Preservation of asymptotic stability in the numerical solution of Delay Differential Equations restated as Partial Differential Equations

Stefano Maset
maset@univ.trieste.it
Department of Mathematical Sciences, University of Trieste, Italy

An alternative approach for numerically solving a Delay Differential Equation (DDE), different from the classic step-by-step integration, is based on restating the DDE as a Partial Differential Equation (PDE).

Let us consider the DDE

$$\begin{cases} y'(t) = f(y(t), y(t-\tau)) & t \geq 0 \\ y(t) = \varphi(t) & t \in [-\tau, 0] \end{cases} \tag{1}$$

where $L, M \in \mathbb{C}^{m \times m}$ and $\varphi \in C([-1,0], \mathbb{C}^{m \times m})$. By setting

$$V(t,\theta) = y(t+\theta) \quad t \geq 0,\ \theta \in [-\tau, 0] \tag{2}$$

we can restate (1) as the hyperbolic PDE

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial \theta} \quad t \geq 0,\ \theta \in [-\tau, 0] \tag{3}$$

provided with the boundary condition

$$\frac{\partial U}{\partial t}(t,0) = f(U(t,0), U(t,-\tau)) \quad t \geq 0 \tag{4}$$

and the initial condition

$$U(0,\theta) = \varphi(\theta) \quad \theta \in [-\tau, 0]. \tag{5}$$

Now we have two ways for solving (3). The first one is a semi-discretization in the variable θ. This is the method of lines as applied to the PDE (3) and yields an Ordinary Differential Equation. The second way is a semi-discretization in the variable t. This semi-discretization is given by a Runge-Kutta (RK) method where at every step we have to solve a boundary value problem.

In this talk we present results on the preservation of asymptotic stability when we apply these two semi-discretization to the linear autonomous DDE

$$y'(t) = Ly(t) + My(t-\tau) \tag{6}$$

where $L, M \in \mathbb{C}^{m\times m}$.
The main results are:

- For the scalar equation (6) and for the semi-discretization in the variable θ, we have that an important class of discretization schemes based on RK methods do not preserve the asymptotic stability for every choice of the stepsize $\Delta\theta$. We can recover the preservation of the asymptotic stability by considering a full-discretization in the variables θ and t. In particular there exist constants C and α, which depend on the discretization schemes in the variables θ and t, such that for all stepsizes $\Delta\theta$ and Δt satisfying $\Delta\theta \leq C(\Delta t)^{\alpha}$ the preservation of asymptotic stability is guaranteed.

- For the semi-discretization in the variable t, we have that $A-$stable RK methods preserve the asymptotic stability for all stepsize Δt.