

# Rigorous Shadowing of Numerical Solutions of ODEs by Containment

Wayne B Hayes

`wayne@cs.toronto.edu`

University of Toronto, Canada

An exact trajectory of a dynamical system lying close to a numerical trajectory is called a *shadow*. We present a general-purpose method for proving the existence of finite-time shadows of numerical ODE integrations of arbitrary dimension in which some measure of hyperbolicity is present and there is either 0 or 1 expanding modes, or 0 or 1 contracting modes. Much of the rigor is provided automatically by interval arithmetic and validated ODE integration software that is freely available. The method is a generalization of a previously published *containment* process that was applicable only to two-dimensional maps. We extend it to handle maps of arbitrary dimension with the above restrictions, and finally to ODEs. The method involves building  $n$ -cubes around each point of the discrete numerical trajectory through which the shadow is guaranteed to pass at appropriate times. The proof consists of two steps: first, the rigorous computational verification of an *inductive containment property*; and second, a simple geometric argument showing that this property implies the existence of a shadow. The computational step is almost entirely automated and easily adaptable to any ODE problem. The method allows for the rescaling of time, which is a necessary ingredient for successfully shadowing ODEs. Finally, the method is local, in the sense that it builds the shadow inductively, requiring information only from the most recent integration step, rather than more global information typical of several other methods. The method produces shadows of comparable length and distance to all currently published results.