A New General Solution of Two-Parameter Family for Four-Stage Fourth-Order Explicit Runge-Kutta Methods

Chisato Suzuki
suzuki@cs.sist.ac.jp
Department of Computer Science, Shizuoka Institute of Science and Technology, Jap

A new general solution of four-stage explicit Runge-Kutta methods,

$$\begin{cases} y_{n+1} = y_n + h\sum_{i=1}^{4} \mu_i k_i, \\ \quad k_1 = f(x_n,\, y_n), \quad k_i = f(x_n{+}\alpha_i h,\, y_n{+}h\sum_{j=1}^{i-1} \beta_{ij} k_j), \ (i{=}2,3,4), \end{cases}$$

for the initial value problem of scaler ordinary differential equation,

$$y' = f(x, y), \ y(x_0) = y_0, \quad (x{\geq}x_0),$$

is presented, which is characterized by $0{<}\alpha_2{<}\alpha_3{<}\alpha_4{<}1$, (or $0{<}\alpha_4{<}\alpha_3{<}\alpha_2{<}1$). More exactly speaking, it is shown under certain assumptions that the general solution of four-stage Runge-Kutta of fourth-order is given as

$$\begin{cases} \mu_1 = 1 - \mu_2 - \mu_3 - \mu_4. \\ \mu_2 = -(3 - 4\alpha_3 - 4\alpha_4 + 6\alpha_3\alpha_4)/(12\gamma(1-\gamma)(\gamma\alpha_3 - \alpha_4)\alpha_3^2), \\ \mu_3 = -(2\gamma\alpha_3(2 - 3\alpha_4) - (3 - 4\alpha_4))/(12(1-\gamma)(\alpha_3 - \alpha_4)\alpha_3^2), \\ \mu_4 = -(2\gamma\alpha_3(2 - 3\alpha_3) - (3 - 4\alpha_3))/(12\alpha_4(\gamma\alpha_3 - \alpha_4)(\alpha_3 - \alpha_4)), \\ \beta_{21} = \alpha_2, \quad \beta_{31} = \alpha_3 - \beta_{32}, \\ \beta_{32} = \alpha_3(1-\gamma)(3 - 4\alpha_4)/(2\gamma(3 - 4\alpha_4 - 4\gamma\alpha_3 + 6\gamma\alpha_3\alpha_4)), \\ \beta_{41} = \alpha_4 - \beta_{42} - \beta_{43}, \\ \beta_{42} = \alpha_4(\gamma\alpha_3 - \alpha_4)(\gamma\alpha_3(9{-}4\alpha_3{-}20\alpha_4{+}4\alpha_3\alpha_4{+}12\alpha_4^2) \\ \quad -(5\alpha_3{-}4\alpha_3^2{-}2\alpha_4)(3{-}4\alpha_4))/(2\gamma(1{-}\gamma)(3{-}4\alpha_4)(3 - 4\alpha_3{-}4\gamma\alpha_3{+}6\gamma\alpha_3^2)\alpha_3^2), \\ \beta_{43} = (\gamma\alpha_3 - \alpha_4)\alpha_4(\alpha_3 - \alpha_4)(3 - 4\alpha_4 - 4\gamma\alpha_3 + 6\gamma\alpha_3\alpha_4)/ \\ \qquad ((1-\gamma)(3 - 4\alpha_4)(3 - 4\alpha_3 - 4\gamma\alpha_3 + 6\gamma\alpha_3^2)\alpha_3^2), \end{cases}$$

and $\alpha_2{=}\gamma\alpha_3$, where

$$\gamma = |3 - 4\alpha_4|/\sqrt{9 - 4\alpha_3 - 20\alpha_4 + 4\alpha_3\alpha_4 + 12\alpha_4^2}\,.$$

In this solution, α_3 and α_4 are two free-parameters. Also we can take γ and α_4 as two free-parameters, since α_3 can be expressed by γ and α_4 as

$$\alpha_3 = (\gamma^2(9 - 20\alpha_4 + 12\alpha_4^2) - (3 - 4\alpha_4)^2)/(4\gamma^2(1 - \alpha_4)).$$

Especially, when γ and α_4 are adopted as free-parameters, we can prove the following theorem that gives the characterization of the general solution.

Theorem For γ $(0<\gamma<1)$ and α_4 $(0<\alpha_4<1)$, if $0<\alpha_3<\min(1, \psi(\alpha_4))$, where $\psi(z)=z+\frac{1}{4}(3-z)^2/(1-z)$, then it holds that $0<\alpha_2<\alpha_3<\alpha_4<1$. Moreover, for $\gamma>1$ and α_4 $(0<\alpha_4<1)$, if $0<\gamma\alpha_3<1$, then it holds that $0<\alpha_4<\alpha_3<\alpha_2<1$.

The following is a fourth-order method obtained from the general solution by putting $\gamma=1/2$ and $\alpha_4=5/6$.

$$\begin{cases} y_{n+1} = y_n + \frac{h}{10}(k_1 + 5k_3 + 4k_4), \\ k_1 = f(x_n,\, y_n), \\ k_2 = f(x_n+\frac{1}{6}h,\, y_n+\frac{1}{6}hk_1), \\ k_3 = f(x_n+\frac{1}{3}h,\, y_n+\frac{1}{3}hk_2), \\ k_4 = f(x_n+\frac{5}{6}h,\, y_n+\frac{5}{8}hk_1-\frac{5}{3}hk_2+\frac{15}{8}hk_3). \end{cases}$$