Bias-robust L-estimators of Scale

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Optimal bias-robust L-estimators of scale parameters are derived. The model is that $X_1, \dots, X_n$ is a random sample from $F((x-\theta)/\sigma)$, where: (i) σ ($\sigma > 0$) is an unknown scale parameter to be estimated; (ii) θ ($-\infty < \theta < \infty$) is an unknown location parameter, to be regarded as a nuisance parameter in the scale estimation problem; and (iii) F is an unknown distribution which lies in the following ε-contamination neighborhood of a known distribution F_0:

$$\mathcal{F}_\varepsilon = \{F : F = (1-\varepsilon)F_0 + \varepsilon G, \text{ for some distribution } G\}, \tag{1}$$

where F_0 is assumed to have a unimodal density f_0 which is symmetric about 0.

We consider a general class $\mathcal{S}$ of L-estimators of σ, which are location- and scale-equivariant, and which are Fisher-consistent at F_0. The class $\mathcal{S}$ is large enough to include (normalized versions of) all mixtures of interquantile ranges as well as all trimmed variances centred at the sample median. The defining scale functional is

$$S_{M,q}(F) = \frac{\{\int_0^1 [F^{-1}(t) - F^{-1}(\frac{1}{2})]^q M(dt)\}^{1/q}}{c(F_0)}, \tag{2}$$

where q is a positive integer, M is a signed measure on (0,1) (subject to a natural symmetry condition but otherwise arbitrary), and $c(F_0)$ is a constant which makes $S_{M,q}$ Fisher-consistent at F_0.

Two types of bias-robustness problems are considered:

(i) to find, for fixed ε, $0 < \varepsilon < \frac{1}{2}$, the L-estimator in $\mathcal{S}$ which minimizes the maximal asymptotic bias as F ranges over $\mathcal{F}_\varepsilon$; and

(ii) to find the L-estimator in $\mathcal{S}$ with minimum gross error sensitivity (GES) at F_0.

Complete results are found for problem (ii), which is the limiting case of problem (i) as the proportion of contamination ε approaches 0. Under mild regularity conditions, the following expression is derived for the GES of an arbitrary member of $\mathcal{S}$ at F_0:

$$\mathrm{GES}(S_{M,q}, F_0) = \frac{\max\left\{\int_{\frac{1}{2}}^1 \frac{(s-\frac{1}{2})[F_0^{-1}(s)]^{q-1}}{f_0[F_0^{-1}(s)]} M(ds), \int_{\frac{1}{2}}^1 \frac{(1-s)[F_0^{-1}(s)]^{q-1}}{f_0[F_0^{-1}(s)]} M(ds)\right\}}{\int_{\frac{1}{2}}^1 [F_0^{-1}(s)]^q M(ds)}. \tag{3}$$

The main result is that the L-estimator which minimizes the GES is a mixture of at most two α-interquantile ranges, normalized to be Fisher-consistent at F_0. The key idea of the proof of this result is to note that the GES functional (3) is a function of three linear functionals of the mixing distribution on the quantiles, so that the generalized method of moment spaces [Collins and Portnoy (1981)] can be applied. In the special case that F_0 is strongly unimodal, the solution is the normalized interquartile range. This is also the case for some other F_0 which are not strongly unimodal, including the Cauchy distribution. These special-case solutions coincide with solutions previously obtained in classes of M-estimators of scale [see Hampel et al. (1986), Martin and Zamar (1993) and Collins (1999)].

In the general case where F_0 does not have a strongly unimodal density, graphical methodology can be used to used to find the minimum GES L-estimator. An example is given in which the unique Fisher-consistent L-estimator attaining minimum GES is a (normalized) *proper* mixture of two distinct α-interquantile ranges.

Similar results also hold for problem (i) of finding L-estimators of scale with minimax asymptotic bias in ε-contamination neighborhoods of F_0 when $\varepsilon > 0$. The minimax bias estimators, which depend upon both F_0 and the fixed value of ε, are seen to be mixtures of at most two α-interquantile ranges.

References

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