Pacific Institute for the Mathematical Sciences

An analytical study on the dynamical contact angle of a drop in steady-state motion

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Preprint number: PIMS-04-12
Received on July 2, 2004

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Abstract

In this paper we study the dynamical contact angle at three phase (fluid-fluid-solid) contact points for the 2 dimensional steady-state moving liquid drop. We present a result giving the speeds at the contact points as a function of the dynamical contact angle, in the form of a simple algebraic expression. The only physical constants involved are the viscosities of the fluids and the surface tension.

Our approach is based only on the hypotheses that the fluids obey Navier-Stokes equation and that on the fluid-fluid interface the free stress equation holds. So, here the dynamical contact angle is defined as the angle in the region where the Navier-Stokes model is valid. Under these hypotheses it follows that at the contact points the local speeds are related through simple equations and the stress is singular, but well-defined in the Cauchy sense. The leading order singularity at the contact point is identified using a blow-up technique which leads to a reduced problem that contains the most singular behavior. This problem is a Stokes one in a sector plane which has a simple solution.

The result we obtain is based on a dynamical force balance equation which leads to a simple contact point speed versus contact angle relation. The originality of this work is that it is based on simple hypotheses and it deals with the singularity, without additional assumption on the stress.

1 Introduction

This work is a contribution on the analysis of the relationship steady-state drop speed versus dynamical contact angle. Typically, a liquid drop at steady state motion in a flat horizontal surface is considered. It is assumed that the dynamical contact angles at three phase contact points (from now on we will refer to them simply as contact points) depend on the steady-state drop motion speed. This problem has been the subject of research for many years and as a result there exist many different experimental studies and theoretical models.

Several generally accepted fact have been derived from the experimental observations. It is observed that in general the receding contact angle tends to 0, and advancing one tends to π, when the steady state drop speed tends to ∞. However, there are cases not obeying to this rule.

Also, it is observed that the drop fluid rolls, but close to the contact points there is a small region where the drop fluid rolls in the inverse sense to the overall rolling, [4]. The other fluid has simpler dynamics, it simply rolls.

Experimentally it has been found that at the solid interface, receding contact point side, there exists a thin layer of drop fluid. This shows that not all the drop fluid is moving - a very small amount of it is stuck to the solid interface, possibly being responsible for modification of the fluid-solid interaction, and consequently fluid-solid surface tension.
Another interesting phenomena observed by experiments is the so-called hysteresis: the drop must pass a certain threshold before it will begin to move, or stated differently, the dynamical contact angle does not tend to the statical contact angle when the drop speed tends to zero.
It has been argued from physical point of view the presence of singularity is more attractable by considering the speed as a multi-valued function at contact points, see [4]. This is the approach we consider.
From theoretical point of view (a very good bibliography of existing models is given in [12]), basically, the dynamical contact angle is found by postulating a law for dynamical contact angle, [3], [8] (the law is motivated by experimental observations and of course this approach often involves empirical constants); by computing the dynamical contact angle using the force balance equation. When only hydrodynamics is used to explain the phenomena, this approach leads to a stress singularity and up to our knowledge, the analysis does not go further. In other works, when the phenomena is modeled using hydrodynamics and diffusion theory, like in [11], [12], there is no stress singularity on fluid-solid interfaces. In general these last models involve several physical constants that require additional measurements. An important fact pointed in [12] is the change of density close to the contact points, which implies a change of the surface tension.
Using the non-slip condition approach and only hydrodynamics, the moving drop problem will lead to non-integrable stresses (in the classical sense). To avoid such a situation several authors, [2], [7], propose that the area enclosing contact points should be divided into three regions. In the inner region the so-called slip condition is given, which consists of imposing tangential stress proportional to the tangential speed at the boundary. In this case, the non-integrable stress singularity is removed. In the outer region the no-slip condition is imposed (the fluids stick to the solid). In order to match these two regions, an intermediate region is required. The regions are matched asymptotically. The slip condition is introduced not from physical considerations but simply in order to eliminate the singularity.
The most important issue arising when modeling the drop motion is the boundary conditions on fluid-solid interfaces. While it is accepted that the fluids stick on this interface when sufficiently far from the contact points, the boundary conditions on a fluid-solid interface close to the contact point is a major subject of discussion. This will be discussed is this work as well as the computation of the stress singularity arising close to the contact points.

The assumptions we make are: the fluids obey to Navier-Stokes equations and the viscosities are constants; the fluid-fluid interface is a C^2 curve having a surface tension σ; the stress free equation holds on the fluid-fluid interface.
The dynamical contact angle we consider is the one between the fluid-fluid interface and the fluid-solid interface in the region where Navier-Stokes equations are valid. With the assumption that the fluid-fluid interface is a C^2 curve, for the speed versus dynamical contact angle relation it is sufficient to consider only a local analysis close to contact points.
Based on these assumptions it follows that at the contact points the stress is non-integrable in L^1 sense, but perfectly defined in the Cauchy sense (which we will consider). From the stress

free equation on the fluid-fluid interface it follows that the speed has opposite directions on the two sides of the fluid-solid interface defined at the contact points, meaning that at least one of the fluids must slip. In the same spirit of the well-known Young's equation, a local force balance yields an equation satisfied by the dynamical contact angle and the contact point speeds. Speed versus dynamical contact angle relation involves the surface tension force f acting on the contact point. If the surface tensions are constant and the same as in the statical case, we obtain simple formulas, which explain the behavior of fluids close to the contact point, in qualitatively agreement with the experiments.
But, several facts suggest the existence of a surface tension variation, which is not included in this work. Indeed, the behavior of the dynamics close to fluid-solid interface at contact points (fluid dissociate or meet each other), the presence of a thin layer of drop fluid between the other fluid and the solid is responsible for the existence of a gradient surface tension, as stated in [12]. The force f then may be found by analysis at molecular level or by stating a constitutive relation in the thin layer between the fluid and the solid, like in [13]. We plan to address these questions in the upcoming work.

In the next section, the basic mathematical problem will be posed. In Section 3, the reduced problem near contact points will be derived followed by the main result stating speed versus contact angle relationship. In Section 5, comparison to experimental results are made followed by a short summary.

2 Mathematical description of the problem

Let $D \subset \mathbb{R}^2$ be an infinite band domain defined by two parallel lines of distance L. Let also $\Omega \subset D$ (the drop fluid drop) be an open connected set, $\tilde{\Omega} = D \backslash \overline{\Omega}$ (the air), such that $\partial(\partial D \cap \partial\Omega) = \{O, O'\}$ (the contact points) as shown in Figure 1. In the rest of the paper we will focus the analysis on the contact points. It is assumed that $\Gamma = \partial\Omega \backslash \partial D$ is a C^2 boundary. The origin is placed on O and x_1 axis is chosen tangent to $\partial\Omega$ at O. Let ξ be a C^2 function such that $\Gamma = \{(x_1, \xi(x_1)\}$ and let $\omega < 0$, resp. $\tilde{\omega} > 0$, be the angle at O that ∂D makes with $\partial\Omega$ inside of Ω, resp. $\tilde{\Omega}$.
Let assume that the droplet is moving with steady-state speed U. With respect to a coordinate system moving with the steady-state speed U, the fluid obeys the following standard equations for viscous immiscible incompressible flow

$$\begin{aligned}
-\boldsymbol{\mu}\Delta\mathbf{u} + \boldsymbol{\rho}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= g, \quad in \quad D\backslash\Gamma, && (1)\\
\nabla\cdot\mathbf{u} &= 0, \quad in \quad D\backslash\Gamma, && (2)\\
[\mathbf{u}] &= 0, \quad on \quad \Gamma, && (3)\\
\left[\boldsymbol{\mu}\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n}\right] &= \sigma\mathcal{H}\mathbf{n}, \quad on \quad \Gamma, && (4)\\
\mathbf{u}\cdot\mathbf{n} &= 0, \quad on \quad \Gamma, && (5)
\end{aligned}$$

where $\mathbf{u}$ is the speed, p the pressure, $g \in \mathbb{L}^2(\mathbb{R}^2;\mathbb{R}^2)$ a given function, $\boldsymbol{\mu} = \mu\mathbb{1}_\Omega + \tilde{\mu}\mathbb{1}_{\tilde{\Omega}}$ is the viscosity ($\mu, \tilde{\mu} > 0$), $\boldsymbol{\rho} = \rho\mathbb{1}_\Omega + \tilde{\rho}\mathbb{1}_{\tilde{\Omega}}$ is the density ($\rho, \tilde{\rho} > 0$), σ is the surface tension coefficient, $\mathbf{n}$ the outward unit normal vector to Γ, $\mathcal{H}$ is the mean curvature of Γ and $[\cdot] = \cdot|_\Omega - \cdot|_{D\backslash\Omega}$ denotes the jump across Γ.

We assume the following boundary conditions

$$\mathbf{u} = \boldsymbol{\alpha} \quad on\ \partial D \cap \partial\Omega, \tag{6}$$

$$\mathbf{u} = \tilde{\boldsymbol{\alpha}} \quad on\ \ \partial D \backslash \partial\Omega, \tag{7}$$

$$\mathbf{u} = O(1), \quad at\ \ \infty, \tag{8}$$

$$p(x_1, \cdot) = \pm p_0, \quad at\ \ x_1 = \pm\infty, \tag{9}$$

where $p_0 \in \mathbb{R}$ and $\boldsymbol{\alpha} \in C^0(D \cap \partial\Omega)$, $\tilde{\boldsymbol{\alpha}} \in C^0(D \cap \partial\tilde{\Omega})$ are two continuous functions such that $\lim\limits_{r\to 0} \boldsymbol{\alpha}(\omega, r) = \alpha\,(\cos\omega, \sin\omega)$, $\lim\limits_{r\to 0} \tilde{\boldsymbol{\alpha}}(\tilde{\omega}, r) = \tilde{\alpha}\,(\cos\tilde{\omega}, \sin\tilde{\omega})$, $\alpha, \tilde{\alpha} \in \mathbb{R}$ and $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} = -U(\cos\omega, \sin\omega)$ out of an arbitrary small neighborhood of contact points. Our main concern in this paper is the relation between U and the angle ω at the point O.

Remark 2.1 *If we assume that fluids stick to the solid boundary then $\alpha = -\tilde{\alpha}$ is the speed of the drop U. As it will be clear in the Proposition 4.1, for consistent mathematical reasons we cannot assume that both fluids stick to the solid boundary, at least close to the fluid-fluid-solid contact points.*

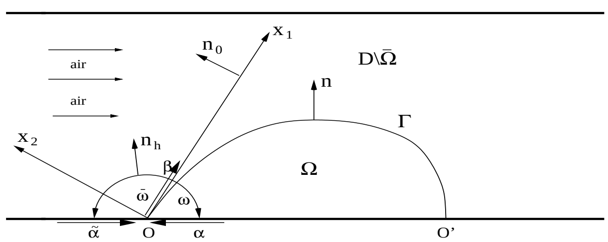

Figure 1:

Up to our knowledge there are no results giving the existence and regularity of $\mathbf{u}$ and p formulated here-above. The main difficulty is that $\mathbf{u} \notin H^1(\Omega \cup \tilde{\Omega})$ (see Remark 4.3). However, there are reasons to assume (see again Remark 4.3) that

1. $\mathbf{u} \in L^\infty(\Omega)$, $\mathbf{u} \in H^1(\Omega^h)$, $p \in L^2(\Omega^h)$ where $\Omega^h = \Omega \backslash B(O, h)$ for any $h > 0$ and (1)-(9) are satisfied in the distribution sense, where $B(O, h) = \{|x - O\| < h\}$.
2. $\exists \eta > 0$ such that for any $0 < h < 1$, if $\Omega_h = \Omega \cap B(O, h)$ then
$$\||x|^\eta(|\nabla\mathbf{u}(x)| + |p(x)|)\|_{L^2(\Omega_h)} < Ch^\eta, \tag{10}$$
3. $\mathbf{u}$ is continuous at O in any of the directions $\theta = 0$, resp. $\omega, \tilde{\omega}$, and the limit is $\alpha(\cos\omega, \sin\omega)$, resp. $\beta(1, 0)$, $\tilde{\alpha}(\cos\tilde{\omega}, \sin\tilde{\omega})$,
4. Similar hypothesis for $\mathbf{u}$, p are considered in $\tilde{\Omega}$, $\tilde{\Omega}_h = \tilde{\Omega} \cap B(O, h)$ and $\tilde{\Omega}^h = \tilde{\Omega} \backslash B(O, h)$.

With these assumptions, we can obtain the following equation for local force balance at contact points

$$f + \lim_{h\to 0} \int_{\partial D_h} T(\mathbf{u}) \cdot \mathbf{n}_h = 0. \tag{11}$$

Indeed, a net force f at the contact point O arises if the angle ω is not equal to the equilibrium contact angle ω_0, a material parameter. The measure force equal to $\sigma\mathcal{H}\mathbf{n}$ acts on Γ. In $D_h := D \cap B(O,h)$ act the inertia forces $\int_{D_h} \rho\mathbf{u}\cdot\nabla\mathbf{u}$, the gravitational forces $\int_{D_h} \rho g$,the tension forces $-\int_{\Gamma\cap D_h} \sigma\mathcal{H}\mathbf{n}_h$, and in ∂D_h act the stress forces equal to $\int_{\partial D_h} T(\mathbf{u})\cdot\mathbf{n}_h$ where $\mathbf{n}_h$ is the outward unit normal vector to ∂D_h, $\boldsymbol{\rho}$ is the density function of fluids and $T = \{T_{ij}(\mathbf{u})\}$ is the stress tensor given by

$$T_{ij}(\mathbf{u}) = -p\delta_{ij} + \boldsymbol{\mu}\frac{\partial \mathbf{u}_i}{\partial x_j}, \quad i,j = 1,2, \quad (\delta_{ij} \textit{ is Kronecker's symbol}). \tag{12}$$

Writing the Cauchy constitutive equation in D_h we have

$$\int_{D_h} \boldsymbol{\rho}\mathbf{u}\cdot\nabla\mathbf{u} = f + \int_{\partial D_h} T(\mathbf{u})\cdot\mathbf{n}_h + \int_{D_h} \boldsymbol{\rho} g - \int_{\Gamma\cap D_h} \sigma\mathcal{H}\mathbf{n}.$$

Letting h tend to 0 we obtain

$$\lim_{h\to 0}\int_{D_h} \boldsymbol{\rho}\mathbf{u}\cdot\nabla\mathbf{u} = f + \lim_{h\to 0}\int_{\partial D_h} T(\mathbf{u})\cdot\mathbf{n}_h. \tag{13}$$

Note that under the regularity assumptions (10) we have

$$\begin{aligned}\left|\int_{D_h} \mathbf{u}\cdot\nabla\mathbf{u}\right| &\le \|\mathbf{u}\|_\infty \int_{D_h} |\nabla\mathbf{u}| = \|\mathbf{u}\|_\infty \int_{D_h} |x|^{-\eta}|x|^{\eta}\,|\nabla\mathbf{u}| \\ &\le \|\mathbf{u}\|_\infty \||x|^{-\eta}\|_{L^2(D_h)}\||x|^{\eta}\nabla\mathbf{u}\|_{L^2(D_h)} \le Ch\|\mathbf{u}\|_\infty,\end{aligned} \tag{14}$$

which with (13) proves (11).

Remark 2.2 *In general $\mathbf{u} \notin H^2(\Omega)$, $p \notin H^1(\Omega)$. This means that the integral in (11) on ∂D_h needs particular attention. In fact, the integral on $D \cap \partial B(O,h)$ is well defined and we need only to deal with integral on $\partial D \cap B(O,h)$. As we will see later it is well defined in the Cauchy sense.*

Remark 2.3 *In the case when $\mathbf{u}$ and p are sufficiently regular functions then the limit integrals in (11) are zero. For example, if $\mathbf{u}$, resp. p, is in $H^2(\Omega\cup\tilde{\Omega})$, resp. $H^1(\Omega\cup\tilde{\Omega})$, then these limits vanish and we obtain the usual statical angle condition $f = 0$. But in general we have to consider these limit integrals because, a priori, $\mathbf{u}$ and p are not sufficiently regular.*
The singularity of stresses near point O has been pointed out by several authors, see for example [4]. Up to our knowledge, none of the authors has dealt with the singularity - they impose conditions on the stress in the close neighborhood of contact point in order to eliminate the singularity.

Equation (11) defines a balance between the singular stresses and the point forces arising from a contact angle different from the equilibrium value. The balance is made in both the tangential and normal directions. The next section of the paper will deal with a Stokes problem obtained from (1)-(9) using an asymptotic technique near the contact point. The resulting equation is simple and can be resolved by hand. This allows easy computation of the relationship (11) as h tends to 0. This will give the relationship between the speed and the dynamical angle at contact points.

3 Reduced asymptotic problem at contact points

Let be $0 < h < 1$. Under the transformation $y = h^{-1}x$ the domain Ω_h, resp. $\tilde{\Omega}_h$, is transformed in some curved domain which, when $h \to 0$ tends to the sector domains, $S \cap B(O, 1)$, resp. $\tilde{S} \cap B(O, 1)$, where $S = \{(r, \theta), r > 0, \omega < \theta < 0\}$ and $\tilde{S} = \{(r, \theta), r > 0, 0 < \theta < \tilde{\omega}\}$. Let us

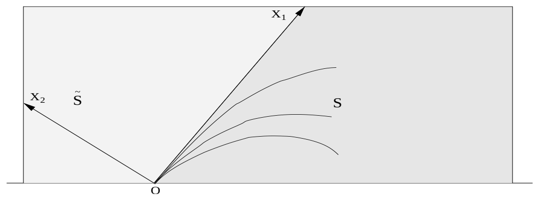

Figure 2:

define the following functions

$$\mathbf{u}_h(y) = \mathbf{u}(x), \qquad p_h(y) = hp(x).$$

If $A \subset S \cup \tilde{S}$ is an arbitrary bounded open set, it is clear that $\mathbf{u}_h$ and p_h are well defined in A for a sufficiently small h. Moreover, according to hypothesis (10) we have

$$\|\mathbf{u}_h\|^2_{L^2(A)} = \int_A |\mathbf{u}_h(y)|^2 dy = h^{-2} \int_{hA} |\mathbf{u}(x)|^2 dx \leq |A| \|\mathbf{u}\|^2_{L^\infty(D)},$$

$$\||y|^\eta \nabla \mathbf{u}_h(y)\|^2_{L^2(A)} = \int_A (|y|^\eta \nabla \mathbf{u}_h(y))^2 dy = h^{-2\eta} \int_{hA} (|x|^\eta \nabla \mathbf{u}(x))^2 dx \leq C(A),$$

$$\||y|^\eta p_h\|^2_{L^2(A)} = \int_A (|y|^\eta p_h(y))^2 dy = Ch^{-2\eta} \int_{hA} (|x|^\eta p(x))^2 dx \leq C(A).$$

These inequalities are easily proven using hypothesis (10). It follows that there exists a subsequence $(\mathbf{u}_h, p_h)$ and a $(\mathbf{v}, q)$ such that

$$\begin{array}{llll} \mathbf{u}_h & \rightharpoonup & \mathbf{v} & in \quad L^2_{loc}(S \cup \tilde{S}), \\ |y|^\eta \nabla \mathbf{u}_h & \rightharpoonup & |y|^\eta \nabla \mathbf{v} & in \quad L^2_{loc}(S \cup \tilde{S}), \\ |y|^\eta p_h & \rightharpoonup & |y|^\eta q & in \quad L^2_{loc}(S \cup \tilde{S}), \end{array} \tag{15}$$

where $L^2_{loc}(S \cup \tilde{S}) = \bigcap_{r>0} L^2((S \cup \tilde{S}) \cap B(O, r))$.

Proposition 3.1 *The functions* $\mathbf{v}$, q *satisfy in the sense of distributions the following equations*

$$-\boldsymbol{\mu}\Delta \mathbf{v} + \nabla q = 0 \quad in \quad S \cup \tilde{S}, \tag{16}$$

$$\nabla \cdot \mathbf{v} = 0 \quad in \quad S \cup \tilde{S}, \tag{17}$$

$$\left[\boldsymbol{\mu}\frac{\partial \mathbf{v}}{\partial \mathbf{n}_0} - q\mathbf{n}_0\right] = 0 \quad on \quad \{\theta = 0\}, \tag{18}$$

$$\mathbf{v} = \beta(\cos\theta, \sin\theta) \quad on \quad \{\theta = 0\}, \tag{19}$$

$$\mathbf{v} = \alpha(\cos\theta, \sin\theta) \quad on \quad \{\theta = \omega\}, \tag{20}$$

$$\mathbf{v} = \tilde{\alpha}(\cos\theta, \sin\theta) \quad on \quad \{\theta = \tilde{\omega}\}, \tag{21}$$

$$\mathbf{v} \;=\; O(1) \;\; at \;\; \infty. \tag{22}$$

where $\mathbf{n}_0$ *is the unit normal vector to* $\{\theta = 0\}$ *directed to* $\tilde{S}$ *and* α*, resp.* $\tilde{\alpha}, \beta$*, is the limit of* $\mathbf{u}$ *in the direction* $\theta = \omega$*, resp.* $\theta = \tilde{\omega}, 0$*.*

Proof. The proof of these equations is simple and can be worked out using (15) and (1)-(9). The relation (18) requires some special attention and we present its proof. Let $\varphi \in \mathcal{D}(S \cup \tilde{S} \cup \{\theta = 0\}; \mathbb{R}^2)$. Then, from the fact that $\mathbf{v}$ and q satisfy (16) and (17) in the sense of distributions, the left hand-side of (18) is also well defined in the sense of distributions. Let set $x = hy$ and $\varphi_h(x) = \varphi(y)$. Then we have

$$\begin{aligned}
\int_{\{\theta=0\}} [T(\mathbf{v}) \cdot \mathbf{n}_0)] \cdot \varphi \;&=\; \int_{\{\omega<\theta<\tilde{\omega}\}} \boldsymbol{\mu} \nabla \mathbf{v}(y) \cdot \nabla \varphi(y) - q(y) \nabla \cdot \varphi(y) dy \\
&=\; \lim_{h \to 0} \int_{h^{-1}(\Omega \cup \tilde{\Omega})} \boldsymbol{\mu} \nabla \mathbf{u}_h(y) \cdot \nabla \varphi(y) - p_h(y) \nabla \cdot \varphi(y) dy \\
&=\; \lim_{h \to 0} \Big(\int_{\Omega} \mu \nabla \mathbf{u}(x) \cdot \nabla \varphi_h(x) - p(x) \nabla \cdot \varphi_h(x) dx \; + \\
&\qquad \int_{\tilde{\Omega}} \tilde{\mu} \nabla \mathbf{u}(x) \cdot \nabla \varphi_h(x) - p(x) \nabla \cdot \varphi_h(x) dx \Big) \\
&=\; \lim_{h \to 0} \int_{\Gamma} \left[\boldsymbol{\mu} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} \right] \cdot \varphi_h(x) dx \\
&= - \lim_{h \to 0} \int_{\Gamma} \sigma \mathcal{H}(x) \mathbf{n} \cdot \varphi_h(x) d\Gamma(x) = 0,
\end{aligned}$$

which proves (18). Now we will deal with the solution of (16)-(22). We have

Proposition 3.2 *The problem (16)-(22) may have at most one solution.*

Proof. Let assume that $(\mathbf{v}_1, q_1)$, $(\mathbf{v}_2, q_2)$ are two solutions of (16)-(22). Then $\mathbf{v}_0 = \mathbf{v}_1 - \mathbf{v}_2$, $q_0 = q_1 - q_2$ satisfy (16)-(22) with $\alpha = \tilde{\alpha} = \beta = 0$. According to (18) it is easy to prove that $\mathbf{v}_0$, q_0 satisfy in distribution sense

$$-\boldsymbol{\mu} \Delta \mathbf{v}_0 + \nabla q_0 = 0, \quad \nabla \cdot \mathbf{v}_0 = 0, \quad in \quad \{r > 0\} \times \{\omega < \theta < \tilde{\omega}\}. \tag{23}$$

Then, from classical regularity results (see for example [10]) , $(\mathbf{v}_0, q_0)$ satisfy (16)-(22) in classical sense in $\{r > 0, \omega < \theta < \tilde{\omega}\}$. Now we can extend $\mathbf{v}_0$, q_0 in $\mathbb{R}^2 \backslash \{0\}$ by formulas

$$(\mathbf{v}_e(r,\theta), q_e(r,\theta)) = \left\{ \begin{array}{cc} (\mathbf{v}_0(r,\theta), q_0(r,\theta)) & \omega \le \theta \le \tilde{\omega} \\ (\mathbf{v}_0(r,\theta-\pi), q_0(r,\theta-\pi)) & \tilde{\omega} \le \theta \le \tilde{\omega} + \pi \end{array} \right. .$$

Due to equations (16)-(22), $\mathbf{v}_e \in H^1(A)$, $q_e \in L^2(A)$ for any A open, bounded set $A \not\ni O$. Moreover $-\Delta \mathbf{v}_e + \nabla \frac{q_e}{\boldsymbol{\mu}} = 0$ in $\mathbb{R}^2 \backslash \{0\}$. To prove this we have to multiply $-\Delta \mathbf{v}_e + \nabla \frac{q_e}{\boldsymbol{\mu}} = 0$ in $\{\omega < \theta < \tilde{\omega}\}$ by $\varphi \in \mathcal{D}(\mathbb{R}^2 \backslash O)$ with $\varphi(r, \omega) = \varphi(r, \tilde{\omega})$. Using Fourier transform and the fact that $\mathbf{v}_e = 0$ on $\theta = 0$ it turns out that $\mathbf{v}_e = q_e = 0$. □

It is well-known that the speed $\mathbf{v}$ of 2D incompressible flow may be given by the stream function ψ whose properties are well known. It satisfies

Proposition 3.3 *The solution of (16)-(17), (19)-(22) is given by* $\mathbf{v} = (D_2\psi, -D_1\psi)$ *satisfying the following equations in polar coordinates*

$$\begin{aligned} -\Delta^2\psi &= 0, \quad \text{in } \{\omega < \theta < 0\} \cup \{0 < \theta < \tilde{\omega}\}, \\ \frac{\partial\psi}{\partial r} &= 0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \alpha \text{ on } \{\theta = \omega\}, \end{aligned}$$

$$\begin{aligned} \frac{\partial\psi}{\partial r} &= 0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \beta \text{ on } \{\theta = 0^+, 0^-\}. \\ \frac{\partial\psi}{\partial r} &= 0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \tilde{\alpha} \text{ on } \{\theta = \tilde{\omega}\}. \end{aligned}$$

Proof. To find the solution of this problem, consider first the case $\omega < \theta < 0$.We look for ψ in the form $\psi(r, \theta) = rz(\theta)$. The boundary conditions for ψ are just the conditions (19)-(21). By simple computations we fine $\Delta^2\psi = r^{-3}(z + 2z'' + z^{(iv)})$. Then

$$\begin{cases} z + 2z'' + z^{(iv)} = 0, \qquad \omega < \theta < 0, \\ z(\omega) = 0, \quad z'(\omega) = \alpha, \\ z(0) = 0, \quad z'(0) = \beta. \end{cases} \tag{24}$$

There exist constants A, B, C, D such that

$$z(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta.$$

It turns out that

$$\begin{aligned} A &= \frac{-\alpha\omega\sin\omega - \beta\omega^2}{d(\omega)}, \quad & C &= \frac{\alpha(\sin\omega - \omega\cos\omega) + \beta(\omega - \sin\omega\cos\omega)}{d(\omega)}, \\ B &= 0, & D &= \frac{\alpha\omega\sin\omega + \beta\sin^2\omega}{d(\omega)}, \end{aligned} \tag{25}$$

where $d(\omega) = \sin^2\omega - \omega^2$.
Let now consider ψ in $\{0 < \theta < \tilde{\omega}\}$. If $\psi = r\tilde{z}(\theta)$ then $\tilde{z}(\theta) = \tilde{A}\sin\theta + \tilde{B}\cos\theta + \tilde{C}\theta\sin\theta + \tilde{D}\theta\cos\theta$ and

$$\begin{aligned} \tilde{A} &= \frac{-\tilde{\alpha}\tilde{\omega}\sin\tilde{\omega} - \beta\tilde{\omega}^2}{d(\tilde{\omega})}, \quad & \tilde{C} &= \frac{\tilde{\alpha}(\sin\tilde{\omega} - \tilde{\omega}\cos\tilde{\omega}) + \beta(\tilde{\omega} - \sin\tilde{\omega}\cos\tilde{\omega})}{d(\tilde{\omega})}, \\ \tilde{B} &= 0, & \tilde{D} &= \frac{\tilde{\alpha}\tilde{\omega}\sin\tilde{\omega} + \beta\sin^2\tilde{\omega}}{d(\tilde{\omega})}, \end{aligned} \tag{26}$$

where $d(\tilde{\omega}) = \sin^2\tilde{\omega} - \tilde{\omega}^2$.

It is easy to find that $\mathbf{v} = (D_2\psi, -D_1\psi)$ is given by

$$\mathbf{v} = \begin{cases} (z(\theta)\sin\theta + z'(\theta)\cos\theta, -z(\theta)\cos\theta + z'(\theta)\sin\theta), \quad \omega < \theta < 0, \\ (\tilde{z}(\theta)\sin\theta + \tilde{z}'(\theta)\cos\theta, -\tilde{z}(\theta)\cos\theta + \tilde{z}'(\theta)\sin\theta), \quad 0 < \theta < \tilde{\omega}. \end{cases} \tag{27}$$

In the same way we can compute the gradient of $\mathbf{v}$ by $(\nabla\mathbf{v})_{ij} = \partial_j \mathbf{v}_i$

$$\nabla\mathbf{v} = \frac{z(\theta) + z''(\theta)}{r} \begin{pmatrix} -\sin\theta\cos\theta, & \cos^2\theta \\ -\sin^2\theta, & \sin\theta\cos\theta \end{pmatrix}, \ \omega < \theta < 0, \tag{28}$$

$$\nabla \mathbf{v} = \frac{\tilde{z}(\theta) + \tilde{z}''(\theta)}{r} \begin{pmatrix} -\sin\theta\cos\theta, & \cos^2\theta \\ -\sin^2\theta, & \sin\theta\cos\theta \end{pmatrix}, \quad 0 < \theta < \tilde{\omega}. \tag{29}$$

Finally, we can compute the pressure q, basically using the equation (16) multiplied by $(\cos\theta, \sin\theta)$. This will give $\partial q/\partial r$ which after integration gives q as follows

$$q(r,\theta) = \begin{cases} -\dfrac{\mu}{r}[z'(\theta) + z'''(\theta)] + c_q, & \omega < \theta < 0, \\ -\dfrac{\tilde{\mu}}{r}[\tilde{z}'(\theta) + \tilde{z}'''(\theta)] + c_{\tilde{q}}, & 0 < \theta < \tilde{\omega}, \end{cases} \tag{30}$$

where c_q and $c_{\tilde{q}}$ are two constants.
In the next section, further relationships between α, $\tilde{\alpha}$, β and ω are derived. This leads to the dynamical contact angle relationship.

4 Main Results

Proposition 4.1 *The speed* $\mathbf{v}$ *given by (27) and the pressure* q *given by (30) satisfy (16)-(22) if and only if* α, $\tilde{\alpha}$ *and* β *satisfy*

$$\begin{aligned} \alpha &= \frac{\beta}{\pi\sin\omega}\left(-\frac{\tilde{\mu}}{\mu}d(\omega) + \sin^2\omega - \omega\tilde{\omega}\right) := \frac{k_\alpha}{\pi\sin\omega}\beta, \\ \tilde{\alpha} &= \frac{\beta}{\pi\sin\omega}\left(-\frac{\mu}{\tilde{\mu}}d(\tilde{\omega}) + \sin^2\omega - \omega\tilde{\omega}\right) := \frac{k_{\tilde{\alpha}}}{\pi\sin\omega}\beta. \end{aligned} \tag{31}$$

Proof. It is clear that $\mathbf{v}$ and q satisfy (16)-(20) except (18). But

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}_0} = \frac{1}{r}\begin{cases} \begin{pmatrix} z + z'' \\ 0 \end{pmatrix} \\ \begin{pmatrix} \tilde{z} + \tilde{z}'' \\ 0 \end{pmatrix} \end{cases} = \frac{1}{r}\begin{cases} \begin{pmatrix} 2C \\ 0 \end{pmatrix} & \text{on } \theta = 0^-, \\ \begin{pmatrix} 2\tilde{C} \\ 0 \end{pmatrix} & \text{on } \theta = 0^+, \end{cases}$$

We have also these relations for the pressure

$$q\mathbf{n}_0 = \begin{cases} \begin{pmatrix} 0 \\ -\frac{\mu}{r}(z' + z''') + c_q \end{pmatrix} \\ \begin{pmatrix} 0 \\ -\frac{\tilde{\mu}}{r}(\tilde{z}' + \tilde{z}''') + c_{\tilde{q}} \end{pmatrix} \end{cases} = \begin{cases} \begin{pmatrix} 0 \\ 2\frac{\mu}{r}D + c_q \end{pmatrix} & \text{on } \theta = 0^-, \\ \begin{pmatrix} 0 \\ 2\frac{\tilde{\mu}}{r}\tilde{D} + c_{\tilde{q}} \end{pmatrix} & \text{on } \theta = 0^-, \end{cases}$$

Then the condition (18) is equivalent to

$$\begin{cases} \mu C = \tilde{\mu}\tilde{C}, \\ 2\dfrac{\mu}{r}D + c_q = 2\dfrac{\tilde{\mu}}{r}\tilde{D} + c_{\tilde{q}}. \end{cases}$$

It follows that $c_q = c_{\tilde{q}}$ and

$$\mu C = \tilde{\mu}\tilde{C}, \qquad \mu D = \tilde{\mu}\tilde{D}. \tag{32}$$

Replacing the values of $C, \tilde{C}, D, \tilde{D}$ we obtain that this system is equivalent to

$$\begin{cases} \mu \dfrac{\alpha(\sin\omega - \omega\cos\omega) + \beta(\omega - \sin\omega\cos\omega)}{d(\omega)} & = \tilde{\mu}\dfrac{\tilde{\alpha}(\sin\tilde{\omega} - \tilde{\omega}\cos\tilde{\omega}) + \beta(\tilde{\omega} - \sin\tilde{\omega}\cos\tilde{\omega})}{d(\tilde{\omega})}, \\ \mu \dfrac{\alpha\omega\sin\omega + \beta\sin^2\omega}{d(\omega)} & = \tilde{\mu}\dfrac{\tilde{\alpha}\tilde{\omega}\sin\tilde{\omega} + \beta\sin^2\tilde{\omega}}{d(\tilde{\omega})}, \end{cases}$$

which after some computations easily leads to values of α, $\tilde{\alpha}$ and β claimed in the lemma.

Remark 4.2 *Based on the model we have presented, if we assume that the fluids stick on the solid surface, then we have to impose $\alpha = -\tilde{\alpha}$. In such a case ω is solution of $k_\alpha = -k_{\tilde{\alpha}}$. This implies*

$$\frac{\mu}{\tilde{\mu}} d(\omega) + \frac{\tilde{\mu}}{\mu} d(\tilde{\omega}) = 2(\sin^2\omega - \omega\tilde{\omega}).$$

As $\omega\tilde{\omega} < 0$ and $d(\omega),\ d(\tilde{\omega}) < 0$, for $\omega \neq 0$ it follows that the equation $k_\alpha = -k_{\tilde{\alpha}}$ has no roots different from zero. This shows that in general $\alpha \neq -\tilde{\alpha}$. Thus, one of the two fluids must slip locally close to the contact point. In Section 5 we will discuss more about the local behavior of the speeds close to the contact points.

Remark 4.3 *As we have seen, the problem (16)-(22) has solution which is computed by hand. Near O, $\nabla \mathbf{v}$ and q are like $|x|^{-1}$ which means that $\nabla \mathbf{v}, q \notin L^2(S)$. In the hypothesis (10) for $\mathbf{u}$ and p we cannot assume, for example, $\nabla \mathbf{u}, p \in L^2(D)$ because this would imply that $\nabla \mathbf{v}, q \in L^2(S)$! The same reasoning holds for $\mathbf{u}$, p in $\tilde{S}$.*

Remark 4.4 *In the analysis of the drop motion many authors impose the following boundary slip condition*

$$\gamma \frac{\partial \mathbf{u}_t}{\partial \mathbf{n}_s} = \mathbf{u}_t, \quad \gamma \in \mathbb{R},\ \gamma \neq 0, \tag{33}$$

where $\mathbf{n}_s$ is the unit normal to the solid boundary and $\mathbf{u}_t = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n}_s)\mathbf{n}_s$ is the tangential part of $\mathbf{u}$. The equalities (27), (28) show that (33) cannot be satisfied, unless $\gamma = 0$. Indeed, if it were true then (assuming that $\mathbf{u}$ satisfies (10)) we end up that $\mathbf{v}$ should satisfy $\gamma \dfrac{\partial \mathbf{v}_t}{\partial \mathbf{n}_s} = 0$. But if $\gamma \neq 0$ this is impossible because the normal derivative of $\mathbf{v}$ is like r^{-1}.

We turn now to the force balance equation at O given by (11). Let us split the integral in (11) into two terms

$$\int_{\partial D \cap B(O,1)} T(\mathbf{u}_h)\cdot\mathbf{n}_1 = \int_{D\cap\partial B(O,1)} T(\mathbf{u}_h)\cdot\mathbf{n}_1 + \int_{\partial D\cap B(O,1)} T(\mathbf{u}_h)\cdot\mathbf{n}_1.$$

Using (15) and the state equation of $\mathbf{u}_h$ it is not difficult to see that

$$\lim_{h\to 0}\int_{D\cap\partial B(0,1)} T(\mathbf{u}_h)\cdot\mathbf{n}_1 dy = \int_{D\cap\partial B(0,1)} T(\mathbf{v})\cdot\mathbf{n}_1 dy.$$

Thus we need to identify only $\int_{\partial D\cap B(O,h)} T(\mathbf{u})\cdot\mathbf{n}_h$.

The analysis of this integral is more delicate. It requires that the properties of $\mathbf{u}_h$ and p_h be known more precisely, which is beyond the scope of this paper. We will assume that

The condition (11) is equivalent to

$$f + \int_{D\cap\partial B(0,1)} T(\mathbf{v})\cdot\mathbf{n}_1 dy + \int_{\partial D\cap B(0,1)} T(\mathbf{v})\cdot\mathbf{n}_1 dy = 0, \tag{34}$$

where the integral on $\partial D \cap B(O,1)$ is understood in the sense of Cauchy

It is easy to compute the integral in (34). Let first set

$$F_1 := \int_{D\cap B(O,h)} T(\mathbf{v})\cdot\mathbf{n}_1, \qquad F_2 := \int_{\partial D\cap B(O,h)} T(\mathbf{v})\cdot\mathbf{n}_1,$$

and $F := F_1 + F_2$.

Lemma 4.5 *The total force F created from the singular fluid stresses at O is given by*

$$F = -\pi\mu \begin{pmatrix} D \\ C \end{pmatrix}. \tag{35}$$

Proof. Using (28) and (30) we have

$$\begin{aligned}
I_1 &:= \int_{\{r=1\}\cap\{\omega<\theta<0\}} T(\mathbf{v})\cdot\mathbf{n}_1 = -\int_\omega^0 [-\mu(z'(\theta)+z'''(\theta))+c_q] \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta \\
&= -\int_\omega^0 [2\mu(C\sin\theta + D\cos\theta)+c_q] \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta \\
&= \left[\frac{\mu}{2}\begin{pmatrix} C\cos(2\theta) - D(2\theta+\sin(2\theta)) \\ -C(2\theta-\sin(2\theta)) + D\cos(2\theta) \end{pmatrix} - c_q \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}\right]_\omega^0 \\
&= \mu \begin{pmatrix} C\sin^2\omega + D(\omega+\sin\omega\cos\omega) \\ C(\omega-\sin\omega\cos\omega) + D\sin^2\omega \end{pmatrix} + c_q \begin{pmatrix} +\sin\omega \\ -\cos\omega + 1 \end{pmatrix}
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
\tilde{I}_1 &:= \int_{\{r=1\}\cap\{0<\theta<\tilde\omega\}} T(\mathbf{v})\cdot\mathbf{n}_1 \\
&= -\tilde\mu \begin{pmatrix} \tilde C\sin^2\tilde\omega + \tilde D(\tilde\omega+\sin\tilde\omega\cos\tilde\omega) \\ \tilde C(\tilde\omega-\sin\tilde\omega\cos\tilde\omega) + \tilde D\sin^2\tilde\omega \end{pmatrix} - c_{\tilde q} \begin{pmatrix} +\sin\tilde\omega \\ -\cos\tilde\omega + 1 \end{pmatrix}
\end{aligned}$$

Taking into account (32) we obtain

$$F_1 := I_1 + \tilde{I}_1 = -\pi\mu \begin{pmatrix} D \\ C \end{pmatrix} + 2c_q \begin{pmatrix} \sin\omega \\ -\cos\omega \end{pmatrix}. \tag{36}$$

Now we turn to the computation of F_2. It is easy to see that the stress on the fluid-solid interface is singular (in L^1 sense). However, it is well defined in the Cauchy sense. Let $\epsilon > 0$.

Taking into account that on $\{\theta=\omega\}$, $\mathbf{n}_1=(\sin\omega,-\cos\omega)$, from (28), (29), (30) we obtain

$$\begin{aligned}
I_{2\epsilon} &= \int_{\{\epsilon<r<1\}\cap\{\theta=\omega\}} T(\mathbf{v})\cdot\mathbf{n}_1\\
&= \int_\epsilon^1 \frac{\mu}{r}\begin{pmatrix} -(z(\omega)+z''(\omega))\cos\omega+(z'(\omega)+z'''(\omega))\sin\omega\\ -(z(\omega)+z''(\omega))\sin\omega-(z'(\omega)+z'''(\omega))\cos\omega\end{pmatrix} - c_q\begin{pmatrix}\sin\omega\\ -\cos\omega\end{pmatrix} dr\\
&= \int_\epsilon^1 2\frac{\mu}{r}\begin{pmatrix}-C\\ D\end{pmatrix} - c_q\begin{pmatrix}\sin\omega\\ -\cos\omega\end{pmatrix} dr,
\end{aligned}$$

and in similar way we find out that

$$\tilde{I}_{2\epsilon} := \int_{\{\epsilon<r<1\}\cap\{\theta=\tilde{\omega}\}} T(\mathbf{v})\cdot\mathbf{n}_1 = -\int_\epsilon^1 2\frac{\tilde{\mu}}{r}\begin{pmatrix}-\tilde{C}\\ \tilde{D}\end{pmatrix} - c_{\tilde{q}}\begin{pmatrix}\sin\tilde{\omega}\\ -\cos\tilde{\omega}\end{pmatrix} dr$$

Then it turns out that

$$F_2 := \lim_{\epsilon\to 0}(I_{2\epsilon}+\tilde{I}_{2\epsilon}) = -2c_q\begin{pmatrix}\sin\omega\\ -\cos\omega\end{pmatrix}, \tag{37}$$

which proves the lemma. □

The horizontal stress force exerted on the contact point is given by (using the condition (32))

$$\begin{pmatrix}D\\ C\end{pmatrix}\cdot\begin{pmatrix}\cos\omega\\ \sin\omega\end{pmatrix} = \frac{1}{d(\omega)}\left(\alpha\sin^2\omega+\beta\omega\sin\omega\right).$$

If $f=(f_1,f_2)$ is the total surface tension force acting on the contact point, from the horizontal force balance equation $(f+F)\cdot(\cos\omega,\sin\omega)=0$, we obtain the following equation

$$\frac{\pi\mu}{d(\omega)}\left(\alpha\sin^2\omega+\beta\omega\sin\omega\right) = f_1\cos\omega+f_2\sin\omega. \tag{38}$$

Theorem 4.6 *Let assume that $f=(f_1,f_2)$ is the total surface tension force acting on the contact point. If $\mathbf{u}$ and p satisfy (1)-(9) and the hypothesis (10), (34) are fulfilled then the speed α the angle ω are related by the following algebraic equation*

$$\alpha = \frac{1}{\pi}\frac{-\frac{\tilde{\mu}}{\mu}(\sin^2\omega-\omega^2)+\sin^2\omega-\omega\tilde{\omega}}{(\mu-\tilde{\mu})\sin^2\omega}(f_1\cos\omega+f_2\sin\omega_0), \qquad (\tilde{\omega}=\pi+\omega). \tag{39}$$

Proof. The proof follows immediately from (38). Indeed, replacing $\beta=\frac{\pi\sin\omega}{k_\alpha}\alpha$ we get

$$\alpha = \frac{\sigma}{\pi\mu}\frac{(\cos\omega-\cos\omega_0)(\sin^2\omega-\omega^2)k_\alpha}{(k_\alpha+\pi\omega)\sin^2\omega}.$$

On the other hand we have

$$\begin{aligned}
k_\alpha &= -\frac{\tilde{\mu}}{\mu}(\sin^2\omega-\omega^2)+\sin^2\omega-\omega\tilde{\omega},\\
k_\alpha+\pi\omega &= (1-\frac{\tilde{\mu}}{\mu})(\sin^2\omega-\omega^2),
\end{aligned}$$

which implies the formula (39) stated in the theorem.

Corollary 4.7 *With the same hypotheses as in Theorem 4.6 the speeds $\tilde{\alpha}$, β are given by:*

$$\tilde{\alpha} = \frac{1}{\pi}\frac{-\frac{\mu}{\tilde{\mu}}(\sin^2\omega - \tilde{\omega}^2) + \sin^2\omega - \omega\tilde{\omega}}{(\mu - \tilde{\mu})\sin^2\omega}(f_1\cos\omega + f_2\sin\omega), \tag{40}$$

$$\beta = \frac{1}{\mu - \tilde{\mu}}\frac{f_1\cos\omega + f_2\sin\omega}{\sin\omega}. \tag{41}$$

Proof. It is immediate using (39) and (31). □

The local speeds given by (39), (40), (41) depend strongly of the surface tension force f. It is commonly accepted that $f = \sigma\mathbf{n}(0) + \sigma_\mu\mathbf{n}(\omega) + \sigma_{\tilde{\mu}}\mathbf{n}(\tilde{\omega})$ where $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$ and σ_μ, resp. $\sigma_{\tilde{\mu}}$, is the surface tension at drop fluid-solid interface, resp. other fluid-solid interface. Even in the statical case f is not known completely. It is easy to measure σ, but not σ_μ and $\sigma_{\tilde{\mu}}$. In dynamical case it is reported a gradient of σ close to contact points, [12], and so the determination of f is difficult.

5 Analysis of local speeds

Here, we will give a form of f based on simple considerations. We know that for drop speed tending to zero, the hydrodynamical force tends to zero. Because $F + f = 0$, f also must tend to zero as well. If σ is considered constant, this means that the surface tension force due to fluids-solid interaction is equal to $-\sigma\cos\omega_0$, where ω_0 is the limit angle when drop speed tends to zero. We will consider that for any other dynamical contact angle, the fluids-solid surface tension is given by $-\sigma\cos\omega_0$, thus $f = \sigma\mathbf{n}(0) - \sigma\cos\omega_0\mathbf{n}(\omega)$. We think that this form of f is not exact, but it allows to predict fluid dynamics behavior at the contact points. The formula (39) then becomes

$$\alpha = \frac{\sigma}{\pi}\frac{-\frac{\tilde{\mu}}{\mu}(\sin^2\omega - \omega^2) + \sin^2\omega - \omega\tilde{\omega}}{(\mu - \tilde{\mu})\sin^2\omega}(\cos\omega - \cos\omega_0). \tag{42}$$

From (40), (41) we can write similar formulas for $\tilde{\alpha}$, β.

Let us consider the case $\mu > \tilde{\mu}$ and the receding contact point. It is easy to check that the sign of α is positive. Indeed, let us first point out that the choice of our coordinate system implies $\omega < 0$, $\tilde{\omega} > 0$ and $\omega_0 < 0$. As $\mu > \tilde{\mu}$ and $\omega_0 < \omega < 0$, it follows that

$$-\frac{\tilde{\mu}}{\mu}(\sin^2\omega - \omega^2) + \sin^2\omega - \omega\tilde{\omega} > 0, \tag{43}$$

$$\cos\omega - \cos\omega_0 > 0. \tag{44}$$

This implies $\alpha > 0$. From (31) it follows that $\beta < 0$, $\tilde{\alpha} > 0$.
A similar analysis may be done at the advancing point. We have still (43) and (44). As this time μ and $\tilde{\mu}$ must permute their places, at advancing contact point we have $\alpha > 0$. From (31) it follows that $\beta < 0$, $\tilde{\alpha} > 0$.
The direction of speeds at contact points and at solid interface is given in Figure 3 (left). It follows that the distribution of speeds must look like in Figure 3 (right). From this figure we can see that the drop fluid, essentially rolls, but very close to the receding and advancing

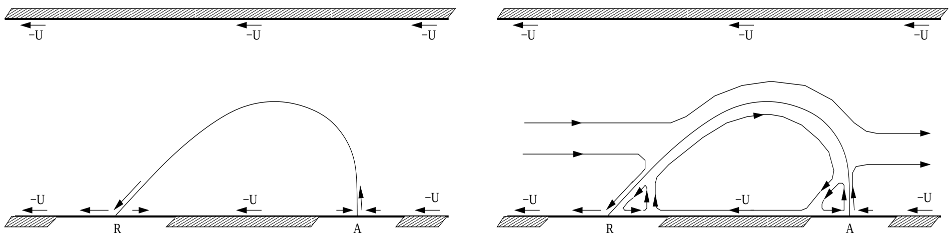

Figure 3: Speeds at the contact points (left) and expected speed behavior (right) ($\mu > \tilde{\mu}$)

points there is a small region where it rolls in the inverse direction. The other fluid simply rolls. This behavior is observed in experiments, [4]; close to the receding and advancing points there are two stagnation points.
In the case $\mu < \tilde{\mu}$ a similar analysis shows that on the fluids-solid interface the fluids meet, resp. dissociate, at the receding, resp. advancing contact point. So, this time the drop fluid simply rolls and the other fluid basically rolls but close to contact points there is a small region where it rolls in the opposite direction.

Now we will address the key question: *"What is the connection between α, $\tilde{\alpha}$ and the drop speed?"* Let focus our analysis at the receding point. Concerning α, it is clear that it has no connection with the drop speed U because $\alpha > 0$ (remember that $\mathbf{v}(r,\omega) = \alpha\mathbf{n}(\omega)$, $\mathbf{v}(r,\tilde{\omega}) = \tilde{\alpha}\mathbf{n}(\tilde{\omega})$, $\mathbf{v}(r,0) = \beta\mathbf{n}(0)$). Thus, the drop fluid close to the receding point must slip. Regarding $\tilde{\alpha}$, which is the speed of the fluid on the other side of the drop, we proved that it is positive, so $\mathbf{v}(r,\tilde{\omega})$ has the same direction as $(-U,0)$. This fluid may slip as well, and in this case possibly the fluid-solid surface tension must change as well. Here, we will postulate that this fluid sticks to the solid interface, i.e. $\tilde{\alpha} = U$.

At the advancing contact point, by a very similar analysis like at the receding point, we find that the drop fluid must slip close to the contact point. We postulate that the other fluid sticks to the solid interface, i.e. $\alpha = U$.
In summary, for $\mu > \tilde{\mu}$, it follows that close to the contact points the drop fluid must slip. We assume that the other fluid sticks on the wall. If $\mu < \tilde{\mu}$, close to the contact points the other fluid (not the drop fluid) must slip. We assume that the drop fluid sticks on the wall.

5.1 Comparison with the experiments

We will consider two different examples for which we have found experimental data in the literature. In this paper we take the hysteretic contact angle behavior as given. Thus we consider an interval (ω_r, ω_a) including the static contact equilibrium angle such that the drop does not move as long as the contact angle is in (ω_r, ω_a). We will take then $\omega_0 = \omega_r$ for the receding point and $\omega_0 = \omega_a$ for the advancing point and $f = \sigma\mathbf{n}(0) - \cos\omega_0\mathbf{n}(\omega)$ as discussed at the beginning of this section.

1. We consider the steady state motion of a glycerol/water drop in a PET (polyethylene terephthalate) horizontal surface. According to [6] we have these data: $\mu = 8.286 \cdot 10^{-2}$ kg/m/s (glycerol/water at 20^oC), $\tilde{\mu} = 1.810 \cdot 10^{-5}$ kg/m/s (air 20^oC), $\sigma = 67.9 \cdot 10^{-3}$, kg/s/s and $\omega_a = 70^o$ and $\omega_r = 45^o$. In Figure 4 (left) is given the graph of the speeds α, $\tilde{\alpha}$.

2. We consider the steady state motion of a water drop in a PET (polyethylene terephthalate) horizontal surface. The other fluid is air. According to [6] we have these data: $\mu = 1.002\cdot 10^{-3}$ kg/m/s (water 20^{o}C), $\tilde{\mu} = 1.810\cdot 10^{-5}$ kg/m/s (air 20^{o}C), $\sigma = 72.8\cdot 10^{-3}$, kg/s/s and $\omega_a = 82^o$ and $\omega_r = 35^o$. In Figure 4 (right) is given the graph of the speeds α and $\tilde{\alpha}$.

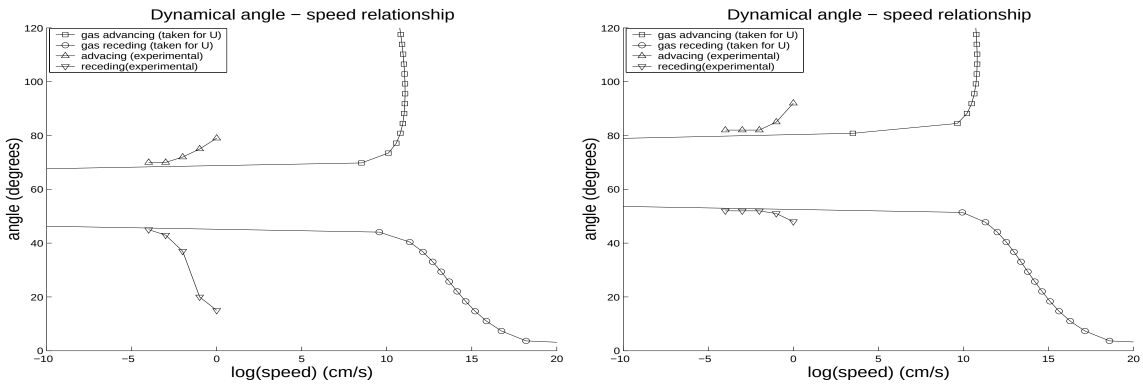

Figure 4: Results for glycerol-water/air (left) and water/air (right)

Although the speed versus contact angle relationship agrees quite good qualitatively with experimental data, it does not agree quantitatively. We think that this is due to the variation of the surface tension forces at contact points (due to many factors: a thin drop fluid layer between the other fluid and the solid, fluids dissociate, fluids meet each other).

6 Summary

In this work we have presented an analytical result relating dynamical contact point speeds, dynamical contact angle and the surface tension force acting on the contact point. We deal with the singular stress and the only parameters involved in the analysis are the viscosities and the surface tension. We have shown that it is possible to predict the qualitative behavior of fluids close to the contact points based only on the hydrodynamics and we have found the drop speed agreeing qualitatively with the experimental data.
We are considering how to describe the exact values of dynamical surface tensions at contact points, which we think should improve our results in agreement with the experimental data even quantitatively.

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