Distance

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Distance

Let $K, D \in \mathbb{R}^n$ be convex bodies.

$$d(K,D) = \inf\{\lambda \mid K + x \subset TD \subset \lambda \cdot (K+x)\}$$

where the infimum is taken over all $x \in \mathbb{R}^n$ and all affine operators $T : \mathbb{R}^n \to \mathbb{R}^n$.

$$d(K,D) \geq 1$$
$$d(K,D) \leq d(K,L) \cdot d(L,D)$$

Symmetric bodies

Theorem. *(John) Let D be a convex symmetric body in $\mathbb{R}^n$. Assume that B_2^n is the ellipsoid of maximal volume inscribed in D. Then*

$$B_2^n \subset D \subset \sqrt{n} B_2^n$$

$$d(D, B_2^n) \leq \sqrt{n} \;\Rightarrow\; d(K,D) \leq n$$

Theorem. *(Gluskin) Let*

$$B(\omega) = \operatorname{conv}(\pm\omega_1 \ldots \pm \omega_{3n})$$

where $\omega_1 \ldots \omega_{3n} \in S^{n-1}$ be independent random vectors uniformly distributed over the unit sphere S^{n-1}. Then with probability close to 1

$$d(B(\omega), B(\omega')) \geq cn.$$

General convex bodies

Theorem. *(John) Let D be a convex body in $\mathbb{R}^n$. Assume that B_2^n is the ellipsoid of maximal volume inscribed in D. Then*

$$B_2^n \subset D \subset nB_2^n$$

$d(D, B_2^n) \le n \;\Rightarrow\; d(K, D) \le n^2$

Gluskin's bodies: $d(K(\omega), K(\omega')) \le Cn$ with probability close to 1.

Lassak (1998):

(1) Let K be a convex body and let D be a convex symmetric body. Then

$$d(K, D) \le 2n - 1$$

(2) Let Δ_n be an n-dimensional simplex. Then

$$d(K, \Delta_n) \le n + 1$$

Theorem. *Let K, D be n-dimensional convex bodies. Then*

$$d(K, D) \le Cn^{4/3} \log^a n.$$

The proof uses random rotations and the MM^*-estimate.

General Dvoretzky Theorem

Theorem. *(Milman – Schechtman, 1996)*
Let $D \subset \mathbb{R}^n$ be a convex symmetric body. For any $m \leq n$ there exists a subspace $E \subset \mathbb{R}^n$ of dimension m such that

$$d_{D \cap E} \leq c \cdot \sqrt{\frac{m}{\log(1 + n/m)}}.$$

General convex bodies:

Gordon, Guedon and Meyer (1998): $m \leq \frac{n}{\log^2 n}$ random.

Litvak and Tomczak-Jaegeremann (1999): $m \leq cn$ non-random.

Litvak, Mankiewicz and Tomczak-Jaegeremann (2002): $m \leq cn$ random.

Problem: $m \geq cn$.

Quotient Subspace Theorem

Theorem. *(Milman) Let* $D \subset \mathbb{R}^n$ *be a convex symmetric body. For any* $\varepsilon < 1$ *there exists a section* B *of a projection of* D *with* $\dim B \geq (1-\varepsilon)n$ *such that*

$$d_B \leq \frac{C}{\varepsilon} \log \frac{1}{\varepsilon}$$

Notation: $U \prec V$ if $U \leq CV \log^a V$.

Theorem. *Let* K *be a convex body in* $\mathbb{R}^n$ *. Then for any* $\varepsilon > 0$ *there exist linear subspaces* $E_2 \subset E_1 \subset \mathbb{R}^n$ *such that* $\dim E_2 > (1-\varepsilon)n$ *and for a body* $D = P_{E_2}(K \cap E_1)$ *one has*

$$d_D \prec \frac{1}{\varepsilon^2}.$$

Volume estimates

$$K \cap (-K) \subset K \subset \mathrm{conv}(K, (-K)).$$

Denote $K - K = \{x - y \mid x, y \in K\}$.

Theorem. *(Rogers–Shephard, 1957)*

$$\mathrm{vol}\,(K - K) \le \binom{2m}{m} \mathrm{vol}\,(K)$$

Lemma. *(Milman and Pajor, 2000) Let K be an n-dimensional convex body and suppose that 0 is the center of mass of K. Then*

$$\left(\frac{\mathrm{vol}\,(\mathrm{conv}\,(K, -K))}{\mathrm{vol}\,(K \cap (-K))} \right)^{1/n} \le 2.$$

Lemma. *Let K be an n-dimensional convex body and suppose that 0 is the Santalo point of K. Then*

$$\left(\frac{\mathrm{vol}\,(\mathrm{conv}\,(K, -K))}{\mathrm{vol}\,(K \cap (-K))} \right)^{1/n} \le C.$$

M-ellipsoids

Theorem. *(Milman) Let K be a convex symmetric body. There exists an elipsoid $\mathcal{E}$ such that*

$$\left(\frac{\mathrm{vol}\,(K+\mathcal{E})}{\mathrm{vol}\,(K\cap\mathcal{E})}\cdot\frac{\mathrm{vol}\,(K^\circ+\mathcal{E}^\circ)}{\mathrm{vol}\,(K^\circ\cap\mathcal{E}^\circ)}\right)^{1/n}\leq C$$

$$\mathrm{vol}\,(K+D)^{1/n}\sim\mathrm{vol}\,(\mathcal{E}_K+\mathcal{E}_D)^{1/n}$$

$$\mathrm{vol}\,(\mathrm{conv}(K,D))^{1/n}\sim\mathrm{vol}\,(\mathrm{conv}(\mathcal{E}_K,\mathcal{E}_D))^{1/n}$$

$$\mathrm{vol}\,(K\cap D)^{1/n}\sim\mathrm{vol}\,(\mathcal{E}_K\cap\mathcal{E}_D)^{1/n}$$

Flatness

Let K be a convex body and let L be a lattice in $\mathbb{R}^n$.

Denote

$$\widetilde{\text{Flat}}(K, L) = \inf \#\{x \in E^{\perp}\},$$

where $(x + E)$ is a lattice hyperplane and $(x + E) \cap K \neq \emptyset$.

Let L^* be a dual alttice:

$$L^* = \{y \in \mathbb{R}^n \mid \forall x \in \mathbb{R}^n \langle x, y\rangle \in \mathbb{Z}\}$$

$$\text{Flat}(K, L) = \inf_{y \in L^* \setminus \{0\}} \left(\sup_{x \in K} \langle x, y\rangle - \inf_{z \in K} \langle z, y\rangle \right)$$

Theorem. *(Khinchinne) Let K be a convex body and let L be a lattice in $\mathbb{R}^n$. If $K \cap L = \emptyset$, then*

$$Flt(K, L) \leq flt(n).$$

$\mathrm{flt}(n) \leq n!$ – Khinchinne (1948)
$\mathrm{flt}(n) \leq e^{cn}$ – Babai (1986)
$\mathrm{flt}(n) \leq Cn^2$ – Kannan and Lovász (1988)

Theorem. *(Banaszczyk, 1993-1996)*
If $K \cap L = \emptyset$, then

$$Flt(K, L) \leq C \cdot \inf_{T} M(TK)M^*(TK)$$

over all affine transforms T.

MM^*-estimate

$$M(K) = \frac{1}{\sqrt{n}} \mathbb{E} \, \|g\|_K \, ,$$
$$M^*(K) = M(K^\circ) = \frac{1}{\sqrt{n}} \mathbb{E} \sup_{x \in K} \langle g, x \rangle$$

Here g is the standard Gaussian vector in $\mathbb{R}^n$.

Denote $K_x = K - x$ and let

$$R(K) = \inf_{T,x} M(TK_x) \cdot M^*(TK_x),$$

Let B be a convex symmetric body.

Figiel, Tomczak-Jaegermann (1979): $R(B) \leq K(B) = \|Rad : L_2(B) \to L_2(B)\|$.

Pisier(1980): $R(B) \leq C \log d_B$

$K(\Delta_n) \geq n/2$, while $M(\Delta_n) \cdot M^*(\Delta_n) \leq C \log n$.

Banaszczyk, Litvak, Pajor, Szarek(1995): $R(K) \leq C\sqrt{d_K}$

Theorem 1. *Any n-dimensional convex body K may be embedded in* $\mathbb{R}^n$ *, so that for every* $\varepsilon > 0$ *there exists an* $(1-\varepsilon)n$*-dimensional subspace* $E \subset \mathbb{R}^n$ *such that*

$$M^*(K) \leq C \log d_K \qquad \textit{and}$$
$$M(K \cap E) \prec \frac{C}{\varepsilon} \cdot \log d_K .$$

Theorem 2. *Any n-dimensional convex body K may be embedded in* $\mathbb{R}^n$ *, so that*

$$M(K)M^*(K) \prec n^{1/3}$$

Symmetrization

$$\frac{1}{2}M(K \cap (-K)) \leq M(K) \leq M(K \cap (-K))$$
$$\frac{1}{2}M^*(\mathrm{conv}(K, -K)) \leq M^*(K) \leq M^*(\mathrm{conv}(K, -K))$$

Can we embed K into $\mathbb{R}^n$ so that

$$M(K \cap (-K)) \cdot M^*(\mathrm{conv}(K, -K)) \leq C \log d_K ?$$

Sections of a difference body

Theorem. *Let $K \in \mathbb{R}^n$ be a convex body and let $F \subset \mathbb{R}^n$ be an m-dimensional subspace. Then*

$$\begin{aligned}&\mathrm{vol}\left((K-K)\cap F\right) \\ &\qquad \le \Big(C\phi(m,n)\Big)^m \cdot \sup_{x\in\mathbb{R}^n} \mathrm{vol}\left(K \cap (F+x)\right),\end{aligned}$$

where

$$\phi(m,n) = \min\left(\frac{n}{m}, \sqrt{m}\right).$$

$$\phi(m,n) \ge \min\left(\log\left(1+\frac{n}{m}\right), \sqrt{m}\right).$$

Problem. *Let $F \subset \mathbb{R}^n$ be an m-dimensional subspace and let $Z \subset \mathbb{R}^n$ be a convex body. Assume that*

$$P_{F^\perp} Z = Z \cap F^\perp$$

Find the minimal φ such that

$$\mathrm{vol}\,(P_F(Z)) \leq \varphi^m \cdot \sup_{y \in E} \mathrm{vol}\,((Z + y) \cap F)$$

Rogers – Shephard $\Rightarrow$ $\varphi \leq c \cdot n/m$

Inverse Brascamp – Lieb
(F. Barthe) $\Rightarrow$ $\varphi \leq \sqrt{m}$