Conservative Methods for Wave and Fluid Dynamics

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Many physical processes, such as wave and inviscid fluid dynamics, can be described to good approximation by conservative PDEs with an underlying Hamiltonian structure. In this framework, a Hamiltonian PDE is often viewed as an "infinite-dimensional" ODE and one transfers the properly generalized geometric concepts such as symplecticity, energy conservation, symmetries and first integrals to the PDE setting. A different, in some sense more traditional, approach is to take a local point of view and to derive the conservation laws associated with each conserved quantity. For example, the total energy of the semi-linear wave equation

$$u_{tt} = u_{xx} - V'(u)$$

is given by

$$H = \frac{1}{2}\int \Big[(u_t)^2 + (u_x)^2 + 2V(u)\Big]\, dx,$$

which is a first integral of motion. However, there is an underlying energy conservation law

$$\frac{1}{2}\Big[(u_t)^2 + (u_x)^2 + 2V(u)\Big]_t - [u_x u_t]_x = 0.$$

Similarly, the standard symplectic form is $\Omega = \int [du \wedge dv] dx$, $v = u_t$. The associated conservation law of symplecticity is given by

$$[du \wedge dv]_t - [du \wedge dw]_x = 0,$$

where $w = u_x$ is the "spatial" momentum. An example of great practical significance is provided by potential vorticity (PV) $q = (v_x - u_y + f_0)/h$ which is materially advected in the rotating shallow-water equations. Here we have the first integrals $I_f = \int hf(q)dxdy$, where h is the layer-depth and f is *any* function. The associated conservation laws are

$$[hf(q)]_t + \nabla_{\mathbf{x}}(hf(q)\mathbf{u}) = 0.$$

More generally, each conservation law gives rise to a first integral via spatial integration. The converse is not true in particular when it comes to numerical integration. Hence it seems quite appropriate to search for numerical methods that respect underlying conservation laws. There are a number of questions associated with such an approach:

- Which underlying conservation laws are fundamental to the physical process and, hence, should be preserved numerically.
- What do we mean by maintaining a conservation law under numerical discretization?
- Can we make progress toward a systematic design methodology for conservative numerical methods?

Quite obviously I will have to leave these questions largely un-answered during my talk. But I will present some case studies based on the one-dimensional semi-linear wave equation and the two-dimensional rotating shallow-water equations. I will discuss concepts such as multi-symplectic geometry as well as energy, momentum, and potential vorticity conservation.