Some Results on Wavelet Frames

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There have been many results for the system $\{a^{-m/2}\psi(a^{-m}x - nb)\}_{m,n\in Z}$ to be a frame for $L^2(R)$, where $a > 1$ and $b > 0$. In this talk we first study some general conditions for the system $\psi_{j,k}(x) = s_j^{-1/2}\psi(s_j^{-1}x - b_k)$, $j, k \in Z$, to be a frame, where $s_j > 0$ and b_k are reals. First we give a necessary condition which extremely extends the one obtained by Chui and Shi in 1993. Secondly we point out that the necessary condition is also sufficient when ψ is band-limited, moreover we give sufficient conditions, which exclusively depend on ψ, for $\{\psi_{j,k}\}$ to be a frame for a large class of $\{s_j\}$ and $\{b_k\}$. At the last we characterize a bounded measurable E such that ψ is a frame function with $\hat{\psi} = \chi_E$.

The main results of this paper are as follows.

Theorem 1 *If $s_j^{-1/2}\psi(s_j^{-1}x - b_k), j, k \in Z$, is a frame with bounds $A > 0$ and $B < \infty$, where $\psi \in L^2(R), s_j > 0$ and $\{b_k\}$ is a r-Fourier frame sequence ,then*

$$\frac{A}{B_r} \leq \sum_{j\in Z} |\hat{\psi}(s_j w)|^2 \leq \frac{B}{A_r},\ a.e.$$

Theorem 2 *Suppose $\psi \in B_\Omega \bigcap L(R)$ for some $\Omega > 0$, where $B_\Omega = \{f \in L^2(R) : \hat{f}(w) = 0 \text{ for any } |w| > \Omega\}$, and $s_j > 0$. Then the following two assertions are equivalent: (i) There is a r-Fourier frame sequence $\{b_k\}$ with $r > 2\Omega$ such that $\psi_{j,k}(x) = s_j^{-1/2}\psi(s_j^{-1}x - b_k), j, k \in Z$, is a frame for $L^2(R)$; (ii) $a \leq \sum_{j\in Z} |\hat{\psi}(s_j w)|^2 \leq b$, a.e., for some $a > 0$ and $b < \infty$. Furtheremore if (ii) is satisfied, then $\{\psi_{j,k}\}$ is a frame for any r-Fourier frame sequence $\{b_k\}$, where $r > 2\Omega$.*

Theorem 3 *Suppose $\psi \in B_\Omega \bigcap L(R)$ for some $\Omega > 0$. If $|\hat{\psi}(w)| \leq A|w|^\alpha$ near $w = 0$ for some $A > 0$ and $\alpha > 0$ and 0 is an isolated zero of $\hat{\psi}(w)$, then $\{s_j^{-1/2}\psi(s_j^{-1}x - b_k)\}_{j,k\in Z}$ is a frame for $L^2(R)$ for any sequence $\{s_j\}_{j\in Z}$ of positives satisfying $0 < \inf_j \frac{s_j}{s_{j+1}} \leq \sup_j \frac{s_j}{s_{j+1}} < 1$ and any r-Fourier frame sequence $\{b_k\}_{k\in Z}$, where $r > 2\Omega$.*

Theorem 4 *Suppose that E is a bounded set with $0 < m(E) < \infty$, $a > 1$ and $\widehat{\psi}(w) = \chi_E(w)$. Then ψ is a frame function for this a and some $b > 0$, if and only if there is an integer K such that*

$$E_+ \triangleq E \bigcap (0, \infty) = \bigcup_{k=-\infty}^{K} a^k A_k\ ,\ E_- \triangleq E \bigcap (-\infty, 0) = \bigcup_{k=-\infty}^{K} a^k B_k\ ,$$

*where $\{A_k\}_{k\leq K}$ and $\{B_k\}_{k\leq K}$ are finitely disjoint partitions of $(1, a]$ and $[-a, -1)$ respectively. Furthermore when the condition (**??**) is satisfied, ψ is a frame function for this a and any b with $0 < b < \frac{1}{\sup E - \inf E}$.*

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