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Inverse obstacle scattering in the time domain

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Abstract

This work is a study of the extension of an inverse obstacle scattering algorithm for fixed, single-frequency data to multi-frequency time-dependent settings. The inversion algorithm is based on the point source method, which reconstructs scattered fields pointwise with respect to frequency. We use Fourier transforms to obtain the time-dependent scattered fields as superpositions of single-frequency scattered fields. We establish criteria for the correspondence between solutions to the inverse problem in the frequency domain and the scattering problem in the time domain. Numerical examples illustrate the method.

Key words. inverse problems, scattering theory, image processing

AMS subject classifications. 35R30, 35P25, 68U10, 94A08

1 Introduction

In recent years there has been great progress toward reconstructing obstacles and inhomogeneities in a medium from measurements of the acoustic or electromagnetic field around the medium due to an incident pulse. For a review of these techniques see [1, 9]. While the methodologies are diverse, they all focus on single frequency, time-harmonic waves. In this work we consider multi-frequency, nonharmonic waves in time. Our principle aim is to establish the theoretical foundation for the extension of powerful single-frequency techniques to a broader class of applications in the time domain. The numerical technique we shall explore is the the point source

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method proposed by Potthast [12, 13], though other methods can also be fit into this framework.

Scattering theory is a vast area of research with a long history in mathematical physics. Modern mathematical surveys can be found in the works of Leis [6] and Ramm [15], among others. Rather than appealing to these works, however, we find it more convenient to tailor the theory to our purpose so that the analysis is self-contained, and so that criteria for the applicability of the numerical method we explore are simple and easy to verify.

In Section 2 of this work we outline the basic theory of scattering and introduce the tools necessary for our analysis. Section 3 details inverse scattering and the extension of the point source method to time-dependent problems. We conclude with numerical illustrations of the method.

2 Forward Scattering

We begin with waves traveling through a homogeneous, isotropic medium with an inclusion denoted by $\Omega \subset \mathbb{R}^m$, $m = 2, 3$. These are modeled with the homogeneous wave equation

$$\left(\triangle - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right) V(x,t) = 0, \tag{2.1}$$

where $x \in \mathbb{R}^m$ is the spatial variable, $t \in \mathbb{R}$ denotes time, $V : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, $n = const > 0$ and $\triangle$ denotes the spatial Laplacian

$$\triangle V(x,t) = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2}\right) V(x,t), \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \ t \in \mathbb{R}.$$

If the waves are time-harmonic, then the wave V takes the form $V(x,t,\omega) = \mathrm{Re}\{v(x,\omega)\exp[i\omega t]\}$ where ω is the *frequency* of the wave. The spatial component of the wave, $v(\cdot,\omega)$, satisfies the Helmholtz equation

$$(\triangle + k^2 n^2) v(x,\omega) = 0. \tag{2.2}$$

where $k = \omega/c = 2\pi/\lambda$ is the *wave number*, and λ is the *wavelength*. Without loss of generality, we consider only a normalized ($c = 1$), nondimensionalized wave equation in free-space ($n = 1$ on $\overline{\Omega}^c$, the complement of Ω) and write the frequency variable as κ rather than ω.

Very little changes when the time-harmonic assumption is dropped, however we need some further assumptions on the behavior of the waves. If we assume that, for x fixed, $V(x,\cdot) \in L^2(\mathbb{R})$ and that the wave and its first derivative with respect to time decay sufficiently fast as time approaches infinity, $V(x,t)$ and $\frac{\partial}{\partial t}V(x,t) \to 0$

as $|t| \to \infty$, then taking the Fourier transform with respect to time of both sides of Eq.(2.1) yields Eq.(2.2). Here $v(x,\kappa) = (\mathcal{F}_t V)(x,\kappa)$ where

$$(\mathcal{F}_t V)(x,\kappa) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa t} V(x,t)dt,$$

and

$$(\mathcal{F}_\kappa v)(x,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\kappa t} v(x,\kappa)dk.$$

Note that $\mathcal{F}_\kappa \mathcal{F}_t V = V$.

2.1 The frequency domain

The time-domain wave is a *real-valued* mapping $V : \mathbb{R}^m \setminus \Omega \times \mathbb{R} \to \mathbb{R}$. Thus, its Fourier transform is a *Hermitian* function of the frequency $\kappa \in \mathbb{R}$, that is it satisfies the property $v(x,k) = \overline{v(x,-k)}$ where $\overline{v}$ denotes the complex conjugate of v. In this case, the results surveyed below hold for *all* wave numbers $\kappa \neq 0$.

HYPOTHESIS 2.1 *The obstacle is described by the bounded domain $\Omega \subset \mathbb{R}^m$ $(m = 2,$ or $3)$ with a connected, piecewise C^2 (twice continuously differentiable) boundary $\partial\Omega$ with outward unit normal ν where it is defined, and corners satisfying the exterior cone condition [3].*

For a fixed κ, given a continuous function $f(\cdot,\kappa) : \partial\Omega \to \mathbb{C}$, we seek the field $v(\cdot,\kappa) \in C^2(\overline{\Omega}^c) \cap C(\Omega^c)$ that satisfies the Helmholtz equation with one of the following boundary conditions:

$$(\triangle + \kappa^2)v(x,\kappa) = 0, \quad x \in \overline{\Omega}^c, \tag{2.3}$$

$$v(x,\kappa) = f(x,\kappa), \quad x \in \partial\Omega \quad \text{(sound-soft obstacle)}, \tag{2.4}$$

$$\frac{\partial v(x,\kappa)}{\partial \nu(x)} = f(x,\kappa), \quad x \in \partial\Omega \quad \text{(sound-hard obstacle)}, \tag{2.5}$$

$$\frac{\partial v(x,\kappa)}{\partial \nu(x)} + i\lambda(\kappa)v(x,\kappa) = f(x,\kappa), \quad x \in \partial\Omega \quad \text{(impedance obstacle with } \lambda : \mathbb{R} \to \mathbb{R}_+ \text{)}. \tag{2.6}$$

For a fixed κ, a solution $v(\cdot,\kappa)$ to Eq.(2.3) whose domain of definition contains the exterior of some sphere is called *radiating* if it satisfies the *Sommerfeld Radiation Condition*:

$$|x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) v(x,\kappa) \to 0, \quad |x| \to \infty, \tag{2.7}$$

uniformly in all directions.

The frequency-domain scattering problem that is central to this work is stated as follows.

> *Let the sound-soft scatterer Ω satisfying Hypothesis 2.1 be embedded in a homogeneous medium. Given a single-frequency* incident field $v^i(\cdot,\kappa):\mathbb{R}^m\to\mathbb{C}$ *that solves Eq.(2.3) on all of* $\mathbb{R}^m$*, find the* total field $v(\cdot,\kappa):\Omega^c\to\mathbb{C}$ *satisfying Eq.(2.3) on* $\overline{\Omega}^c$ *with* $f:=0$ *in Eq.(2.4), and with* $v=v^i+v^s$*, where* $v^s(\cdot,\kappa)$ *is the* scattered field *satisfying Eq.(2.3) and the radiation condition Eq.(2.7).*

For simplicity, we only treat problems with the Dirichlet boundary condition Eq.(2.4). The other boundary conditions Eq.(2.5)-(2.6) are handled similarly.

It is well known, that the scattering problem has a unique solution [2]. The scattered field v^s is a radiating solution to Eq.(2.3) and has the asymptotic behavior

$$v^s(x,\kappa) \;=\; \frac{e^{i\kappa|x|}}{|x|^{\frac{(m-1)}{2}}}\left\{v^\infty(\hat{x},\kappa)+O\left(\frac{1}{|x|}\right)\right\},\quad |x|\to\infty, \tag{2.8}$$

where the function $v^\infty(\cdot,\kappa):\mathbb{S}\to\mathbb{C}$ is known as *far field pattern*,

$$\mathbb{S}:=\{x\in\mathbb{R}^m \mid |x|=1\}\quad\text{and}\quad \hat{x}:=\frac{x}{|x|}.$$

Green's formula, stated below, represents fields in terms of fundamental solutions to the Helmholtz equation. In particular, let $\Omega\subset\mathbb{R}^m$ satisfy Hypothesis 2.1 with unit outward normal ν; let $v(\cdot,\kappa)\in C^2(\overline{\Omega}^c)\cap C(\Omega^c)$ satisfy Eq.(2.3) and Eq.(2.7) (i.e., v is a radiating solution to the Helmholtz equation), with normal derivative on $\partial\Omega$ in the sense of Gâteaux. Then

$$v(x,\kappa)=\int_{\partial\Omega}\left\{v(y,\kappa)\frac{\partial\Phi(y,x,\kappa)}{\partial\nu(y)}-\frac{\partial v}{\partial\nu}(y,\kappa)\Phi(y,x,\kappa)\right\}ds(y),\quad x\in\overline{\Omega}^c, \tag{2.9}$$

where $\Phi(\cdot,x,\kappa)$ is the free space fundamental solution to Eq.(2.3) in $\mathbb{R}^m\setminus\{x\}$. To accommodate negative frequencies, the fundamental solution is defined so that it is a Hermitian function of κ:

$$\kappa>0\qquad \Phi(x,z,\kappa)\equiv\begin{cases}\frac{i}{4}H_0^{(1)}(\kappa|x-z|), & x\neq z,\ \text{and } m=2\\ \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}, & x\neq z,\ \text{and } m=3,\end{cases} \tag{2.10}$$

$$\kappa<0\qquad \Phi(x,z,\kappa)\equiv\begin{cases}\frac{-i}{4}H_0^{(2)}(|\kappa||x-z|), & x\neq z,\ \text{and } m=2\\ \frac{1}{4\pi}\frac{e^{-i|k||x-z|}}{|x-z|}, & x\neq z,\ \text{and } m=3,\end{cases} \tag{2.11}$$

where $H_0^{(n)}$ denotes the zero-th order Hankel function of the n-th kind [2, Eq.(3.60) and Eq(2.1)].

The representation in Eq.(2.9) gives the radiating solution to the Helmholtz equation as a combination of acoustic single- and double-layer potentials defined respectively by

$$(2.12)\qquad (\mathcal{S}_\kappa\varphi)(x,\kappa) = \int_{\partial\Omega} \Phi(x,y,\kappa)\varphi(y)ds(y) \quad x \in \mathbb{R}^m \setminus \partial\Omega$$

and

$$(2.13)\qquad (\mathcal{K}_\kappa\varphi)(x,\kappa) = \int_{\partial\Omega} \frac{\partial\Phi(x,y,\kappa)}{\partial\nu(y)}\varphi(y)ds(y) \quad x \in \mathbb{R}^m \setminus \partial\Omega.$$

For Ω satisfying Hypothesis 2.1, assuming that v is a solution to the scattering problem for a sound-soft scatterer with incident wave v^i satisfying Eq.(2.3) on all of $\mathbb{R}^m$, then the first part of the integrand in Eq.(2.9) is zero and we have $v(x,\kappa) = v^i(x,\kappa) + v^s(x,\kappa), \quad x \in \overline{\Omega}^c, \quad \kappa \in \mathbb{R} \setminus \{0\}$, where the scattered field is given by Huygens' principle as

$$(2.14)\qquad v^s(x,\kappa) = -\int_{\partial\Omega} \frac{\partial v(y,\kappa)}{\partial\nu(y)}\Phi(x,y,\kappa)ds(y), \quad x \in \overline{\Omega}^c.$$

The corresponding far field pattern is given by

$$(2.15)\qquad v^\infty(\hat{x},\kappa) = -\gamma \int_{\partial\Omega} \frac{\partial v}{\partial\nu}(y,\kappa)e^{-i\kappa\hat{x}\cdot y}ds(y), \quad \hat{x} \in \mathbb{S},$$

which follows by passing to the far field pattern of $\Phi(\cdot,y,\kappa)$.

The *exterior Dirichlet problem* is the fundamental problem we address in what follows. This is stated precisely as:

> *Given a continuous function f on $\partial\Omega$, find a radiating solution $v \in C^2(\overline{\Omega}^c) \cap C(\Omega^c)$ to Eq.(2.3) that satisfies Eq.*(2.4).

THEOREM 2.2 *For $\kappa \in \mathbb{R} \setminus \{0\}$ fixed, the exterior Dirichlet problem has a unique solution that depends continuously on the boundary data with respect to uniform convergence of the solution on Ω^c and all its derivatives on closed subsets of $\overline{\Omega}^c$.*

Proof. For positive κ the result can be found in [2, Theorem 3.9]. There, the proof is given by the construction of a solution $v(x,\kappa)$ which is a potential with the density $\varphi(y,\kappa)$,

$$(2.16)\qquad v(x,\kappa) = (\mathcal{K}_\kappa\varphi)(x,\kappa) - i\beta(\mathcal{S}_\kappa\varphi)(x,\kappa), \ \ x \in \mathbb{R}^m\backslash\partial\Omega, \ \ \kappa \in \mathbb{R}_+\backslash\{0\},$$

for the real coupling parameter $\beta \neq 0$. The density φ is the unique solution to the integral equation

$$(2.17)\qquad \big((I + K_\kappa - i\beta S_\kappa)\varphi\big)(\cdot,\kappa) = -2f(\cdot,\kappa), \quad x \in \partial\Omega,$$

where

$$(S_\kappa \varphi)(x,\kappa) = 2\int_{\partial\Omega} \Phi(x,y,\kappa)\varphi(y)ds(y), \quad x \in \partial\Omega, \tag{2.18}$$

and

$$(K_\kappa \varphi)(x,\kappa) = 2\int_{\partial\Omega} \frac{\partial \Phi(x,y,\kappa)}{\partial \nu(y)}\varphi(y)ds(y), \quad x \in \partial\Omega. \tag{2.19}$$

Thus we have the representation

$$v(\cdot,\kappa) = -2\left((\mathcal{K}_\kappa - i\beta\mathcal{S}_\kappa)(I + K_\kappa - i\beta S_\kappa)^{-1} f\right)(\cdot,\kappa), \tag{2.20}$$

for all $\kappa > 0$.

The same theory applies to solutions with $\kappa < 0$, since $\Phi(x,y,\kappa)$ is Hermitian with respect to κ. Indeed, $\overline{S_\kappa} = S_{-\kappa}$ and $\overline{K_\kappa} = K_{-\kappa}$, and likewise for the double- and single-layer potentials $\mathcal{K}_\kappa$ and $\mathcal{S}_\kappa$ defined by Eq.(2.13)-(2.12). Moreover, we have

$$(I + K_{-\kappa} + i\beta S_{-\kappa})^{-1} = (I + \overline{K}_\kappa + i\beta\overline{S}_\kappa)^{-1} = \overline{(I + K_\kappa - i\beta S_\kappa)^{-1}}.$$

The sign of the coupling parameter β is irrelevant to the representation Eq.(2.20), thus the case $\kappa < 0$ is treated exactly as the case $\kappa > 0$. □

Corollary 2.3 *If the boundary value $f(x,\kappa)$ is n-times smoothly differentiable with respect to $\kappa \in \mathbb{R} \setminus \{0\}$, then so is the corresponding solution to the exterior Dirichlet problem. Likewise, if $f(x,\kappa)$ is Hermitian with respect to $\kappa \in \mathbb{R}\setminus\{0\}$, then so is the corresponding solution to the exterior Dirichlet problem.*

Proof. Consider the representation

$$v(\cdot,\kappa) = -2(\mathcal{K}_\kappa - i\ sgn(\kappa)\beta\mathcal{S}_\kappa)(I + K_\kappa - i\ sgn(\kappa)\beta S_\kappa)^{-1} f(\cdot,\kappa), \tag{2.21}$$

where $\beta \neq 0$ and $sgn(\kappa) = 1$ for $\kappa > 0$ and -1 for $\kappa < 0$. At a fixed $\kappa \neq 0$ the double- and single-layer operators defined by Eq.(2.19) and Eq.(2.18), as well as the double- and single-layer potentials given by Eq.(2.13) and Eq.(2.12), are bounded linear operators, Hermitian and analytic with respect to κ. By the chain and product rules, the solution v will be exactly as smooth with respect to κ as the boundary value $f(x,\kappa)$. To prove that v is Hermitian if f is, one need only show that $\overline{v(x,\kappa)} = v(x,-\kappa)$ in the representation above. Indeed,

$$\begin{aligned}
\overline{v(\cdot,\kappa)} &= -2\overline{(\mathcal{K}_\kappa - i\ sgn(\kappa)\beta\mathcal{S}_\kappa)(I + K_\kappa - i\ sgn(\kappa)S_\kappa)^{-1} f(\cdot,\kappa)} \\
&= -2(\overline{\mathcal{K}_\kappa} + i\ sgn(\kappa)\beta\overline{\mathcal{S}_\kappa})(I + \overline{K_\kappa} + i\ sgn(\kappa)\beta\overline{S_\kappa})^{-1}\overline{f}(\cdot,\kappa) \\
&= -2(\mathcal{K}_{-\kappa} - i\ sgn(-\kappa)\beta\mathcal{S}_{-\kappa})(I + K_{-\kappa} - i\ sgn(-\kappa)\beta S_{-\kappa})^{-1} f(\cdot,-\kappa) \\
&= v(\cdot,-\kappa). \qquad (2.22)
\end{aligned}$$

With this, the proof is complete. □

Corollary 2.4 *Let $f \in C(\partial\Omega)\times C(\mathbb{R}\setminus\{0\})$ be the boundary value for the exterior Dirichlet problem parameterized by $\kappa \in C(\mathbb{R}\setminus\{0\})$. Then the corresponding solution $v(x,\kappa)$ exists and satisfies the radiation condition Eq.(2.7) uniformly in κ on compact subsets of $\mathbb{R}\setminus\{0\}$.*

Proof. Since f is continuous, Theorem 2.3 guarantees that $v(x,\kappa)$ exists pointwise in κ, and, by definition, satisfies Eq.(2.7) pointwise in κ. Moreover, by Corollary 2.3 v is continuous in κ on $\mathbb{R}\setminus\{0\}$, therefore v satisfies Eq.(2.7) uniformly in κ on compact subsets of $\mathbb{R}\setminus\{0\}$. □

2.2 The time-domain

We now turn our attention to scattering in the time-domain, where the waves $V : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ satisfy the wave equation Eq.(2.1) with $n = c = 1$ on $x \in \overline{\Omega}^c$, that is,

$$\left(\triangle - \frac{\partial^2}{\partial t^2}\right) V(x,t) = 0, \quad x \in \overline{\Omega}^c, \tag{2.23}$$

$$V(x,t) = F(x,t), \ \forall\, t \text{ and } x \in \partial\Omega \quad \text{(sound-soft)} \tag{2.24}$$

Radiating solutions to the wave equation whose domain of definition contains the exterior of some sphere satisfy the *time-domain Sommerfeld Radiation Condition*:

$$\left\| |x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - \frac{\partial}{\partial t}\right) V(x,\cdot)\right\|_{L^2(\mathbb{R})} \to 0, \ \ |x| \to \infty \tag{2.25}$$

uniformly in all directions.

Note that, in contrast to the frequency domain radiation condition Eq.(2.7), the radiation condition Eq.(2.25) is satisfied *in norm* rather than pointwise with respect to t. Also notice that no initial value is specified in the system above. The boundary-value problem Eq.(2.23)-(2.25) is under-determined. To remedy this we impose further restrictions on the decay of V and its derivatives. These are given explicitly by

$$\frac{\partial^n}{\partial t^n}\frac{\partial^m}{\partial x^m} V(x,\cdot) \in L^1(\mathbb{R}), \quad m,n \in \{0,1,2\}, \tag{2.26}$$

and

$$\int_{|t|>r} \left| \frac{\partial^n}{\partial t^n}\frac{\partial^m}{\partial x^m} V(x,t)\right| \to 0, \quad r \to \infty, \quad m,n \in \{0,1,2\}, \tag{2.27}$$

uniformly in x on compact sets. The motivation for these decay conditions will become apparent in the proof to the correspondence between the time-domain scattering problem and the frequency domain problem, Theorem 2.5.

We would like to point out the contrast between the system of equations Eq.(2.23)-(2.25) together with the decay conditions Eq.(2.26)-(2.27) and systems for which the limiting amplitude principle [10] holds. The limiting amplitude principle gives the steady state solution to the inhomogeneous wave equation with zero initial and boundary values in terms of radiating solutions to the inhomogeneous Helmholtz equation with zero boundary values. This is very different from the situation above, where, rather than the steady state, we are interested in the behavior of the waves *for all time*. Indeed, the steady state for our system of equations is not very interesting since it is simply zero. Instead, it is what happens inbetween that allows us to reconstruct the obstacle.

The time-domain Dirichlet scattering problem that we consider is the following.

> *For a scatterer Ω satisfying Hypothesis 2.1, given an* incident field V^i : $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ *that solves Eq.(2.23) on all of* $\mathbb{R}^m \times \mathbb{R}$, *find the* total field $V : \Omega^c \to \mathbb{R}$ *satisfying Eq.(2.23) on* $\overline{\Omega}^c$ *with* $F := 0$ *in Eq.(2.24) and with* $V = V^i + V^s$, *where* V^s *is the* scattered field *satisfying Eq.(2.23) and the radiation conditions Eq.(2.25)-(2.27).*

The next theorem establishes the correspondence between this scattering problem and the classical scattering problem in the frequency-domain.

Theorem 2.5 (Correspondence of time- and frequency-domain.) *Let* $f(x,\kappa) : \partial\Omega \times \mathbb{R} \to \mathbb{C}$ *satisfy* $f \in C(\partial\Omega) \times C^2_c(\mathbb{R})$, *with* $f(x,\cdot)$ *Hermitian and* $\operatorname{supp} f = \partial\Omega \times \mathbb{K}$, *where* $\mathbb{K} \subset \mathbb{R} \setminus \{0\}$ *is compact. Then the function* v *satisfies the exterior Dirichlet problem at almost every* $\kappa \in \mathbb{K}$ *if and only if the Fourier dual* $V : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $V \in C^2(\overline{\Omega}^c) \times C^2(\mathbb{R})$, *satisfies the time-domain boundary value problem Eq.(2.23)-(2.27), with* $F(x,t) := (\mathcal{F}_\kappa f)(x,t)$ *in Eq.(2.24).*

Proof. Suppose that $v(x,\kappa)$ satisfies the exterior Dirichlet problem with boundary values f at almost every $\kappa \in \mathbb{K}$. Then, by Corollary 2.3, v is almost everywhere equivalent to $v_* \in (C^2(\overline{\Omega}^c) \cap C(\Omega^c)) \times C^2(\mathbb{R})$, with $\operatorname{supp} v_*(x,\cdot) \subset \mathbb{K}$ and $v_*(x,-k) = \overline{v_*(x,k)}$. Also, by Corollary 2.4, v_* satisfies Eq.(2.7) uniformly in κ on $\mathbb{K}$, thus

$$\left\| |x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) v_*(x,\cdot) \right\|_{L^2(\mathbb{R})} \to 0, \quad |x| \to \infty. \tag{2.28}$$

We show next that the Fourier transform of v_* and it's derivatives are absolutely integrable with respect to κ uniformly in x on compact sets. To see this, recall that, for $m = 0, 1$ or 2, the partial derivative $\frac{\partial^m v_*}{\partial x^m}$ is Hermitian and belongs to $(C^{2-m}(\overline{\Omega}^c) \cap C(\Omega^c)) \times C^2(\mathbb{R})$. The Fourier transform of $\frac{\partial^m v_*}{\partial x^m}$, call it V_m

where $V_m(x,t) = (\mathcal{F}_\kappa \frac{\partial^m v_*}{\partial x^m})(x,t)$, is therefore a bounded, real-valued function with $\mathcal{F}_\kappa (i\kappa)^n \frac{\partial^m v_*}{\partial x^m} = \frac{\partial^n V_m}{\partial t^n}$, satisfying $\left|\frac{\partial^n V_m(x,t)}{\partial t^n}\right| \le M_m(x)\,(1+|t|)^{-2}$ by standard Fourier analysis, and, in particular, $\frac{\partial^n V_m(x,\cdot)}{\partial t^n} \in L^1(\mathbb{R})$, for $n \ge 0$ uniformly in x on compact sets. Now, since v_* is bandlimited with respect to κ and twice continuously differentiable with respect to x then $\frac{\partial^m v_*}{\partial x^m} \in L^1(\mathbb{R})$ uniformly in x on compact sets. Thus, by [5, Theorem 53.5] we can differentiate under the integral in the Fourier transform to yield $V_m = \frac{\partial^m V}{\partial x^m}$ ($m = 0, 1$ or 2). The Fourier dual V therefore satisfies Eq.(2.26)-(2.27) and the boundary value problem Eq.(2.23)-(2.25) where the radiation condition Eq.(2.25) follows from Parseval's relation.

Conversely, suppose that the bounded, real-valued function $V : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $V \in C^2(\overline{\Omega}^c) \times C^2(\mathbb{R})$, satisfies Eq.(2.26)-(2.27) uniformly in x on compact sets, in addition to the time-domain boundary value problem Eq.(2.23)-(2.25), with $F(x,t) := (\mathcal{F}_\kappa f)(x,t)$ in Eq.(2.24). Then [5, Theorem 53.5] can be applied to show $\mathcal{F}_t \frac{\partial V}{\partial |x|} = \frac{\partial v}{\partial |x|}$ and $\mathcal{F}_t \triangle V = \triangle v$, where $v = \mathcal{F}_t V$. These, together with the decay conditions Eq.(2.27) yield

$$
\begin{aligned}
(\triangle + \kappa^2) v(x,\kappa) &= 0, \quad x \in \overline{\Omega}^c, \\
v(x,\kappa) &= f(x,\kappa), \quad x \in \partial\Omega, \\
\text{and} \quad \left\| |x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) v(x,\cdot) \right\|_{L^2(\mathbb{R})} &\to 0, \quad |x| \to \infty.
\end{aligned} \tag{2.29}
$$

To complete the proof, we must achieve a pointwise decay condition from Eq.(2.29). To do this, define $\mathbb{S}_j := \{x \in \mathbb{R}^m \mid |x| = r_j\}$, where the sequence of scalars $r_j \to \infty$. The corresponding sequence of functions is given by

$$
\psi_j(\kappa) := \max_{x \in \mathbb{S}_j} \left| |x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) v(x,\kappa) \right|.
$$

The radiation condition Eq.(2.29) implies that $\|\psi_j\|_{L^2} \to 0$, thus there exists a subsequence $\psi_{j_i}(\kappa)$ converging pointwise almost everywhere to zero in κ [16, Theorem 3.12]. Since $v(\cdot,\kappa)$ satisfies Eq.(2.3), $\psi_{j_i}(\kappa) \to 0$ implies that

$$
\left| |x|^{\frac{m-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) v(x,\kappa) \right| \to 0, \quad |x| \to \infty,
$$

that is, $v(x,\kappa)$ satisfies the exterior Dirichlet Problem at almost every κ. □

Corollary 2.6 (Uniqueness and Existence for time-domain scattering.) *Under the hypotheses of Theorem 2.5, solutions V to Eq.(2.23)-(2.27) are unique.*

Proof. This follows directly from Theorem 2.2 and Theorem 2.5. □

3 Inverse Scattering

The point source method seeks to construct a backprojection operator from the far field to the near field of a scatterer that is illuminated by an incident field that can be represented as a superposition of plane waves. In the frequency domain the far field pattern u^∞ describes the asymptotic behavior of the scattered field u^s when the modulus $|x|$ tends to infinity as described by equation (2.8). In the time domain we define the far field U^∞ to be the Fourier transform $\mathcal{F}_\kappa u^\infty$of u^∞. In practical applications, U^∞ is approximately the time dependent scattered field U^s on the surface of a large ball centered at the origin with radius $R \gg 1$, $\partial\mathbb{B}_R(0)$, normalized with the appropriate decay factor $e^{i\kappa R}/R^{(m-1)/2}$.

The backprojection operator as introduced in [14] relies on the duality of point sources and plane waves. We denote the total field generated by an incident plane wave by u,

$$u(x,\kappa,\hat\eta) = u^i(x,\kappa,\hat\eta) + u^s(x,\kappa,\hat\eta) \tag{3.1}$$

where $u^i(x,\kappa,\hat\eta) \;:=\; e^{i\kappa x\cdot\hat\eta},\;\; x\in\mathbb{R}^m,\; \kappa\in\mathbb{R}$, and $\hat\eta\in\mathbb{S}$ denotes the *direction of incidence*. The field resulting from excitation by a point source $\Phi(x,,\kappa,z)\;\;(x\neq z)$ is given by $w(\cdot,\kappa,z) : \overline{\Omega}^c \to \mathbb{C}$:

$$w(\cdot,\kappa,z) := w^i(\cdot,\kappa,z) + w^s(\cdot,\kappa,z), \tag{3.2}$$

where $w^i(\cdot,\kappa,z) := \Phi(\cdot,\kappa,z),\;\; z\in\overline{\Omega}^c$, and $\kappa\neq 0$. The field w is a solution to the scattering problem with an incident point source. This field is the *Green function* for the boundary value problem Eq.(2.3), Eq.(2.4) (or Eq.(2.5) or Eq.(2.6)) and Eq.(2.7), and is symmetric: $w(x,\kappa,z) = w(z,\kappa,x)\quad x,z\in\overline{\Omega}^c,\quad x\neq z$. The corresponding scattered field $w^s(\cdot,\kappa,z)$ satisfies Eq.(2.3)-(2.7) with $f = -\Phi(\cdot,\kappa,z)$ on $\partial\Omega$.

3.1 The time-dependent point source method

Consider $\Omega_a\subset\mathbb{R}^m$ a bounded domain with simply connected piecewise C^2 boundary with corners satisfying the cone conditions [3]. We first note that, for $\overline{\Omega}\subset\Omega_a$, a straight forward argument using Green's theorem and the boundary conditions for the fields u and w (see [4]) shows that

$$w^\infty(\hat x,\kappa,z) \;=\; \gamma(\kappa)u^s(z,,\kappa,-\hat x),\;\; \hat x\in\mathbb{S},\; z\in\overline{\Omega}_a^c, \tag{3.3}$$

where

$$\gamma(\kappa) = \left\{\begin{array}{ll} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8\pi\kappa}}, & m=2 \\ \frac{1}{4\pi} & m=3 \end{array}\right. . \tag{3.4}$$

Equation Eq.(3.3) is referred to as the *mixed reciprocity relation* and is discussed in further detail in [14, Theorem 2.1.4]. Second, by the principle of superposition for far field patterns [2, Theorem 3.16], the far field pattern due to scattering from any incident field (in particular an incident point source) can be expressed as a superposition of far field patterns due to scattering from incident plane waves. The point source method uses these two facts to reconstruct the scattered field on some region $\mathbb{E}$ outside of Ω. This technique has been explored in [7, 8, 12–14] where it is applied to frequency-domain problems. Here we extend this methodology to nonharmonic, time-dependent waves.

Let $\Lambda \subset \mathbb{S}$ denote an open set of directions on $\mathbb{S}$. Here, Λ models the aperture on which our sensors lie. In our numerical experiments, this is a symmetric interval of the unit sphere centered with respect to the direction of the incident field. The far field u^∞ due to an incident plane wave with direction $\hat{\eta} \in \mathbb{S}$ is measured at points $\hat{y} \in \Lambda$. Define the Herglotz wave operator $H_\kappa : L^2(-\Lambda) \to L^\infty(\mathbb{R}^m)$ by

$$(H_\kappa g)(x) := \int_\Lambda e^{i\kappa x\cdot(-\hat{y})} g(-\hat{y})\, ds(\hat{y}), \quad x \in \mathbb{R}^m,\ g \in L^2(-\Lambda). \tag{3.5}$$

The corresponding family of Herglotz wave functions parameterized by κ and mapping $\mathbb{R}^m$ to $\mathbb{C}$, $h_g(\cdot,\kappa) := (H_\kappa g)(\cdot)$, consists of entire solutions to the Helmholtz equation for fixed κ. Of particular interest is the Herglotz wave operator restricted to some surface $\mathbb{X} \subset \mathbb{R}^m$. The adjoint of H_κ with κ fixed, denoted $H_\kappa^* : L^2(\mathbb{X}) \to L^2(\Lambda)$, is given by

$$(H_\kappa^* \psi)(d) \ := \int_{\mathbb{X}} e^{-i\kappa x\cdot(-d)} \psi(x)\, ds(x), \ \ d \in \Lambda.$$

Let Ω_a be a bounded domain for which κ^2 is not a Dirichlet eigenvalue of $-\triangle$ on the interior of Ω_a, and whose boundary $\partial\Omega_a$ is simply connected, piecewise C^2. It can be shown that in this case H_κ and H_κ^* restricted to $\partial\Omega_a$ are injective with dense range. Thus one can choose the Herglotz wave function h_g with density g to approximate arbitrarily closely any convenient incident field v^i on $\partial\Omega_a$. The incident field we shall approximate is an incident point source located at a point $z \in \overline{\Omega}_a^c$. To this end, we define the Herglotz wave function as a function of the spatial variable $x \in \Omega_a$ and wavenumber $\kappa \in \mathbb{R}$ parameterized by the point z as

$$h_g(x,\kappa,z) := \int_\Lambda e^{i\kappa x\cdot(-\hat{y})} g(-\hat{y},\kappa,z)\, ds(\hat{y}). \tag{3.6}$$

The backprojection operator that is at the heart of the point source method is built upon the integral operator $\mathcal{B}_g : L^2(\Lambda \times \mathbb{R} \times \mathbb{S}) \to L^2(\mathbb{R}^m \times \mathbb{R} \times \mathbb{S})$ with kernel $g(\cdot,\cdot,z)$. For a function $\psi \in L^2(\Lambda \times \mathbb{R} \times \mathbb{S})$, the operator $\mathcal{B}_g$ is defined by

$$(\mathcal{B}_g \psi)(z,\kappa,\hat{\eta}) := \int_\Lambda \psi(\hat{x},\kappa,\hat{\eta}) \frac{g(-\hat{x},\kappa,z)}{\gamma(\kappa)}\, ds(\hat{x}), \tag{3.7}$$

for $\gamma(\kappa)$ given by Eq.(3.4). The corresponding time-domain operator, denoted $B_g : L^2(\Lambda \times \mathbb{R} \times \mathbb{S}) \to L^2(\mathbb{R}^m \times \mathbb{R} \times \mathbb{S})$, is defined by

$$B_g := \mathcal{F}_\kappa \mathcal{B}_g \mathcal{F}_t \tag{3.8}$$

The next theorem states that, provided $\Omega \subset \Omega_a$, the density $g(\cdot, \kappa, z)$ for which the Herglotz wave function approximates $\Phi(\cdot, \kappa, z)$ on $\partial\Omega_a$ allows one directly to calculate an approximation to the scattered field u^s at the point z from fixed frequency far field data u^∞. For multifrequency fields in the time-domain, we are interested in the uniformity of such approximations over all κ. For this we need the following lemma.

LEMMA 3.1 *Let $\Omega_a \subset \mathbb{R}^m$ be a bounded domain with simply connected, piecewise C^2 boundary. The set of Dirichlet eigenvalues of the negative Laplacian on the interior of Ω_a has measure zero with respect to Lebesgue measure on $\mathbb{R}$.*

Proof. The spectrum of $-\triangle$ on bounded domains has a countably infinite spectrum [6, Theorem 4.1], and any countable set of points in $\mathbb{R}$ has measure zero with respect to Lebesgue measure. □

Before stating the main result of this section, we introduce the far field mapping $\mathcal{T}_{\Omega'_a} : v^s|_{\Omega'_a} \to v^\infty$ mapping the scattered field v^s restricted to any compact subset $\Omega'_a \subset \overline{\Omega}^c$ containing open subsets to the far field v^∞. This is a continuous mapping. The time domain counterpart to this – also a continuous mapping – is denoted T with $\mathcal{T}_{\Omega'_a} : v^s|_{\Omega'_a} \to v^\infty$. The next theorems show how to approximate the inverse of the far field mapping of the scattered field due to an incident plane wave.

THEOREM 3.2 (NORM CONVERGENCE IN FREQUENCY) *Let $\Omega_a \subset \mathbb{R}^m$ be a bounded domain with simply connected, piecewise C^2 boundary satisfying $\overline{\Omega} \subset \Omega_a$ and let $\mathbb{K} = \mathbb{R} \setminus \mathbb{B}(0, \epsilon')$ where $\mathbb{B}(0, \epsilon')$ is the closed ball of radius ϵ' centered at the origin. Given any $\delta > 0$ and any fixed $z \in \overline{\Omega}_a^c$, there exists an $\epsilon > 0$ such that, for all $\hat{\eta} \in \mathbb{S}$,*

$$\left\| \Phi(\cdot, \cdot, z) - h_g(\cdot, \cdot, z) \right\|_{C(\partial\Omega_a) \times L^2(\mathbb{K})} < \epsilon \tag{3.9}$$

implies

$$\left\| u^s(z, \cdot, \hat{\eta}) - (\mathcal{B}_g u^\infty)(z, \cdot, \hat{\eta}) \right\|_{L^2(\mathbb{K})} < \delta. \tag{3.10}$$

Here u^s and u^∞ are the scattered field and far field pattern due to an incident plane wave with direction $\hat{\eta}$, $\mathcal{B}_g$ is defined by Eq.(3.7), and $h_g(\cdot, \cdot, z)$ is defined by Eq.(3.6).

Proof. Consider any g satisfying Eq.(3.9) (we know that such a g exists since H_κ is injective with dense range). Both h_g and Φ solve Eq.(2.3) on the interior of Ω_a, thus both are analytic with respect to the spatial variable [2, Theorem 2.2]. Let $\mathbb{A} = \{\kappa \mid \kappa^2 \text{ is an eigenvalue of } -\triangle \text{ on } \Omega_a\}$. For all $\kappa \in \mathbb{K} \setminus \mathbb{A}$ solutions to Eq.(2.3) with boundary values on $\partial\Omega_a$ are unique. Thus on $\Omega_a \times \mathbb{K} \setminus \mathbb{A}$, given any δ' there is an ϵ such that Eq.(3.9) implies

$$\left|\left|\Phi(\cdot,\cdot,z) - h_g(\cdot,\cdot,z)\right|\right|_{C(\overline{\Omega}_a)\times L^2(\mathbb{K}\setminus\mathbb{A})} \leq \delta'$$

and hence

$$\left|\left|\Phi(\cdot,\cdot,z) - h_g(\cdot,\cdot,z)\right|\right|_{C(\partial\Omega)\times L^2(\mathbb{K}\setminus\mathbb{A})} \leq \delta'. \tag{3.11}$$

By Lemma 3.1, the set $\mathbb{A}$ has measure zero, thus the inequalities above hold when $\mathbb{K} \setminus \mathbb{A}$ is replaced by $\mathbb{K}$ in the norm above.

Now consider the scattered field associated with the incident field $v^i(x,\kappa,z) = h_g(x,\kappa,z)$ due to scattering from the sound-soft obstacle Ω. Likewise, consider the scattered field w^s due to an incident point source $\Phi(x,\kappa.z)$, $z \in \Omega_a^c$. On $\partial\Omega$ we have $v^i = -v^s$ and $w^s = -\Phi$, thus Eq.(3.11) can be rewritten as

$$\left|\left| - w^s(\cdot,\cdot,z) + v^s(\cdot,\cdot,z)\right|\right|_{C(\partial\Omega)\times L^2(\mathbb{K})} \leq \delta'.$$

Since the far field mapping is $\mathcal{T}_{\partial\Omega}$ is a continuous mapping, given any $\delta > 0$ there is a $\delta' > 0$ such that $\|w^s(\cdot,\cdot,z) - v^s(\cdot,\cdot,z)\|_{C(\partial\Omega)\times L^2(\mathbb{K})} \leq \delta'$ implies

$$\left|\left|w^\infty(-\hat{\eta},\cdot,z) - v^\infty(-\hat{\eta},\cdot,z)\right|\right|_{L^2(\mathbb{K})} \leq \delta, \quad \forall\, \hat{\eta} \in \mathbb{S}.$$

By [2, Lemma 3.16], $v^\infty(-\hat{\eta},\kappa,z) = \int_\Lambda u^\infty(-\hat{\eta},\kappa,-\hat{y})g(-\hat{y},\kappa,z)ds(\hat{y})$, where u^∞ is the far field pattern due to scattering of an incident plane wave. Now, the mixed reciprocity relation Eq.(3.3) together with the standard reciprocity relation $u^\infty(-\hat{\eta},\kappa,-\hat{y}) = u^\infty(\hat{y},\kappa,\hat{\eta})$ (see [2, Theorem 3.13]) yield the result

$$\left\|u^s(z,\cdot,\hat{\eta}) - \frac{1}{\gamma(\kappa)}\int_\Lambda u^\infty(\hat{y},\cdot,\hat{\eta})g(-\hat{y},\cdot,z)ds(\hat{y})\right\|_{L^2(\mathbb{K})} \leq \delta, \quad \forall\, \hat{\eta} \in \mathbb{S}.$$

□

Corollary 3.3 (Pointwise convergence in time) *In addition to the assumptions of Theorem 3.2, let $u^i(\cdot,\kappa,\hat{\eta}) = 0$ for all $\kappa \in \mathbb{B}(0,\epsilon')$. Then, for any sequence $\{g_i\}$ with*

$$\left\|\Phi(\cdot,\cdot,z) - h_{g_i}(\cdot,\cdot,z)\right\|_{C(\partial\Omega_a)\times L^2(\mathbb{K})} \to 0. \tag{3.12}$$

there exists a subsequence $\{g_{i_j}\}$ *such that, for any* $z \in \overline{\Omega}_a^c$

$$(B_{g_{i_j}} U^\infty)(z, \hat{\eta}, t) \to U^s(z, \hat{\eta}, t), \quad \forall\ \hat{\eta} \in \mathbb{S},$$

where $B_{g_{i_j}}$ *is defined by Eq.(3.8).*

Proof. Since $u^i(\cdot, \kappa, \hat{\eta}) = 0$ for $\kappa \in \mathbb{B}(0, \epsilon')$, then the scattered field and far field pattern at these wavenumbers are also zero. Without loss of generality define $g(\hat{y}, \kappa, z) := 0$ for $\kappa \in \mathbb{B}(0, \epsilon')$. Thus, the inequality Eq.(3.10) can be extended to

$$\left\| u^s(z, \cdot, \hat{\eta}) - (\mathcal{B}_g u^\infty)(z, \cdot, \hat{\eta}) \right\|_{L^2(\mathbb{R})} < \delta. \tag{3.13}$$

By Parseval's identity, we have

$$\left\| U^s(z, \cdot, \hat{\eta}) - (B_g U^\infty)(z, \cdot, \hat{\eta}) \right\|_{L^2(\mathbb{R})} < \delta, \tag{3.14}$$

where $U^s = \mathcal{F}_\kappa u^s$, $U^\infty = \mathcal{F}_\kappa u^\infty$ and B is defined by Eq.(3.8). Now let δ_i be a sequence with $\delta_i \to 0$. By Theorem 3.2 there exists a sequence $\epsilon_i \to 0$, such that, for any corresponding sequence $\{g_i\}$ satisfying

$$\left\| \Phi(\cdot, \cdot, z) - h_{g_i}(\cdot, \cdot, z) \right\|_{C(\partial\Omega_a) \times L^2(\mathbb{K})} < \epsilon_i \to 0,$$

we have

$$\left\| U^s(z, \cdot, \hat{\eta}) - (B_{g_i} U^\infty)(z, \cdot, \hat{\eta}) \right\|_{L^2(\mathbb{R})} < \delta_i \to 0, \quad \forall\ \hat{\eta} \in \mathbb{S}.$$

Thus by [16, Theorem 3.12] there exists a subsequence $\{g_{i_j}\}$ such that

$$\lim_{j\to\infty} (B_{g_{i_j}} U^\infty)(z, t, \hat{\eta}) = U^s(z, t, \hat{\eta}), \quad \forall\ \hat{\eta} \in \mathbb{S} \text{ and for a.e. } t \in \mathbb{R}.$$

Since $U^s(z, \cdot, \hat{\eta})$ and $(B_{g_i} U^\infty)(z, \cdot, \hat{\eta}) \in C^2(\mathbb{R})$ convergence is pointwise everywhere. □

We close this section with a few remarks about the regularity of the point source method, by which we mean that the method admits a *regular regularization strategy.* This is defined below.

Definition 3.4 *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be normed spaces and let* $A : \mathbb{X} \to \mathbb{Y}$ *be an injective bounded linear operator.* Regularized inverses of A *are bounded linear operators* $R_\alpha : \mathbb{X} \to \mathbb{Y}$ *parameterized by* $\alpha > 0$, *a* regularization parameter, *with the property of pointwise convergence to* A^{-1} *for all* $\varphi \in \mathbb{X}$,

$$\lim_{\alpha\to 0} R_\alpha A\varphi = \varphi, \quad \forall \varphi \in \mathbb{X}.$$

A regularized solution *to the problem of solving* $A\varphi = \psi$ *for the unknown* φ *(*$\psi \in$ *range* A*) is the approximation* $\varphi_\alpha = R_\alpha \psi$.

Noisy inverse problems are characterized by a mismatch, or error, between the true image ψ of the input φ under the operator A, $\psi = A\varphi$, and the observed image ψ_δ. The next definition establishes the framework for analyzing the convergence properties of regularized inversion in the presence of noise.

DEFINITION 3.5 *A* regularization strategy R_α, *that is a rule for choosing* α *depending on the size of the image error* δ, *is called* regular *if, for all* $\psi \in \operatorname{range} A$ *and all* $\psi_\delta \in \mathbb{Y}$ *with* $\|\psi_\delta - \psi\| \le \delta$ *we have*

$$R_{\alpha(\delta)}\psi_\delta \to A^{-1}\psi, \ \ as \quad \delta \to 0.$$

COROLLARY 3.6 (REGULARITY OF THE TIME-DOMAIN POINT SOURCE METHOD) *The density* $g_\alpha(\hat{y}, \kappa, z)$ *given by*

$$g_\alpha(x, \kappa, z) := (H_\kappa^* H_\kappa + \alpha I)^{-1} H_\kappa^* \Phi(x, \kappa, z) \tag{3.15}$$

is a regularized solution (in fact, the Tikhonov regularized solution) to the inverse problem

$$(H_\kappa g)(x, \kappa, z) = \Phi(x, \kappa, z), \quad x \in \partial\Omega_a.$$

Moreover, under the assumptions of Corollary 3.3, the operator B_{g_α} *defined by Eq.(3.8) is a regularized inverse of the far field mapping* $T_{\Omega_a'}$ *on compact subsets of* $\overline{\Omega}^c$, *which, in the presence of noise, admits a regular regularization strategy.*

PROOF. Recall that H_κ and H_κ^*, restricted to $\partial\Omega_a$ for fixed κ and κ^2 not a Dirichlet eigenvalue of $-\triangle$ on Ω_a, are injective with dense range. Thus the operator $R_\alpha = (\alpha I + H_\kappa^* H_\kappa)^{-1} H_\kappa^*$ is the Tikhonov regularized inverse operator for H_κ on $\partial\Omega_a$, for $\kappa \in \mathbb{K} \setminus \{\text{Dirichlet eigenvalues of } -\triangle \text{ on } \partial\Omega_a\}$ (see for example [2, Ch.4]).

To prove the remainder of the theorem we decompose the error into the regularization error and the data error:

$$||B_{g_\alpha} U^\infty_{\delta'} - T^{-1}_{\Omega_a'} U^\infty|| \ \le \ ||B_{g_\alpha} U^\infty - T^{-1}_{\Omega_a'} U^\infty|| + ||B_{g_\alpha}(U^\infty - U^\infty_{\delta'})|| \tag{3.16}$$

where $U^\infty_{\delta'}$ is a noisy measurement satisfying $\|U^\infty(z, t, \hat{\eta}) - U^\infty_{\delta'}(z, t, \hat{\eta})\| < \delta'(t)$. To show that B_{g_α} is a regularized inverse of $T_{\Omega_a'}$, we need only show that, for all $U^\infty \in \operatorname{range} T_{\Omega_a'}$, the first norm on the right of Eq.(3.16) tends to zero as $\alpha \to 0$. If the second norm on the right of Eq.(3.16) tends to zero as the data error $\|U^\infty(z, t, \hat{\eta}) - U^\infty_{\delta'}(z, t, \hat{\eta})\| < \delta'(t) \to 0$, then by Definition 3.5 B_{g_α} is *regular*.

First, assuming that $u^s(\cdot, \kappa, \hat{\eta}) = 0$ for all $\kappa \in \mathbb{K}^c$, then by Eq.(3.14), given any $\delta > 0$ and any $U^\infty \in \operatorname{range} T_{\{z\}}$, there exists $\epsilon > 0$ and $\alpha > 0$ such that

$$\Big|\Big|\Phi(x, \kappa, z) - (H_\kappa g_\alpha)(x, \kappa, z)\Big|\Big|_{C(\partial\Omega_a)\times L^2(\mathbb{K})} < \epsilon$$

implies

$$\left\| (T^{-1}_{\{z\}} U^\infty)(z,\cdot,\hat\eta) - (B_{g_\alpha} U^\infty)(z,\cdot,\hat\eta) \right\|_{L^2(\mathbb{R})} < \frac{\delta}{2}, \quad \forall\, \hat\eta \in \mathbb{S}. \tag{3.17}$$

Since $(\alpha I + H_\kappa^* H_\kappa)^{-1} H_\kappa^*$ is a regularized inverse of H_κ, which is injective with dense range, the norm above tends to zero as ϵ, and $\alpha \to 0$.

For the second step, recall that for fixed t, the backprojection operator B_{g_α} is a continuous linear operator, thus, for each function $\delta''(t) > 0$ with $\left(\int_{\mathbb{R}} \delta''(t)dt\right)^{1/2} = \delta$, there exists a function $\delta'(t) > 0$ such that

$$\left\| U^\infty(\cdot,t,\hat\eta) - U^\infty_{\delta'}(\cdot,t,\hat\eta) \right\|^2_{L^2(\Lambda)} < \delta'(t), \quad \forall\, \hat\eta \in \mathbb{S}$$

implies

$$\left| (B_{g_\alpha} U^\infty)(z,t,\hat\eta) - (B_{g_\alpha} U^\infty_{\delta'})(z,t,\hat\eta) \right|^2 \le \frac{\delta''(t)}{4}, \quad \forall\, \hat\eta \in \mathbb{S}.$$

Integrating with respect to t and taking the square root yields

$$\left\| B_{g_\alpha} U^\infty(z,\cdot,\hat\eta) - B_{g_\alpha} U^\infty_{\delta'}(z,\cdot,\hat\eta) \right\|_{L^2(\mathbb{R})} \le \frac{\delta}{2}, \quad \forall\, \hat\eta \in \mathbb{S}. \tag{3.18}$$

Together Eq.(3.17) and Eq.(3.18) yield

$$\left\| T^{-1}_{\{z\}} U^\infty(z,\cdot,\hat\eta) - B_{g_\alpha} U^\infty_{\delta'}(z,\cdot,\hat\eta) \right\|_{L^2(\mathbb{R})} < \delta, \quad \forall\, \hat\eta \in \mathbb{S},$$

and thus the regularity of the point source method. □

REMARK 3.7 Corollary 3.6 relies on some strong, but practical, assumptions about the nature of the noise. Specifically, we require that the error in the time dependent far field pattern $U^\infty_{\delta'}$ is integrable with respect to time. This is often enforced in the form of a priori assumptions on the decay of the wave with respect to time.

4 Numerical examples

We conclude with two numerical demonstrations of the method described above. The following images show the actual and reconstructed time progression of the total wave-field (scattered plus incident waves) as it scatters around a sound soft obstacle. The incident wave travels from right to left, and has the frequency profile shown in Figure 1. As this picture shows, the incident field, that is, the boundary condition f in Eq.(2.4), has been constructed so that it satisfies the hypotheses of Theorem 2.5. We demonstrate the point source method when the obstacle is an ellipse shown in Figure 2(a), and, to see the effects of nonconvexity, when the obstacle is a kite shape obstacle shown in Figure 2(b). Note that the wave numbers range from -5 to 5 and the obstacle diameters are around 2, thus the wave numbers are in the resonance region of the obstacles.

The data consists of time-series measurements of the scattered field on a sphere $\mathbb{S}$ in the far field. To generate the forward data we use integral equations as described in [2]. We discretize the integral equations using 160 points on the boundary, ignoring the singularity. This approach yields far field data with approximately 2% error. We evaluate the far field pattern at 80 points on the unit sphere with 66 time slices. The scattered field, reconstructed using the point source method, was computed at 6400 evenly spaced radial points in the computational domain. For reference, we calculated the true scattered field via boundary integral techniques on a 80×80 Cartesian grid.

To implement the point source method in this setting, we apply the Fourier transform to the far field data and solve for the scattered field at the sampled wavenumber κ in the usual way with the point source method (see [7, 12, 14]). We refine our solution for the scattered fields at each frequency by estimating the location of the center of the obstacle via the techniques outlined in [9]. With the location of the center in hand, we then reconstruct the total field at each frequency along radial lines extending from this center. Applying the inverse Fourier transform yields the time-dependent wave. This is shown in the series of snap-shots of the wave displayed in Figures 3(a)-(c) and 4(a)-(c) for the obstacles shown in Figures 2(a) and (b) respectively. Each of these is compared to the true time-dependent wave shown in Figures 3(e)-(h) and 4(e)-(h). The true solution is calculated using boundary integral techniques outlined in [2].

To give some idea of the computational intensity of these experiments, the forward solution for the time-dependent scattered field calculated on the 80×80 Cartesian grid via boundary integral techniques took 206 seconds using MATLAB with the DP Toolbox [11] on a parallel cluster of 6 Linux PC's (2 Ghz, 1280 MB RAM). In contrast, the inverse solution, calculated on a polar grid of 6400 evenly spaced points from 80 simulated far field measurements, took 13 seconds.

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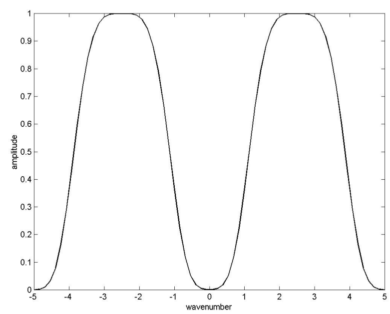
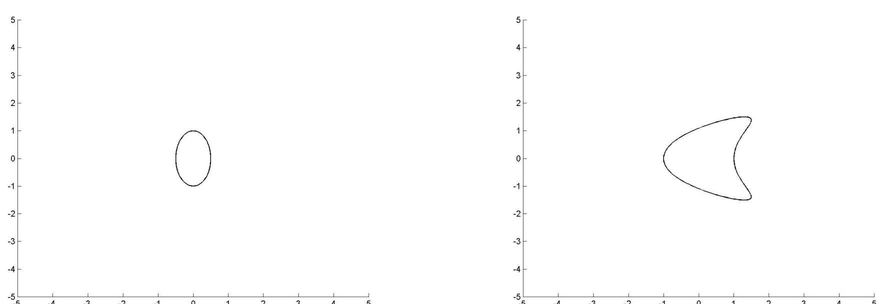


Figure 1: Frequency profile of incident wave.



(a) (b)

Figure 2: Sound-soft obstacles.

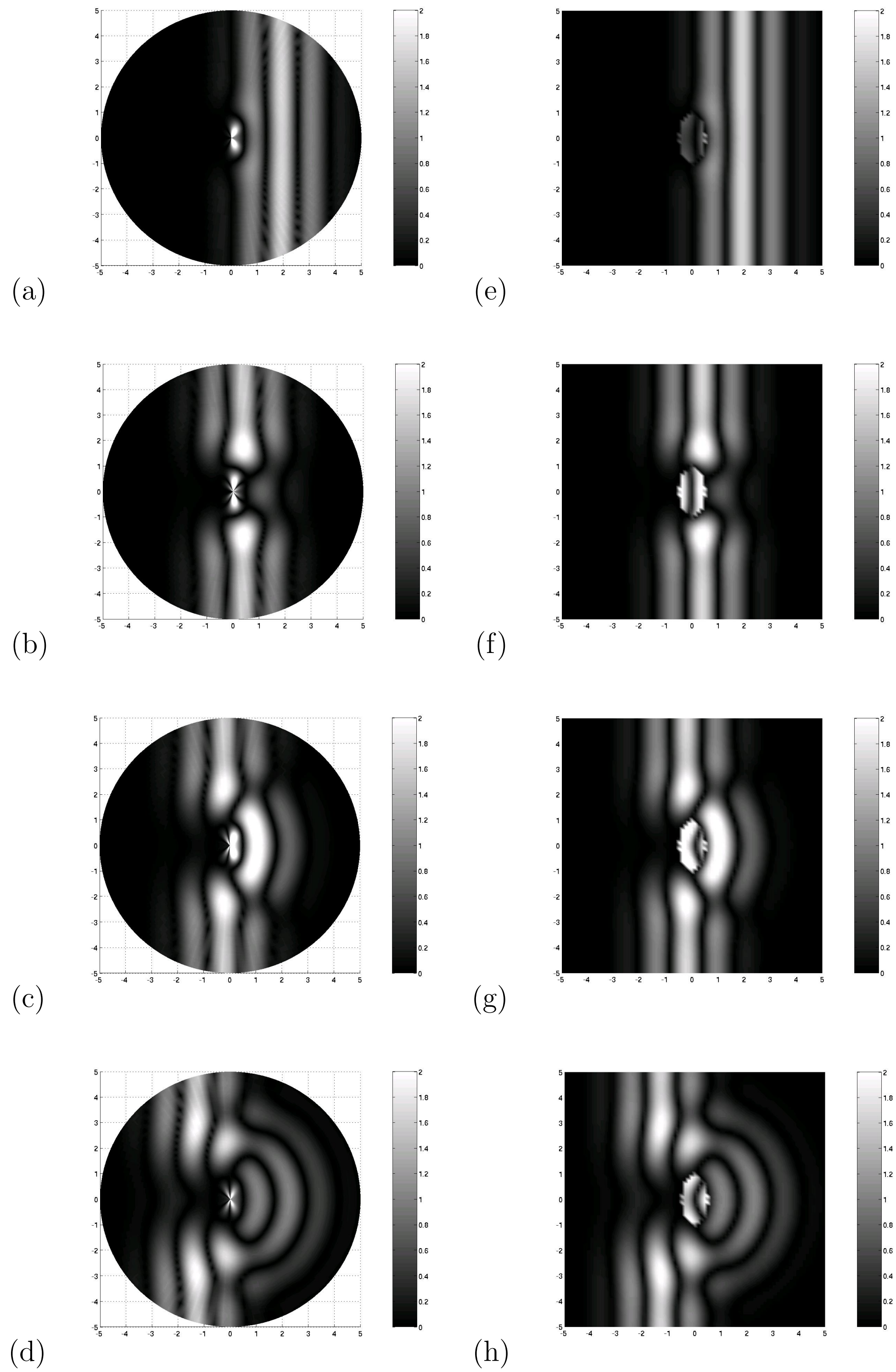


Figure 3: (a)-(d) Amplitude of the total wave calculated via the point source method from far field measurments at times $t = 21, 31, 35$ and 42 for the ellipse shown in Figure 2(a). (e)-(h) Amplitude of total wave at the same times calculated using the forward problem with exact boundary data.

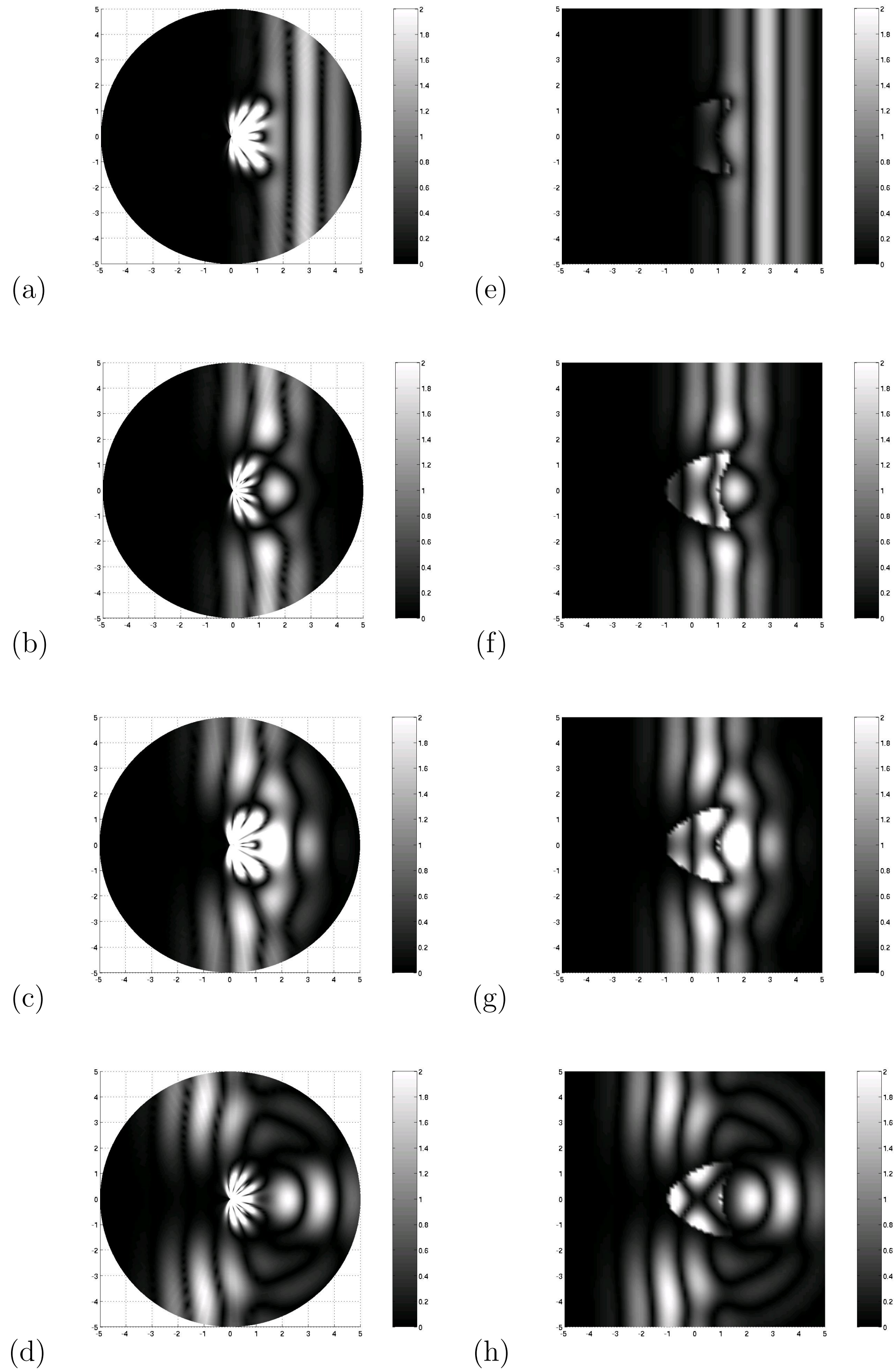


Figure 4: (a)-(d) Amplitude of the total wave calculated via the point source method from far field measurments at times $t = 15, 25, 30$ and 40 for the kite-shaped obstacle shown in Figure 2(b). (e)-(h) Amplitude of total wave at the same times calculated using the forward problem with exact boundary data.

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