Numerical Linear Algebra in the Global Positioning System

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It is now possible to find out where you are.

Gilbert Strang
The Mathematics of GPS,
SIAM News of June 1997

Outline

- Computation of GPS relative position estimates
  - Introduction to GPS
  - The mathematical model
  - A recursive least squares method
  - Real data tests
- Computation of a test statistic in data quality control
  - Generalized likelihood ratio test statistic
  - A numerically stable method

Part I:

Computation of GPS Relative Position Estimates

Collaborator: Lan Yin, Cimmetry Systems, Inc., Montreal.

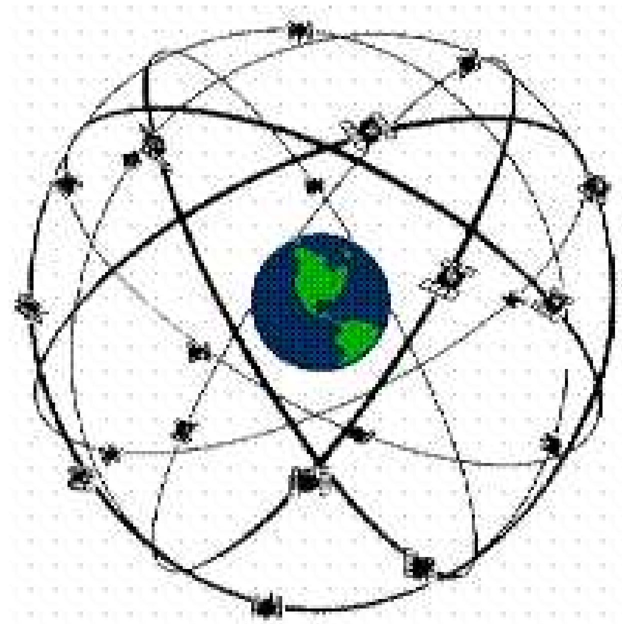
Introduction to GPS (6)

What Is the Global Positioning System (GPS) ?

An all-weather, worldwide, continuous coverage, satellite based navigation system, operated by the U. S. military.

GPS Segments:

- **Space:** 24 Satellites, in 6 orbits at 20,200 km altitude.
- **Control:** 1 master control station, 6 monitor stations, 4 ground antennas.
- **User:** receivers and users.

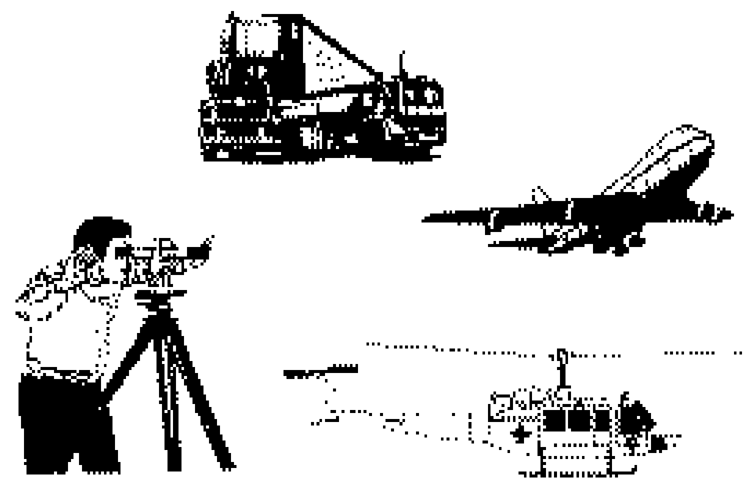


GPS Applications

GPS Military Uses:

GPS has become important for nearly all military operations and weapons systems.

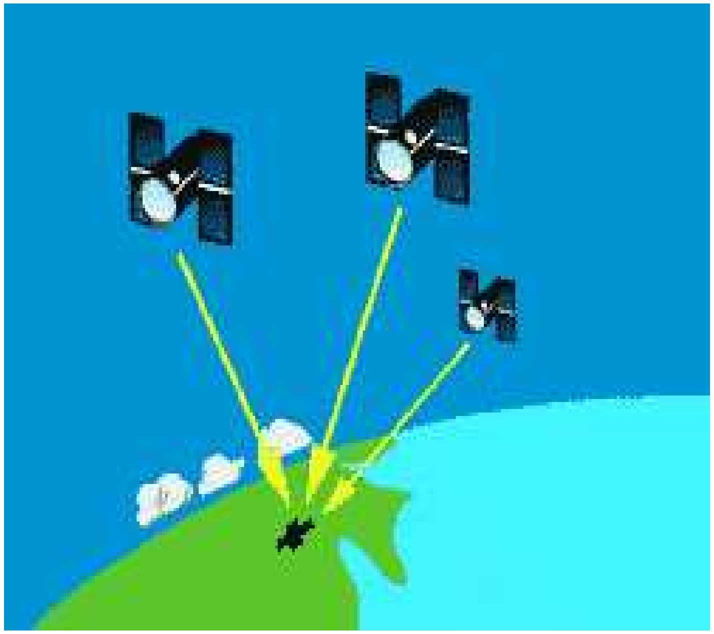
GPS Civil Uses:



- Land/sea/air/space navigation
- Mapping / GIS
- Surveying
- Search and rescue
- Recreation
- Intelligent vehicle highway systems, ...

How GPS Works

- The basis of GPS is "trilateration".
- In principle, if the distances from 3 satellites to a receiver can be measured, the receiver position can be determined.



GPS Signal & Measurements:

Each satellite transmits signals on two frequencies: L1 & L2. Superimposed on the carriers are C/A, P codes & navigation data.

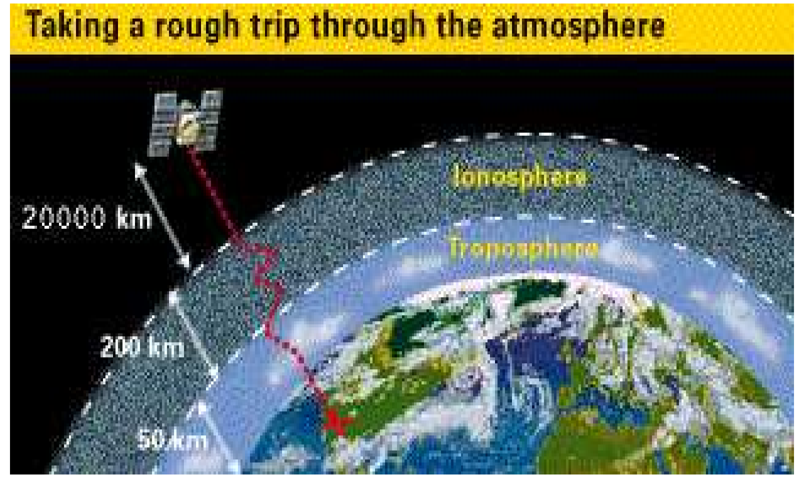
Measurements:

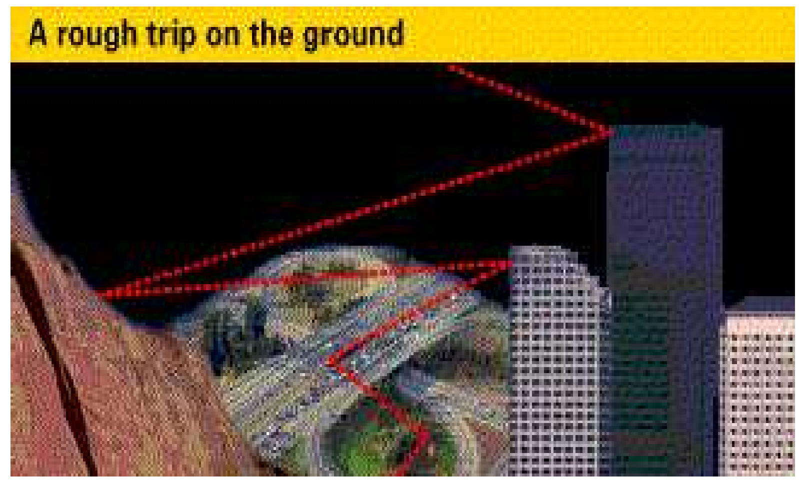
• Code measurements; • carrier phase measurements.

Carrier phase measurements are much more **complicated** but much more **accurate** than code measurements.

GPS Signal Errors

- Satellite clock errors
- Satellite hardware delay
- Satellite orbit errors
- Ionospheric reflection
- Tropospheric reflection
- Multipath errors
- Receiver clock errors
- Receiver hardware delay
- Noise errors

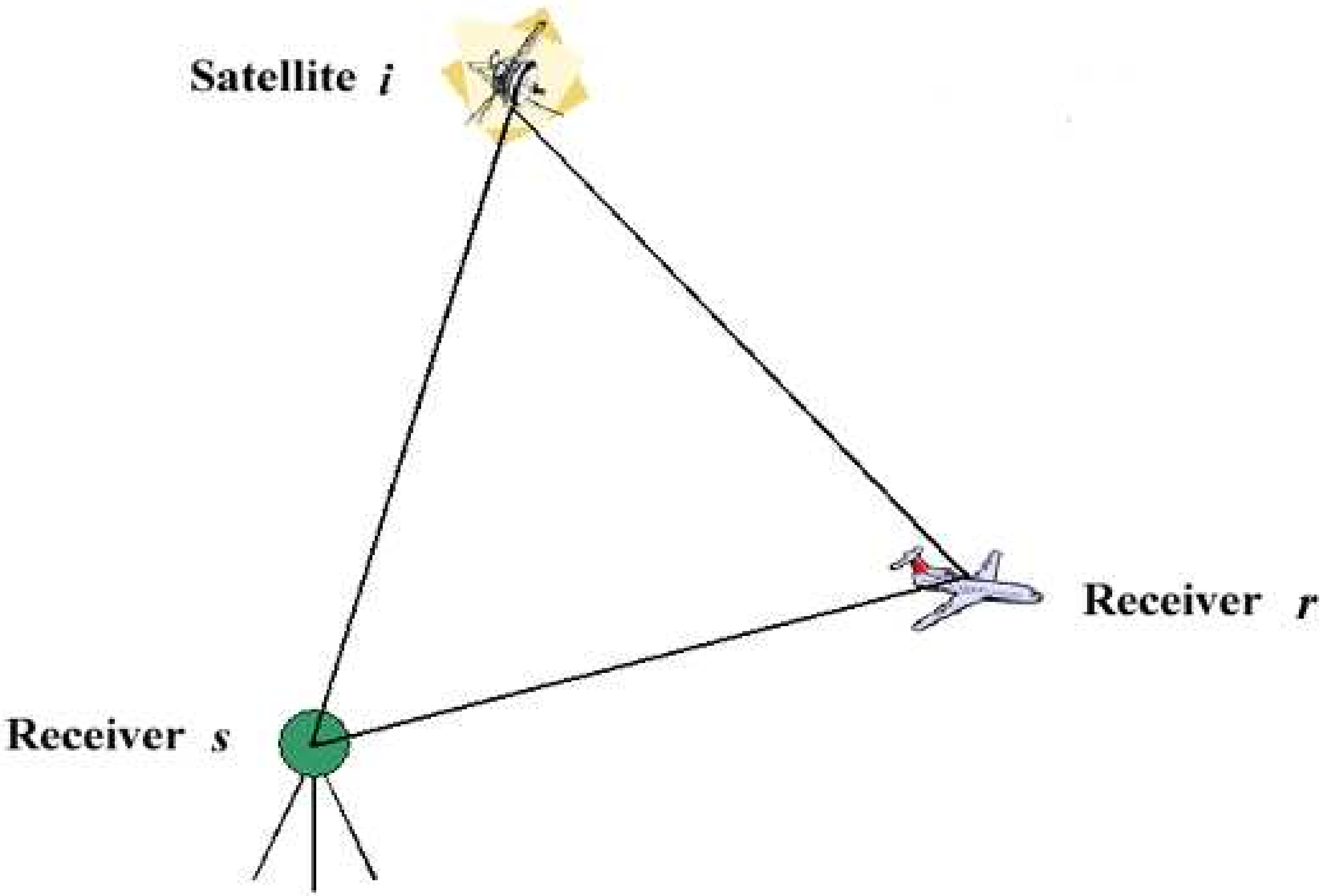




The multipath errors and noise errors of **code** measurements are usually $\sim 100\times$ larger than those of **carrier phase** measurements.

Physical Setting

- **Point positioning**: (simple, but precision is lower)
  - One receiver.
- **Relative positioning**: (complicated, but precision is high)
  - One stationary receiver with a known position; one (roving) receiver with a position to be determined.
  - They track the GPS signals at the same time.
  - They are close enough (10 km say) — the received signals have almost the **same** ionospheric and tropospheric errors.
  - For real time applications, there is a radio link between them.



Geometry for two Receivers and one Satellite.

Motivations and Goal

- Typical approach used in GPS:
  Least squares based on measurement eqn. $\boldsymbol{y}_k = A_k \boldsymbol{x}_k + \boldsymbol{v}_k$;
  Kalman filtering based on

  $$\boldsymbol{y}_k = A_k \boldsymbol{x}_k + \boldsymbol{v}_k \text{ (meas eqn)}, \quad \boldsymbol{x}_{k+1} = F_k \boldsymbol{x}_k + \boldsymbol{w}_k \text{ (dynamic eqn)}$$

  Sometimes the dynamic eqns are **artificially constructed**.

- Most methods given in GPS literature do not address the computer implementation issues.

- Efficiency is important—particularly for real time applications.
  Numerical reliability is important—ill-conditioned problems may arise.

Goal: present an efficient and numerically reliable approach for relative positioning.

The Mathematical Model (3)

Different combinations of measurements can be used for point and relative positioning.

We consider **relative positioning** based on **code and carrier phase measurements** from **L1** signal.

Single difference (SD) technique:
Difference the carrier (and code) measurements from the same satellite at two receivers to eliminate some common errors: *satellite clock error, satellite hardware delay, satellite orbit error, ionospheric reflection and tropospheric reflection.*

Single Difference (SD) Measurement Equations

At time t_k, for signal from satellite i

$$
\begin{aligned}
\text{carrier:}\quad & \phi_k^i = \lambda^{-1}(\boldsymbol{e}_k^i)^T\boldsymbol{x}_k + \alpha^i + \beta_k^\phi + \mu_k^i, && \mu_k^i \sim \mathcal{N}(0, \sigma_\phi^2) \\
\text{code:}\quad & \rho_k^i = \lambda^{-1}(\boldsymbol{e}_k^i)^T\boldsymbol{x}_k \qquad\quad + \beta_k^\rho + \nu_k^i, && \nu_k^i \sim \mathcal{N}(0, \sigma_\rho^2)
\end{aligned}
$$

ϕ_k^i: SD carrier phase measurement; ρ_k^i: SD code measurement;

λ: wavelength; μ_k^i: carrier phase noise; ν_k^i: code noise;

$\boldsymbol{x}_k$: **baseline** vector from receivers s to r, *to be determined*;

$\boldsymbol{e}_k^i$: unit vector from the midpoint of $\boldsymbol{x}_k$ to satellite i;

α^i: **SD integer ambiguity**, *constant but unknown*;

β_k^ϕ: SD receiver clock error and hardware delay for **carrier**;

β_k^ρ: SD receiver clock error and hardware delay for **code**.

Single Difference Measurement Equations, cont

$$\begin{array}{lll} \text{carrier} & \phi_k^i = \lambda^{-1}(\boldsymbol{e}_k^i)^T \boldsymbol{x}_k + \alpha^i + \beta_k^\phi + \mu_k^i, & \mu_k^i \sim \mathcal{N}(0, \sigma_\phi^2) \\ \text{code} & \rho_k^i = \lambda^{-1}(\boldsymbol{e}_k^i)^T \boldsymbol{x}_k + \beta_k^\rho + \nu_k^i, & \nu_k^i \sim \mathcal{N}(0, \sigma_\rho^2) \end{array}$$

Suppose there are m visible satellites. Then we have $2m$ such eqs. Write these in the matrix-vector form ($\boldsymbol{e} = [1, \ldots, 1]^T$):

$$\begin{array}{lll} \text{carrier} & \boldsymbol{y}_k^\phi = \boldsymbol{E}_k \, \boldsymbol{x}_k + \boldsymbol{a} + \boldsymbol{e} \, \beta_k^\phi + \boldsymbol{v}_k^\phi, & \boldsymbol{v}_k^\phi \sim \mathcal{N}(\boldsymbol{0}, \sigma_\phi^2 I_m) \\ \text{code} & \boldsymbol{y}_k^\rho = \boldsymbol{E}_k \, \boldsymbol{x}_k + \boldsymbol{e} \, \beta_k^\rho + \boldsymbol{v}_k^\rho, & \boldsymbol{v}_k^\rho \sim \mathcal{N}(\boldsymbol{0}, \sigma_\rho^2 I_m) \end{array}$$

Usually assume $\boldsymbol{v}_k^\phi$ and $\boldsymbol{v}_l^\rho$ $(k, l = 1, 2, \ldots)$ are uncorrelated.

Note: $\boldsymbol{E}_k$ depends on $\boldsymbol{x}_k$, so the model is nonlinear.
Use our estimate of $\boldsymbol{x}_{k-1}$ to compute an approximation to $\boldsymbol{E}_k$ — usually good enough. From now on, assume $\boldsymbol{E}_k$ is known.

A Recursive LS Method (10)

Background for the LS estimation

A standard linear model (A has full column rank):

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}, \quad \boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}),$$

The best linear unbiased estimate (BLUE) of $\boldsymbol{x}$ solves

$$\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

Solve the LS problem by (Householder) QR factn (see Golub '65):

$$\boldsymbol{Q}^T \boldsymbol{A} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{Q} = [\boldsymbol{Q}_1, \boldsymbol{Q}_2] \text{ orthogonal}, \quad \boldsymbol{R} \text{ upper triangular}$$

The LS solution and residual satisfy

$$\hat{\boldsymbol{x}} = \boldsymbol{R}^{-1}\boldsymbol{Q}_1^T \boldsymbol{y}, \qquad \mathcal{E}\{\hat{\boldsymbol{x}}\} = \boldsymbol{x}, \qquad \operatorname{cov}\{\hat{\boldsymbol{x}}\} = \sigma^2(\boldsymbol{R}^T\boldsymbol{R})^{-1}$$

$$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{Q}_2\boldsymbol{Q}_2^T\boldsymbol{y}, \qquad \mathcal{E}\{\boldsymbol{r}\} = \boldsymbol{0}, \qquad \operatorname{cov}\{\boldsymbol{r}\} = \sigma^2 \boldsymbol{Q}_2^T\boldsymbol{Q}_2.$$

Orthogonal transformation approach for position estimation

Use orthogonal transformations to recursively estimate $\boldsymbol{x}_k$ based on the model

$$
\begin{aligned}
\text{carrier} \quad & \boldsymbol{y}_k^{\phi} = \boldsymbol{E}_k\,\boldsymbol{x}_k + \boldsymbol{a} + \boldsymbol{e}\,\beta_k^{\phi} + \boldsymbol{v}_k^{\phi}, \quad && \boldsymbol{v}_k^{\phi} \sim \mathcal{N}(\mathbf{0}, \sigma_{\phi}^2 I_m), \\
\text{code} \quad & \boldsymbol{y}_k^{\rho} = \boldsymbol{E}_k\,\boldsymbol{x}_k + \boldsymbol{e}\,\beta_k^{\rho} + \boldsymbol{v}_k^{\rho}, && \boldsymbol{v}_k^{\rho} \sim \mathcal{N}(\mathbf{0}, \sigma_{\rho}^2 I_m).
\end{aligned}
$$

$$k = 1, 2, \ldots.$$

- Orthogonal transformations are numerically reliable.
- Orthogonal transformations can keep the noise vectors uncorrelated.

Eliminating β_k^ϕ and β_k^ρ from the model:

Let $\boldsymbol{P}$ be a Householder transformation: $\boldsymbol{P}\boldsymbol{e} = \sqrt{m}\,[1, 0, ..., 0]^T$.

Apply $\quad \boldsymbol{P} \equiv \begin{bmatrix} \boldsymbol{p}^T \\ \bar{\boldsymbol{P}} \end{bmatrix}$ to $\quad \boldsymbol{y}_k^\phi = \boldsymbol{E}_k\,\boldsymbol{x}_k + \boldsymbol{a} + \boldsymbol{e}\,\beta_k^\phi + \boldsymbol{v}_k^\phi$:

$$\begin{bmatrix} \boldsymbol{p}^T \boldsymbol{y}_k^\phi \\ \bar{\boldsymbol{P}}\boldsymbol{y}_k^\phi \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}^T \boldsymbol{E}_k \\ \bar{\boldsymbol{P}}\boldsymbol{E}_k \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \boldsymbol{p}^T \\ \bar{\boldsymbol{P}} \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} \sqrt{m}\beta_k^\phi + \begin{bmatrix} \boldsymbol{p}^T \boldsymbol{v}_k^\phi \\ \bar{\boldsymbol{P}}\boldsymbol{v}_k^\phi \end{bmatrix}.$$

Drop the 1st equation

$$\bar{\boldsymbol{P}}\boldsymbol{y}_k^\phi = \bar{\boldsymbol{P}}\boldsymbol{E}_k\boldsymbol{x}_k + \bar{\boldsymbol{P}}\boldsymbol{a} + \bar{\boldsymbol{P}}\boldsymbol{v}_k^\phi, \qquad \bar{\boldsymbol{P}}\boldsymbol{v}_k^\phi \sim \mathcal{N}(\boldsymbol{0}, \sigma_\phi^2 \boldsymbol{I}_{m-1}).$$

Since $\bar{\boldsymbol{P}}$ is $(m-1) \times m$, $\quad \boldsymbol{a} \in \Re^m \quad$ cannot be determined.

Introduce the double difference **integer** ambiguity vector

$$\boldsymbol{z} \equiv [\alpha^2 - \alpha^1, \alpha^3 - \alpha^1, \ldots, \alpha^m - \alpha^1]^T \in \Re^{m-1},$$

where satellite 1 is chosen to be the **'reference satellite'**.

Can show

$$\bar{\boldsymbol{P}}\boldsymbol{a} = \boldsymbol{F}\boldsymbol{z}, \quad \boldsymbol{F} = \boldsymbol{I}_{m-1} - \frac{1}{m-\sqrt{m}}\boldsymbol{e}\boldsymbol{e}^T \text{ nonsingular.}$$

So $\bar{\boldsymbol{P}}\boldsymbol{y}_k^{\phi} = \bar{\boldsymbol{P}}\boldsymbol{E}_k\boldsymbol{x}_k + \bar{\boldsymbol{P}}\boldsymbol{a} + \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\phi}$ becomes

$$\bar{\boldsymbol{P}}\boldsymbol{y}_k^{\phi} = \bar{\boldsymbol{P}}\boldsymbol{E}_k\boldsymbol{x}_k + \boldsymbol{F}\boldsymbol{z} + \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\phi}, \qquad \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\phi} \sim \mathcal{N}(\boldsymbol{0}, \sigma_\phi^2 \boldsymbol{I}_{m-1}). \tag{1}$$

Similarly, applying $\bar{\boldsymbol{P}}$ to $\boldsymbol{y}_k^{\rho} = \boldsymbol{E}_k\boldsymbol{x}_k + \boldsymbol{e}\,\beta_k^{\rho} + \boldsymbol{v}_k^{\rho}$ gives

$$\bar{\boldsymbol{P}}\boldsymbol{y}_k^{\rho} = \bar{\boldsymbol{P}}\boldsymbol{E}_k\boldsymbol{x}_k + \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\rho}, \qquad \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\rho} \sim \mathcal{N}(\boldsymbol{0}, \sigma_\rho^2 \boldsymbol{I}_{m-1}). \tag{2}$$

Let $\sigma = \frac{\sigma_\phi}{\sigma_\rho}$. Combine (1) and (2):

$$\begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{y}_k^{\phi} \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{y}_k^{\rho} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{E}_k \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{E}_k \end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\phi} \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{v}_k^{\rho} \end{bmatrix}, \quad \begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{v}_k^{\phi} \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{v}_k^{\rho} \end{bmatrix} \sim \mathcal{N}(\boldsymbol{0}, \sigma_\phi^2 \boldsymbol{I})$$

For simplicity, ignore the noise vector and use '≈' to replace '='.

$$\begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{y}_k^{\phi} \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{y}_k^{\rho} \end{bmatrix} \approx \begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{E}_k \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{E}_k \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{z} \tag{3}$$

Transform the coefficient matrix of $\boldsymbol{x}_k$ to upper triangular

Compute the QR factorization:

$$\boldsymbol{Q}_k^T \begin{bmatrix} \bar{\boldsymbol{P}}\boldsymbol{E}_k \\ \sigma\bar{\boldsymbol{P}}\boldsymbol{E}_k \end{bmatrix} = \begin{array}{cl} \begin{bmatrix} \boldsymbol{R}_k \\ \boldsymbol{0} \end{bmatrix} & \begin{array}{l} 3 \\ 2m-5 \end{array} \\ 3 & \end{array}$$

Note: Make full use of the structure of the matrix for efficiency.

Then apply $\boldsymbol{Q}_k^T$ to (3), with obvious notation:

$$\begin{bmatrix} \boldsymbol{y}_k \\ \bar{\boldsymbol{y}}_k \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{R}_k \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \boldsymbol{G}_k \\ \bar{\boldsymbol{G}}_k \end{bmatrix} \boldsymbol{z}$$

$$\begin{bmatrix} \boldsymbol{y}_k \\ \bar{\boldsymbol{y}}_k \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{R}_k \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \boldsymbol{G}_k \\ \bar{\boldsymbol{G}}_k \end{bmatrix} \boldsymbol{z}$$

Combine these for $k = 1, 2, \ldots,$ and reorder:

$$\left[\begin{array}{c} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_k \\ \hline \bar{\boldsymbol{y}}_1 \\ \vdots \\ \bar{\boldsymbol{y}}_k \end{array}\right] \approx \left[\begin{array}{ccc|c} \boldsymbol{R}_1 & & & \boldsymbol{G}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{R}_k & \boldsymbol{G}_k \\ \hline & & & \bar{\boldsymbol{G}}_1 \\ & & & \vdots \\ & & & \bar{\boldsymbol{G}}_k \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_k \\ \hline \boldsymbol{z} \end{array}\right] \tag{4}$$

First estimate $\boldsymbol{z}$ from the lower part of (4);
then estimate $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ from the upper part of (4).

Recursively computing the LS estimate of z $\begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_k \end{bmatrix} z \approx \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_k \end{bmatrix}$

Suppose at t_{k-1} we have obtained the orthogonal transformations:

$$U_{k-1}^T \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_{k-1} \end{bmatrix} = \begin{bmatrix} S_{k-1} \\ \hline 0 \end{bmatrix}, \qquad U_{k-1}^T \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_{k-1} \end{bmatrix} = \begin{bmatrix} b_{k-1} \\ \hline \bar{b}_{k-1} \end{bmatrix},$$

U_{k-1}: orthogonal, S_{k-1}: nonsingular upper triangular.

At t_k, after obtaining $\bar{G}_k$ and $\bar{y}_k$, perform

$$\tilde{U}_k^T \begin{bmatrix} S_{k-1} \\ \bar{G}_k \end{bmatrix} = \begin{bmatrix} S_k \\ \hline 0 \end{bmatrix}, \qquad \tilde{U}_k^T \begin{bmatrix} b_{k-1} \\ \bar{y}_k \end{bmatrix} = \begin{bmatrix} b_k \\ \hline \hat{b}_k \end{bmatrix}$$

$\tilde{U}_k$: orthogonal, S_k : nonsingular upper triangular.

Thus $\begin{bmatrix} \bar{\boldsymbol{G}}_1 \\ \vdots \\ \bar{\boldsymbol{G}}_k \end{bmatrix} \boldsymbol{z} \approx \begin{bmatrix} \bar{\boldsymbol{y}}_1 \\ \vdots \\ \bar{\boldsymbol{y}}_k \end{bmatrix}$ is transformed to $\begin{bmatrix} \boldsymbol{S}_k \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{z} \approx \begin{bmatrix} \boldsymbol{b}_k \\ \hat{\boldsymbol{b}}_k \end{bmatrix}$.

Compute the LS estimate $\boldsymbol{z}_k$ of $\boldsymbol{z}$ at t_k by solving

$$\boldsymbol{S}_k \boldsymbol{z}_k = \boldsymbol{b}_k.$$

It is easy to show

$$\mathrm{cov}\{\boldsymbol{z}_k\} = \sigma_\phi^2 (\boldsymbol{S}_k^T \boldsymbol{S}_k)^{-1}.$$

Remarks:

- Here we regarded $\boldsymbol{z}$ as a real vector.
- To get centimeter accuracy quickly, we have to fix $\boldsymbol{z}$ as a vector of integers. Then $\boldsymbol{z}$ will be regarded as known.
- The LAMBDA method (Teunissen '93) uses $\mathrm{cov}\{\boldsymbol{z}_k\}$ to fix $\boldsymbol{z}$.

Computing the LS estimates of $\boldsymbol{x}_1, \ldots \boldsymbol{x}_k$

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_k \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{R}_1 & & \\ & \ddots & \\ & & \boldsymbol{R}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_1 \\ \vdots \\ \boldsymbol{G}_k \end{bmatrix} \boldsymbol{z}$$

We compute $\boldsymbol{x}_{1|k}, \boldsymbol{x}_{2|k}, \ldots, \boldsymbol{x}_{k|k}$, the LS estimates of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ at time t_k by solving the upper triangular systems

$$\boldsymbol{R}_j \boldsymbol{x}_{j|k} = \boldsymbol{y}_j - \boldsymbol{G}_j \boldsymbol{z}_k, \qquad j = 1, \ldots, k.$$

Remarks:

- These can be solved in any order.
- $\boldsymbol{x}_{j|k}$ for $j \leq k-1$ is called the smoothed estimate of $\boldsymbol{x}_j$.
- For real time applications, we may only want to find $\boldsymbol{x}_{k|k}$.

Computing the error covariance matrices $\mathrm{cov}\{\boldsymbol{x}_{j|k} - \boldsymbol{x}_j\}$

The equations used for estimating $\boldsymbol{x}_j$ and $\boldsymbol{z}$ at time t_k:

$$\begin{bmatrix} \boldsymbol{y}_j \\ \boldsymbol{b}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_j & \boldsymbol{G}_j \\ \boldsymbol{0} & \boldsymbol{S}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_j \\ \boldsymbol{z} \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{w}_k \end{bmatrix}}_{\text{noise}}, \quad \begin{bmatrix} \boldsymbol{u}_j \\ \boldsymbol{w}_k \end{bmatrix} \sim \mathcal{N}(\boldsymbol{0}, \sigma_\phi^2 I). \tag{5}$$

Let $\boldsymbol{Z}_{j|k}^T \begin{bmatrix} \boldsymbol{R}_j & \boldsymbol{G}_j \\ \boldsymbol{0} & \boldsymbol{S}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{j|k} & \boldsymbol{0} \\ \bar{\boldsymbol{R}}_{j|k} & \boldsymbol{S}_{j|k} \end{bmatrix}$, $\quad \boldsymbol{Z}_{j|k}$: Givens rotations; $\boldsymbol{R}_{j|k}, \ \boldsymbol{S}_{j|k}$: U.T.

Apply $\boldsymbol{Z}_{j|k}^T$ to (5):

$$\boldsymbol{Z}_{j|k}^T \begin{bmatrix} \boldsymbol{y}_j \\ \boldsymbol{b}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{j|k} & \boldsymbol{0} \\ \boldsymbol{R}_{j|k} & \boldsymbol{S}_{j|k} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_j \\ \boldsymbol{z} \end{bmatrix} + \boldsymbol{Z}_{j|k}^T \begin{bmatrix} \bar{\boldsymbol{u}}_j \\ \boldsymbol{w}_k \end{bmatrix}.$$

Can show

$$\mathrm{cov}\{\boldsymbol{x}_{j|k} - \boldsymbol{x}_j\} = \sigma_\phi^2 (\boldsymbol{R}_{j|k}^T \boldsymbol{R}_{j|k})^{-1}.$$

Remarks (2)

- **Cycle slip**

  Cycle slip is the change in ambiguities mainly caused by temporary obstructions of the satellite signal.
  We need incorporate an algorithm for cycle slip detection.

- **Number of satellites**

  There may be different number of satellites at different epochs, e.g., satellite rising and setting, cycle slip.
  But we can modify our algorithm to handle such cases.

  Two cases:
  a. The 'reference satellite' at t_{k-1} goes down between t_{k-1} & t_k.
  b. The 'reference satellite' at t_{k-1} remains at t_k.

- **Dual frequency receivers**

  Some receivers can measure carrier phase and code for both L1 and L2 carrier signals.

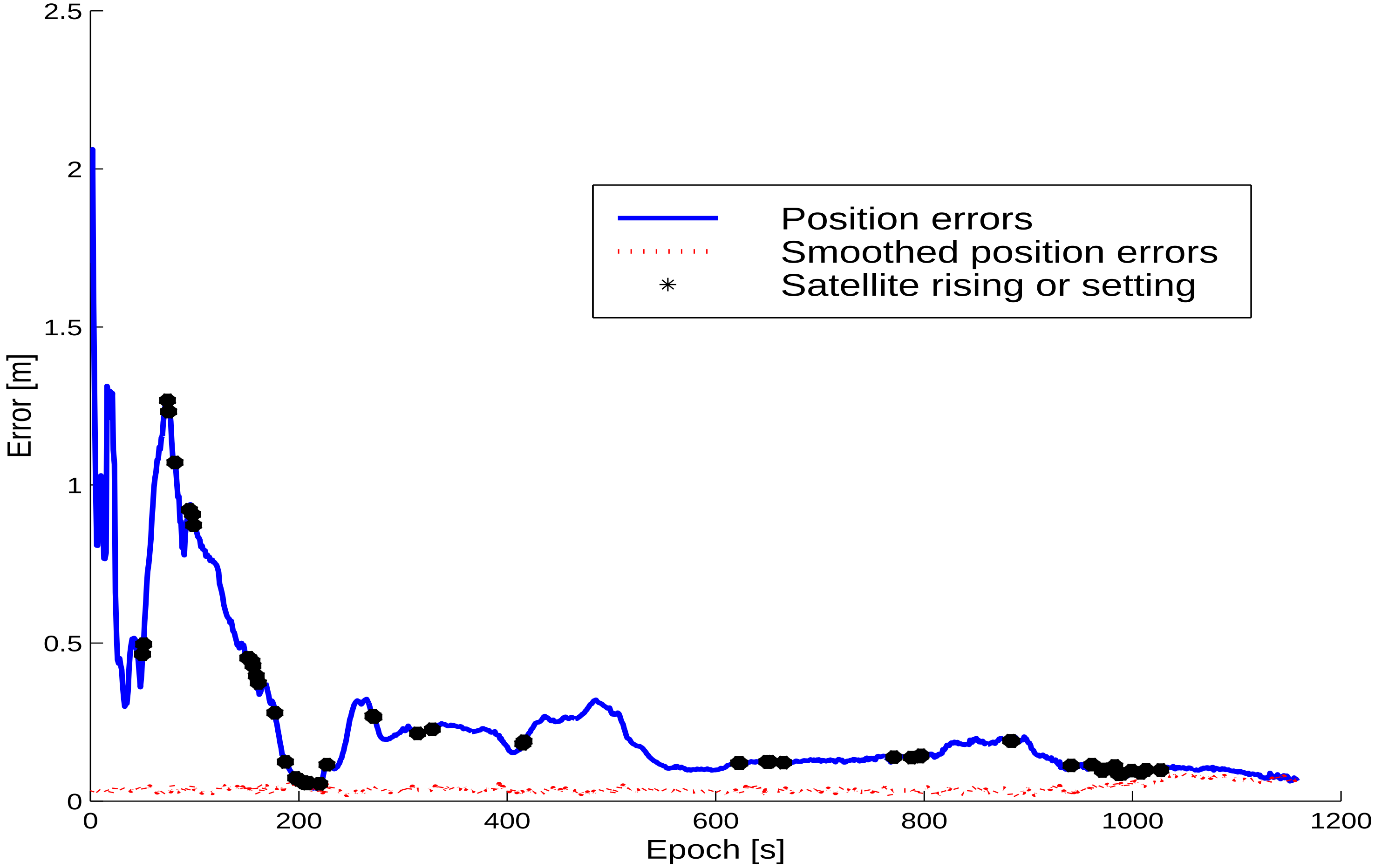
  It is easy to modify our approach to include more measurement equations.

- **Kalman filtering**

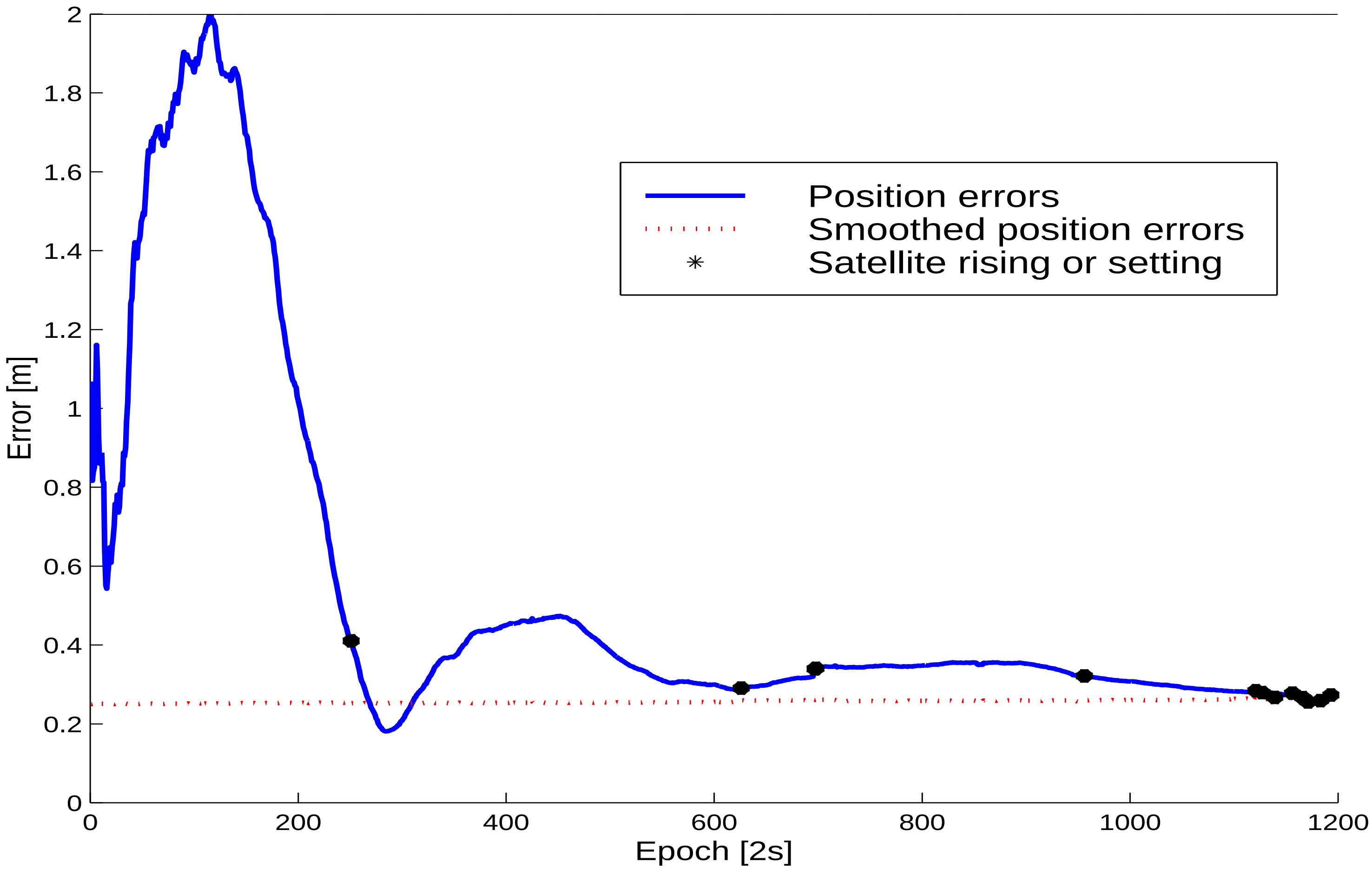
  When a dynamic model for the roving receiver is available, we can modify our approach to handle it.

Real data tests (3)

- Two data sets were provided by VIASAT Geo-Technology Inc.
- The receivers were made by Canadian Marconi Company.
- Data set 1: The user was walking;
  Data set 2: The user was riding a four wheel trail bike;
  Both were in an open sky environment.
- The time interval between two consecutive epochs:
  Data set 1: 1 second, Data set 2: 2 seconds.
- We took $\sigma = \sigma_\phi / \sigma_\rho = 10^{-3}$.
- We used the position estimates obtained by VIASAT software as the "true" positions. The software used a complex positioning algorithm. It is believed the errors are about a few centimeters.



Position errors versus run time for **data set 1**



Position errors versus run time for **data set 2**

Part II:

Computation of a Test Statistic in Data Quality Control

Collaborator: Christian Tiberius, Delft University of Technology.

Data Quality Control

Motivations

- GPS signals may be corrupted.
- Using corrupted data may lead big errors in the position estimates — too dangerous for some applications, such as aircraft landing.
- The data need to be carefully validated by statistical testing.
- Data quality control is useful not only in GPS, but also in many other applications.

Goal:
Provide a numerically stable method to compute the commonly used generalized likelihood ratio test statistic.

Example (2)

Suppose at some epoch, there are 5 visible satellites.

In a normal situation, the code measurements satisfy

$$\mathrm{H}_0 : \quad \underbrace{\begin{bmatrix} \rho^1 \\ \rho^2 \\ \rho^3 \\ \rho^4 \\ \rho^5 \end{bmatrix}}_{\text{meas}} = \begin{bmatrix} \lambda^{-1}(\boldsymbol{e}^1)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^2)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^3)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^4)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^5)^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \beta^\rho \end{bmatrix} + \underbrace{\begin{bmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^4 \\ \nu^5 \end{bmatrix}}_{\text{noise}}.$$

If the signal from satellite 2 has a blunder (outlier), then

$$
\mathrm{H}_a : \quad \begin{bmatrix} \rho^1 \\ \rho^2 \\ \rho^3 \\ \rho^4 \\ \rho^5 \end{bmatrix} = \begin{bmatrix} \lambda^{-1}(\boldsymbol{e}^1)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^2)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^3)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^4)^T & 1 \\ \lambda^{-1}(\boldsymbol{e}^5)^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \beta^\rho \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \nabla + \begin{bmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^4 \\ \nu^5 \end{bmatrix},
$$

$\boldsymbol{c} \equiv [0, 1, 0, 0, 0]^T$ specifies the type of model error, ∇ is unknown.

Do a statistic test to determine whether the data **supports** or **rejects** H_0 on the basis of H_a .

Test Statistic (4)

Consider a very general case.

Null hypothesis $\mathbf{H}_0$: The measurements satisfy the model

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}, \qquad \boldsymbol{v} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{V}).$$

$\boldsymbol{A} \in \mathcal{R}^{m\times n}$ has full column rank, $\boldsymbol{V}$ is symmetric positive definite.

Alternative hypothesis $\mathbf{H}_a$: The corrupted measurements satisfy the model

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\nabla + \boldsymbol{v}, \qquad \boldsymbol{v} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{V}),$$

$\boldsymbol{C} \in \mathcal{R}^{m\times q}$ is known, $[\boldsymbol{A}, \boldsymbol{C}]$ has full column rank, and ∇ is an unknown vector.

Maximum likelihood estimator (MLE)

The **density functions** of $\boldsymbol{y}$:

under H_0:

$$p(\boldsymbol{y}|\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{m}{2}}\sigma|\boldsymbol{V}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^T\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})\right]$$

under H_a:

$$p(\boldsymbol{y}|\boldsymbol{x},\nabla) = \frac{1}{(2\pi)^{\frac{m}{2}}\sigma|\boldsymbol{V}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{C}\nabla)^T\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{C}\nabla)\right]$$

The MLE of x under H_0:

$$\boldsymbol{x}_0 = \arg\max_{\boldsymbol{x}} p(\boldsymbol{y}|\boldsymbol{x})$$

The MLE of $\{\boldsymbol{x},\nabla\}$ under H_0:

$$\{\boldsymbol{x}_a,\nabla_a\} = \arg\max_{\boldsymbol{x},\nabla} p(\boldsymbol{y}|\boldsymbol{x},\nabla).$$

Obviously x_0 and $\{x_a, \nabla_a\}$ are respectively the solutions of the **generalized linear least squares (GLLS)** problems:

$$\begin{aligned} \mathrm{GLLS}_0: &\quad \min(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x})^T\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}); \\ \mathrm{GLLS}_a: &\quad \min(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{C}\nabla)^T\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{C}\nabla). \end{aligned}$$

They are also the **best linear unbiased estimators (BLUE)**.

Define the residuals

$$\boldsymbol{r}_0 \equiv \boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}_0, \qquad \boldsymbol{r}_a \equiv \boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}_a-\boldsymbol{C}\nabla_a.$$

The generalized likelihood ratio is given by

$$l(\boldsymbol{y}) \equiv \frac{p(\boldsymbol{y}|\boldsymbol{x}_0)}{p(\boldsymbol{y}|\boldsymbol{x}_a, \nabla_a)} = \exp\left[-\frac{1}{2\sigma^2}\left(\boldsymbol{r}_0^T\boldsymbol{V}^{-1}\boldsymbol{r}_0-\boldsymbol{r}_a^T\boldsymbol{V}^{-1}\boldsymbol{r}_a\right)\right].$$

The test statistic is defined by

$$\boxed{\delta_{\mathrm{TS}} \equiv -2\log l(\boldsymbol{y}) = \sigma^{-2}(\boldsymbol{r}_0^T\boldsymbol{V}^{-1}\boldsymbol{r}_0-\boldsymbol{r}_a^T\boldsymbol{V}^{-1}\boldsymbol{r}_a)}$$

Distribution of δ_{TS}

$$\delta_{\text{TS}} \sim \chi^2(q, 0), \quad \text{under } \mathrm{H}_0;$$

$$\delta_{\text{TS}} \sim \chi^2(q, \lambda), \quad \lambda = \nabla^T \boldsymbol{C}^T \boldsymbol{V}^{-1} \mathrm{cov}\{\boldsymbol{r}_0\} \boldsymbol{V}^{-1} \boldsymbol{C} \nabla, \quad \text{under } \mathrm{H}_a.$$

where $\quad \mathrm{cov}\{\boldsymbol{r}_0\} = \sigma^2[\boldsymbol{V} - \boldsymbol{A}(\boldsymbol{A}^T \boldsymbol{V}^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^T].$

Statistic testing

When δ_{TS} is larger than a given threshold, reject H_0 in favor of H_a. Otherwise accept H_0.

The threshold is usually determined by the requirement of a specific application.

An Obvious Method to Compute δ_{TS} (2)

Let $\boldsymbol{V}$ have the factorization: $\boldsymbol{V} = \boldsymbol{B}\boldsymbol{B}^T$.

Nonsingular $\boldsymbol{B}$ is given or obtained by the Cholesky factorization.

Define $\bar{\boldsymbol{y}} = \boldsymbol{B}^{-1}\boldsymbol{y}$, $\bar{\boldsymbol{A}} = \boldsymbol{B}^{-1}\boldsymbol{A}$, $\bar{\boldsymbol{C}} = \boldsymbol{B}^{-1}\boldsymbol{C}$.

Transform problems GLLS_0 & GLLS_a to the ordinary LS problems:

$$\boldsymbol{x}_0 = \arg\min(\bar{\boldsymbol{y}} - \bar{\boldsymbol{A}}\boldsymbol{x})^T(\bar{\boldsymbol{y}} - \bar{\boldsymbol{A}}\boldsymbol{x});$$

$$\{\boldsymbol{x}_a, \nabla_a\} = \arg\min(\bar{\boldsymbol{y}} - \bar{\boldsymbol{A}}\boldsymbol{x} - \bar{\boldsymbol{C}}\nabla)^T(\bar{\boldsymbol{y}} - \bar{\boldsymbol{A}}\boldsymbol{x} - \bar{\boldsymbol{C}}\nabla)$$

Compute the QR factn: $[\bar{\boldsymbol{A}},\ \bar{\boldsymbol{C}}] = [\boldsymbol{Q}_A, \boldsymbol{Q}_C, \boldsymbol{Q}_3]\begin{bmatrix} \bar{\boldsymbol{R}}_A & \bar{\boldsymbol{R}}_{AC} \\ \mathbf{0} & \bar{\boldsymbol{R}}_C \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

We can show

$$\boxed{\delta_{\mathrm{TS}} = \sigma^{-2}\|\boldsymbol{Q}_C^T\bar{\boldsymbol{y}}\|_2^2}$$

Two problems with this method:

- The method is not numerically stable.

  When $\boldsymbol{B}$ is ill-conditioned, accuracy may unnecessarily be lost.

- Recall $\boldsymbol{V} = \boldsymbol{B}\boldsymbol{B}^T$. If $\boldsymbol{V}$ is singular, then $\boldsymbol{B}^{-1}$ does not exist. So the method will not work.

  Note: The original formula $\delta_{\text{TS}} = \sigma^{-2}(\boldsymbol{r}_0^T \boldsymbol{V}^{-1} \boldsymbol{r}_0 - \boldsymbol{r}_a^T \boldsymbol{V}^{-1} \boldsymbol{r}_a)$ is not defined.

A Backward Stable Method for Computing δ_{TS} (5)

Idea: Use Paige's approach (1978) to solving GLLS problems.

Recall $\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{V})$ and $\boldsymbol{V} = \boldsymbol{B}\boldsymbol{B}^T$. Write

$$\boldsymbol{v} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{u} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}).$$

Reformulate the GLLS problems:

$$\begin{aligned} &\mathrm{GLLS}_0: \quad \min \|\boldsymbol{u}\|_2^2, \quad \text{s.t. } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, && \text{under } \mathrm{H}_0 \\ &\mathrm{GLLS}_a: \quad \min \|\boldsymbol{u}\|_2^2, \quad \text{s.t. } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\nabla + \boldsymbol{B}\boldsymbol{u}, && \text{under } \mathrm{H}_a \end{aligned}$$

For simplicity, we still assume $\boldsymbol{B}$ is nonsingular.

Let $[\boldsymbol{A}, \boldsymbol{C}]$ and $\boldsymbol{B}$ have the following generalized QR factorization

$$\boldsymbol{P}^T[\boldsymbol{A}, \boldsymbol{C}] = \begin{bmatrix} \boldsymbol{U}_A & \boldsymbol{U}_{AC} \\ \boldsymbol{0} & \boldsymbol{U}_C \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{P}^T \boldsymbol{B} \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{R}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix}$$

Solving Problem GLLS$_a$

Transform problem GLLS$_a$:

$$\boxed{\min \|\boldsymbol{u}\|_2^2, \quad \text{s.t.} \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\nabla + \boldsymbol{B}\boldsymbol{u}}$$

$$\Downarrow$$

$$\boxed{\min \|\boldsymbol{u}\|_2^2, \quad \text{s.t.} \quad \underbrace{\boldsymbol{P}^T\boldsymbol{y}}_{\boldsymbol{z}} = \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{P}^T\boldsymbol{C}\nabla + \boldsymbol{P}^T\boldsymbol{B}\boldsymbol{Q}\underbrace{\boldsymbol{Q}^T\boldsymbol{u}}_{\boldsymbol{w}}}$$

$$\Downarrow$$

$$\boxed{\begin{gathered}\min \|\boldsymbol{w}_A\|_2^2 + \|\boldsymbol{w}_C\|_2^2 + \|\boldsymbol{w}_3\|_2^2 \\ \text{s.t.} \quad \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_A \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{U}_{AC} \\ \boldsymbol{U}_C \\ \boldsymbol{0} \end{bmatrix} \nabla + \begin{bmatrix} \boldsymbol{R}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_A \\ \boldsymbol{w}_C \\ \boldsymbol{w}_3 \end{bmatrix}\end{gathered}}$$

Solving Problem GLLS$_a$, cont.

$$\min \|\boldsymbol{w}_A\|_2^2 + \|\boldsymbol{w}_C\|_2^2 + \|\boldsymbol{w}_3\|_2^2$$
$$\text{s.t.} \quad \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_A \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{U}_{AC} \\ \boldsymbol{U}_C \\ \boldsymbol{0} \end{bmatrix} \nabla + \begin{bmatrix} \boldsymbol{R}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_A \\ \boldsymbol{w}_C \\ \boldsymbol{w}_3 \end{bmatrix}$$

The optimal soln $\boldsymbol{x}_a$, ∇_a, $\boldsymbol{w}_a \equiv [(\boldsymbol{w}_A^a)^T, (\boldsymbol{w}_C^a)^T, (\boldsymbol{w}_3^a)^T]^T$ satisfies

$$\boldsymbol{w}_A^a = 0, \qquad \boldsymbol{w}_C^a = 0, \qquad \begin{bmatrix} \boldsymbol{U}_A & \boldsymbol{U}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{U}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_a \\ \nabla_a \\ \boldsymbol{w}_3^a \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix}.$$

The GLS residual or the optimal $\boldsymbol{u}$: $\quad \boldsymbol{u}_a = \boldsymbol{Q}\boldsymbol{w}_a$.

Solving Problem GLLS$_0$

$$\boxed{\min \|\boldsymbol{u}\|_2^2, \quad \text{s.t.} \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \min(\|\boldsymbol{w}_A\|_2^2 + \|\boldsymbol{w}_C\|_2^2 + \|\boldsymbol{w}_3\|_2^2), \\ \text{s.t.} \quad \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_A \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{R}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_A \\ \boldsymbol{w}_C \\ \boldsymbol{w}_3 \end{bmatrix} \end{array}}$$

The optimal soln $\boldsymbol{x}_0$, $\boldsymbol{w}_0 \equiv [(\boldsymbol{w}_A^0)^T, (\boldsymbol{w}_C^0)^T, (\boldsymbol{w}_3^0)^T]^T$ satisfies

$$\boldsymbol{w}_A^0 = 0, \qquad \begin{bmatrix} \boldsymbol{U}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \boldsymbol{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{w}_C^0 \\ \boldsymbol{w}_3^0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix}$$

The GLS residual or the optimal $\boldsymbol{u}$: $\quad \boldsymbol{u}_0 = \boldsymbol{Q}\boldsymbol{w}_0$.

Computing the Test Statistic δ_{TS}

The GLS residual under H_a: $\quad \boldsymbol{u}_a = \boldsymbol{Q}\,[(\boldsymbol{w}_A^a)^T, (\boldsymbol{w}_C^a)^T, (\boldsymbol{w}_3^a)^T]^T$

$$\boldsymbol{w}_A^a = 0, \quad \boldsymbol{w}_C^a = 0, \quad \begin{bmatrix} \boldsymbol{U}_A & \boldsymbol{U}_{AC} & \boldsymbol{R}_{A3} \\ \mathbf{0} & \boldsymbol{U}_C & \boldsymbol{R}_{C3} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_a \\ \nabla_a \\ \boldsymbol{w}_3^a \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix}$$

The GLS residual under H_0: $\quad \boldsymbol{u}_0 = \boldsymbol{Q}\,[(\boldsymbol{w}_A^0)^T, (\boldsymbol{w}_C^0)^T, (\boldsymbol{w}_3^0)^T]^T$

$$\boldsymbol{w}_A^0 = 0, \quad \begin{bmatrix} \boldsymbol{U}_A & \boldsymbol{R}_{AC} & \boldsymbol{R}_{A3} \\ \mathbf{0} & \boldsymbol{R}_C & \boldsymbol{R}_{C3} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{w}_C^0 \\ \boldsymbol{w}_3^0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_A \\ \boldsymbol{z}_C \\ \boldsymbol{z}_3 \end{bmatrix}$$

Thus we can easily show

$$\boxed{\delta_{\mathrm{TS}} = \sigma^{-2}(\|\boldsymbol{u}_0\|_2^2 - \|\boldsymbol{u}_a\|_2^2) = \sigma^{-2}\|\boldsymbol{w}_C^0\|_2^2}$$

Remarks

- The test statistic δ_{TS} can be computed in parallel with the the optimal estimates $\boldsymbol{x}_0$ under H_0 and $\boldsymbol{x}_a$ under H_a.
- Can easily show our method is numerically stable.
- Can find the covariance matrices of $\boldsymbol{x}_0$ and $\boldsymbol{x}_a$.
- Can extend the method to deal with the non-square $\boldsymbol{B}$ case.
- Can easily handle linear equality constraints.

Summary

- Computing GPS relative position estimates
  - Present a recursive LS approach for relative positioning based on carrier phase and code measurements for L1 carrier signal.
  - The algorithm is efficient—makes full use of structure, & numerically reliable—uses orthogonal transformations.
  - The approach allows: satellite rising/setting, more measurement equations, dynamic equations.
- Computing a test statistic for data quality control
  - Present a backward stable method
  - Can handle the singular noise covariance matrix

Numerical Linear Algebra Is Important and Useful in GPS !!!