Nonlinear PDEs in Economics

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I. Background

Modelization: the economist identifies a stylized fact and proposes a *model*

Falsifiability: he (or preferably others) designs *experiments* with a view to disproving the model

Identifiability: adjust the parameters to *fit observations*; the model is called identifiable if all parameters can be uniquely determined from the data

II. The standard model

The microeconomic theory of the consumer states that each individual is characterized by a concave *utility function* $U : R_+^K \to R$ and a wealth $w \in R_+$, and that given a set of prices $p \in R_+^K$, he will buy the goods bundle $x \in R_+^K$ which maximizes his utility under the *budget constraint*:

$$\max_x U(x)$$
$$px \leq w$$

So the individual decision process is reduced to a concave optimization problem.

Is this a falsifiable model ?

III. Individual data

One cannot observe individual *utility*. But it has been known since Antonelli (1870) and Slutsky (1913) that, if you can observe individual *demand*, the model is falsifiable.

Introduce:

- the individual demand function :

$$x(p) = \arg\max \{U(x) \mid p'x = w\}$$

- the indirect utility function :

$$V(p) = \max \{U(x) \mid p'x = w\}$$

They are related by

$$D_p V(p) = -\lambda(p)\, x(p) \in R^K$$

So that:

$$x(p) = -\frac{1}{\lambda(p)} D_p V$$

- Question 1: when is a given vector field collinear to a gradient ?
- Question 2: when is it *positively* collinear to the gradient of a *convex* function ?

IV. Household data

Two individuals sharing the same budget constraint:

- $U_1(x_1)$ for the first
- $U_2(x_2)$ for the second
- $x = (x_1 + x_2)$ is observable
- $p' x = w$ budget constraint

Modelization: the individuals share the total budget, in an unspecified way, and each one solves his/hers own problem:

$$\begin{array}{cc} \max_x U_1(x) & \max_x U_2(x) \\ px \leq w_1(p) & px \leq w - w_1(p) \end{array}$$

Leading to the indirect utility functions:

$$\begin{aligned} V_1(p) &= \max \{U_1(x) \mid p'x = w_1(p)\} \\ V_2(p) &= \max \{U_2(x) \mid p'x = w - w_1(p)\} \end{aligned}$$

and to the equations:

$$\begin{aligned} D_p V_1(p) &= -\lambda_1(p)\left(x_1(p) - D_p w_1(p)\right) \\ D_p V_2(p) &= -\lambda_2(p)\left(x_2(p) + D_p w_1(p)\right) \end{aligned}$$

$$-\frac{1}{\lambda_1(p)} D_p V_1(p) - \frac{1}{\lambda_2(p)} D_p V_2(p) = x_1(p) + x_2(p) = x(p)$$

- Question 1: When is a given vector field a linear combination of two gradients ?
- Question 2: When is it a *positive* linear combination of the gradients of two *convex* functions ?

V. Market data

N individuals, each one with his own utility function U_n, his own wealth w_n, and his own budget. We observe the w_n and the *aggregate* demand:

$$X(p) = \sum x_n(p)$$

with

$$x_n(p) = \arg\max \{U_n(x) \mid p'x = w_n\}$$

As above, we have:

$$\begin{aligned} x_n(p) &= -\frac{1}{\lambda_n(p)} D_p V_n(p) \\ w_n &= -\frac{1}{\lambda_n(p)} p' D_p V_n(p) \end{aligned}$$

Substituting, we get:

$$X(p) = \sum_{n=1}^{N} \frac{w_n}{p' D_p V_n(p)} D_p V_n(p) \in R^K$$

- Question 1: does this system of nonlinear PDEs have a solution $(V_1, \dots V_N)$?
- Question 2: does it have a solution with *convex* $(V_1, \dots V_N)$ and *positive* $p' D_p V_n(p)$?

VI. The convex Darboux theorem

VI.1. The necessary conditions

Suppose $x : R^K \to R^K$ is such that:

$$x^k(p) = \sum_{n=1}^{N} \lambda_n(p) \frac{\partial V_n}{\partial p_k}$$

with $\lambda_n > 0$ and V_n convex, then:

$$\frac{\partial x^k}{\partial p_j} = \sum_{n=1}^{N} \lambda_n \frac{\partial^2 V_n}{\partial p_k \partial p_j} + \sum_{n=1}^{N} \frac{\partial \lambda_n}{\partial p_j} \frac{\partial V_n}{\partial p_k}$$

so that:

1. $\left(\frac{\partial x^k}{\partial p_j}\right)_{j,k} = S + M_N$, where S is symmetric and M_N has rank N
2. The restriction of $\left(\frac{\partial x^k}{\partial p_j}\right)_{j,k}$ to $\ker M_N$ is positive definite

VI.2. The sufficient conditions.

Theorem (Darboux) Condition (1) $\Longrightarrow x^k(p) = \sum_{n=1}^{N} \lambda_n(p) \frac{\partial V_n}{\partial p_k}$

Theorem (Chiappori-IE, Nirenberg-IE) If conditions (1) and (2) hold, then λ_n can be taken to be positive and V_n convex

As a result, the standard model is falsifiable from individual data.

VII. The Cartan-Kähler theorem

Consider the system of K equations with N unknowns:

$$\sum_{n=1}^{N} \frac{1}{\sum p_j \frac{\partial V_n}{\partial p_j}} \frac{\partial V_n}{\partial p_k} = X^k(p), \ 1 \le k \le K$$

Theorem (Chiappori and IE) If $N \ge K$, if the right-hand side is real analytic, then this system has local solutions: given any $\bar{p}$, and any set of values v_n^k such that

$$\sum_{n=1}^{N} \frac{1}{\sum \bar{p}_j v_n^j} v_n^k = X^k(\bar{p}), \ 1 \le k \le K$$

there exists convex functions V_n which satisfy the system in a neighbourhood of $\bar{p}$ and the initial conditions $\frac{\partial V_n}{\partial p_k}(\bar{p}) = v_n^k$

The proof goes by rewriting this as an exterior differential system.

$$-\sum_{n=1}^{N} \frac{1}{\lambda_n} v_n^k = X^k(\bar{p}), \ 1 \le k \le K$$
$$-\sum_{k=1}^{K} p_k v_n^k = \lambda_n(p), \ 1 \le n \le N$$
$$\sum_{k=1}^{K} dp_k \wedge dv_n^k = 0, \ 1 \le n \le N$$
$$dp_1 \wedge ... \wedge dp_K \ne 0$$

An applying the Cartan-Kähler theorem. This raises a mathematical question: would a similar result hold when $X(p)$ is C^∞ only ? Here is a simple example: find functions $u(x,y,z)$ and $v(x,y,z)$ such that

$$\frac{u_x}{u_z} + \frac{v_x}{v_z} = f(x,y,z)$$
$$\frac{u_y}{u_z} + \frac{v_y}{v_z} = g(x,y,z)$$

where the right-hand sides f, g are C^∞. Nirenberg and I have been working on this problem without success so far.