscript $1 \leq j \leq N$ indicates th j^{th} of N shots. **Cost is** $\sim \mathbf{m^2}$.

(c) A: Angle independent reflectivity $R(p) = R$ results in a very simple expression

$$R = \frac{1}{N}\sum_j \frac{\varphi_R}{\varphi_I}. \tag{73}$$

Equation (73) can be computed in x or p. **Cost is** $\sim \mathbf{m^2}$ - no improvement in cost for a significant reduction in accuracy.

5. Using the FFT provides significant computational savings when q is approximated as $a(x) + b(p)$. Are there any sacrifices using FFT in the computation of approximate U, D? What about exact/approximate R?

   - A: For isotropic media, U and D are unchanged, but for anisotropic media, $v(p)$ needs to be worked out as $v(k, \omega)$. Also, $R(p)$ needs to be worked out as $R(k, \omega)$. So, apart from a little algebra, no and no.