Integration of implicit ODEs around singular equilibria

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In the numerical analysis of ordinary differential equations (ODEs) $\dot{x} = f(x)$, with sufficiently smooth $f : \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa}$, qualitative aspects are of central importance in a variety of applications. Focusing on hyperbolic equilibria, a wide family of numerical schemes yield discrete dynamical systems having a nearby fixed point which inherits the stability properties of the original equilibrium [2]. Our purpose is to discuss conditions under which the discrete mapping displays a *quadratic convergence* to the fixed point x^*, in the sense that the error evolution satisfies $\|x^{(k+1)} - x^*\| \leq c\|x^{(k)} - x^*\|^2$ around x^* for some $c > 0$.

The key requirement in this direction is that the spectrum of x^* be mapped into the origin in the integration process. To achieve this, consider the so-called continuous Newton method (see [1] and references therein) for f:

$$\dot{x} = -J(x)^{-1} f(x), \tag{1}$$

where $J = f'$. If $J(x^*)$ is invertible, the spectrum of x^* has a unique index-1 eigenvalue equal to -1, and Euler discretization of (1), with stepsize 1, leads to the quadratically convergent iteration defined by the classical Newton method for root-finding problems.

If, on the contrary, $f(x^*) = 0$ with singular $J(x^*)$, system (1) must be rewritten in the quasilinear form $-J(x)\dot{x} = f(x)$. It follows that, under generic assumptions, x^* is an equilibrium of a singular ODE or a *singular index-0* differential-algebraic equation (DAE)

$$A(x)\dot{x} = f(x). \tag{2}$$

In this type of problems, convergence may be guaranteed only from restricted regions, and quadratic convergence is lost. Our purpose is to show that this loss of quadratic convergence may be seen as the result of a (in a certain sense) defective transformation of spectra at singular solutions. Quadratic convergence from certain domains may be recovered using explicit Runge-Kutta schemes different from Euler's method.

References

[1] R. Riaza and P. J. Zufiria, Stability of singular equilibria in quasilinear implicit differential equations, *J. Differential Equations* **171** (2001) 24-53.

[2] A. M. Stuart and A. R. Humphries, *Dynamical Systems and Numerical Analysis,* Cambridge University Press, 1996.