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Abstract

We introduce and analyze a new mixed finite element method for the numerical approximation of incompressible magneto-hydrodynamics (MHD) problems in polygonal and polyhedral domains. The method is based on standard inf-sup stable elements for the discretization of the hydrodynamic unknowns and on nodal elements for the discretization of the magnetic variables. In order to achieve convergence in non-convex domains, the magnetic bilinear form is suitably modified using the weighted regularization technique recently developed in [7]. We discuss the well-posedness of this approach and establish a novel existence and uniqueness result for non-linear MHD problems with small data. We further derive quasi-optimal error bounds for the proposed finite element method and show the convergence of the approximate solutions in non-convex domains. The theoretical results are confirmed in a series of numerical experiments for a linear two-dimensional Oseen-type MHD problem, demonstrating that weighted regularization is indispensable for the resolution of the strongest magnetic singularities.

Key words: Incompressible magneto-hydrodynamics, mixed methods, weighted regularization

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1 Introduction

Incompressible magneto-hydrodynamics (MHD) describes the flow of a viscous, incompressible and electrically conducting fluid. The governing partial differential equations are obtained by coupling the incompressible Navier-Stokes equations with Maxwell's equations and arise in several engineering applications such as, for example, liquid metals in magnetic pumps or aluminum electrolysis, see, e.g., [20]. In the stationary case, the resulting multi-field problem is of the form: find the velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the hydrodynamic pressure $p = p(\mathbf{x})$, and the magnetic field $\mathbf{b} = \mathbf{b}(\mathbf{x})$ that satisfy

$$\begin{aligned} - R_e^{-1}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\,\mathbf{b}\times\mathbf{curl}\,\mathbf{b} &= \mathbf{f} && \text{in } \Omega, \\ R_m^{-1}\,S\;\mathbf{curl}(\mathbf{curl}\,\mathbf{b}) - S\;\mathbf{curl}(\mathbf{u}\times\mathbf{b}) &= \mathbf{g} && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\ \operatorname{div}\mathbf{b} &= 0 && \text{in } \Omega, \end{aligned}$$

supplemented with suitable boundary conditions on $\partial\Omega$. Here, Ω is a bounded domain in $\mathbb{R}^3$, and the functions $\mathbf{f}$ and $\mathbf{g}$ are given source terms, with $\mathbf{g}$ being solenoidal. Furthermore, R_e is the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, and S the coupling number. These numbers are defined by

$$R_e = \frac{\varrho U_0 L}{\eta}, \qquad R_m = \mu\sigma U_0 L, \qquad S = \frac{B_0^2}{\mu\varrho U_0^2},$$

with B_0 and U_0 denoting the characteristic values of the magnetic field and the velocity, respectively. The parameter L is the characteristic length scale of the problem. The constants ϱ and η represent the density and the viscosity of the fluid, and μ and σ are the magnetic permeability and the electric conductivity, respectively. In industrial applications, one typically has $R_e \approx 10^2 - 10^5$, $R_m \approx 10^{-1}$ and $S \approx 1$.

Over the last few years, several finite element methods (FEM) to numerically solve the incompressible MHD equations and linearizations thereof have been proposed that are based on nodal (i.e., $H^1(\Omega)$-conforming) finite elements for the magnetic field $\mathbf{b}$, combined with standard discretizations of the hydrodynamic unknowns $\mathbf{u}$ and p. We mention here [1,13,16–18] and the references cited therein. However, it has been known for some time that in non–convex polyhedra of engineering practice the magnetic field components may have regularity below $H^1(\Omega)$ and that nodal FEM discretizations of the magnetic operator, albeit stable, can converge quasi-optimally to a magnetic field that misses certain singular solution components induced by reentrant vertices or edges (for more details, see, e.g., [7] and the references cited therein). Consequently, in non–convex domains, setting the magnetic components of the incompressible MHD equations in $H^1(\Omega)$ leads to a well-posed problem where the magnetic field lacks certain singular (but physical) solution components.

A possible way to overcome these difficulties was recently proposed in [24,25] by the use of Nédélec's elements for the magnetic field $\mathbf{b}$ and by the introduction of an additional Lagrange multiplier related to the constraint $\operatorname{div} \mathbf{b} = 0$.

In this paper, we propose a new mixed finite element approximation for incompressible MHD problems. Our method is also based on nodal elements for the magnetic field $\mathbf{b}$, and employs standard inf-sup stable elements for the unknowns $\mathbf{u}$ and p. However, as opposed to the approaches mentioned above, we modify the magnetic bilinear form using the weighted regularization technique recently developed by Costabel and Dauge in [7]. This allows us to account for the possible low regularity of the magnetic field in non-convex domains. We first discuss the well-posedness of this approach and show the existence and uniqueness of weak solutions for small data. We then carry out an error analysis for the proposed finite element method and show that it leads to quasi-optimal error bounds in the mesh-size. Finally, we show the convergence of the approximate solutions in non-convex domains where the components of the magnetic fields may have regularity below $H^1(\Omega)$. Our theoretical results are confirmed in a series of numerical experiments for a linear Oseen-type MHD problem in two dimensions.

The outline of the paper is as follows. In Section 2, we introduce a weighted regularization approach for incompressible MHD problems and show the well-posedness of the underlying weak formulation. Our finite element approximation is proposed and analyzed in Section 3. A series of numerical results for a two-dimensional MHD problem is presented in Section 4. We end our presentation with concluding remarks in Section 5.

Throughout the paper, we use the following notation: For a Lipschitz domain $D \subset \mathbb{R}^n$, $n = 2, 3$, we denote by $L^p(D)$, $1 \leq p \leq \infty$, the Lebesgue space of p-integrable functions, endowed with the norm $\|\cdot\|_{L^p(D)}$. We write $L^p_{\mathrm{loc}}(\Omega)$ to denote the space of functions that are locally p-integrable. We further make use of the subspace $L^2_0(D)$ of $L^2(D)$ defined by

$$L^2_0(D) = \{q \in L^2(D) \mid \int_D q \, d\mathbf{x} = 0\}.$$

For $s \geq 0$, we denote by $H^s(D)$ the standard L^2-based Sobolev space of order s and write $\|\cdot\|_{H^s(D)}$ for its norm. The closure of $\mathcal{D}(\Omega)$ (smooth functions with compact support) in $H^s(D)$ is denoted by $H^s_0(D)$. We write $H^{-s}(D)$ for the dual space of $H^s_0(D)$, equipped with the dual norm $\|\cdot\|_{H^{-s}(D)}$. For a generic function space $X(D)$ we write $X(D)^n$, $n = 2, 3$, to denote vector fields whose components belong to $X(D)$. Without further specification, these spaces are equipped with the usual product norms which we simply denote in the same way as the norms in $X(D)$.

2 Weighted Regularization of Incompressible MHD Problems

In this section, we introduce the governing equations of stationary incompressible magneto-hydrodynamics, derive a weak formulation using the weighted regularization technique of [7], and establish the existence and uniqueness of weak solutions for small data.

2.1 Incompressible MHD Equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedron. We assume throughout that Ω is simply-connected and that its boundary $\partial\Omega$ is connected. We consider stationary incompressible MHD problems of the following form: Given forcing terms $\mathbf{f}$ and $\mathbf{g}$ in $L^2(\Omega)^3$, find the velocity field $\mathbf{u} = (u_1, u_2, u_3)$, the magnetic field $\mathbf{b} = (b_1, b_2, b_3)$ and the pressure p such that

$$\begin{aligned}
-R_e^{-1}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\,\mathbf{b}\times\mathbf{curl}\,\mathbf{b} &= \mathbf{f} && \text{in } \Omega, && (2.1)\\
R_m^{-1}\,S\ \mathbf{curl}(\mathbf{curl}\,\mathbf{b}) - S\,\mathbf{curl}(\mathbf{u}\times\mathbf{b}) &= \mathbf{g} && \text{in } \Omega, && (2.2)\\
\operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, && (2.3)\\
\operatorname{div}\mathbf{b} &= 0 && \text{in } \Omega, && (2.4)\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, && (2.5)\\
\mathbf{n}\times\mathbf{b} &= \mathbf{0} && \text{on } \partial\Omega. && (2.6)
\end{aligned}$$

Here, $\mathbf{n}$ is the outward normal unit vector on $\partial\Omega$. For simplicity, we have imposed no-slip boundary conditions on the velocity field $\mathbf{u}$ and perfectly insulating magnetic boundary conditions on $\mathbf{b}$. We comment on inhomogeneous boundary conditions in Remark 2.19 below.

By taking the divergence of equation (2.2), we see that the datum $\mathbf{g}$ has to be solenoidal. Thus, we assume throughout that $\mathbf{g} \in H(\mathrm{div};\Omega)$ and

$$\operatorname{div}\mathbf{g} = 0 \quad \text{in } \Omega, \qquad \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\,ds \;=\; 0. \tag{2.7}$$

Here, $H(\mathrm{div};\Omega) \;=\; \{\mathbf{g} \in L^2(\Omega)^3 \mid \operatorname{div}\mathbf{g} \in L^2(\Omega)\}$, endowed with the norm

$$\|\mathbf{g}\|_{\mathrm{div}}^2 \;=\; \|\mathbf{g}\|_{L^2(\Omega)}^2 \;+\; \|\operatorname{div}\mathbf{g}\|_{L^2(\Omega)}^2.$$

Due to [14, Theorem I.3.4], there is a stream function $\Phi \in H^1(\Omega)^3$ such that $\mathbf{g} = \mathbf{curl}\,\Phi$.

Remark 2.1 *We also consider the two-dimensional analogue of the MHD problem (2.1)–(2.6). However, in two dimensions, the definition of the curl-operators requires some care. For vector fields* $\mathbf{w} = (w_1, w_2)$ *and* $\mathbf{r} = (r_1, r_2)$*,* *we define the vector product* $\mathbf{w} \times \mathbf{r} = w_1 r_2 - w_2 r_1$ *and the scalar-valued curl-operator* $\operatorname{curl} \mathbf{w} = \partial_{x_1} w_2 - \partial_{x_2} w_1$. *Furthermore, for a scalar function* c, *we set* $\mathbf{w} \times c = c(\, w_2\,, -w_1\,)$. *The vector-valued curl-operator is given by* $\mathbf{curl}\, c = (\,\partial_{x_2} c\,, -\partial_{x_1} c\,)$. *The two-dimensional analogue of (2.1)–(2.6) then reads as follows: Find the velocity field* $\mathbf{u} = (u_1, u_2)$, *the magnetic field* $\mathbf{b} = (b_1, b_2)$ *and the pressure* p *such that*

$$
\begin{aligned}
- R_e^{-1}\, \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + S\, \mathbf{b} \times \operatorname{curl} \mathbf{b} &= \mathbf{f} && \textit{in } \Omega, && (2.8)\\
R_m^{-1}\, S\, \mathbf{curl}(\operatorname{curl} \mathbf{b}) - S\, \mathbf{curl}(\mathbf{u} \times \mathbf{b}) &= \mathbf{g} && \textit{in } \Omega, && (2.9)\\
\operatorname{div} \mathbf{u} &= 0 && \textit{in } \Omega, && (2.10)\\
\operatorname{div} \mathbf{b} &= 0 && \textit{in } \Omega, && (2.11)\\
\mathbf{u} &= \mathbf{0} && \textit{on } \partial\Omega, && (2.12)\\
\mathbf{n} \times \mathbf{b} = \mathbf{b} \cdot \mathbf{t} &= 0 && \textit{on } \partial\Omega. && (2.13)
\end{aligned}
$$

Here, we denote by $\mathbf{t}$ *the counterclockwise oriented unit tangent vector on* $\partial\Omega$.

Note that, by identifying the two-dimensional vector field $\mathbf{b} = (b_1, b_2)$ *with its extension* $\widetilde{\mathbf{b}} = (b_1, b_2, 0)$ *in* $\mathbb{R}^3$, *it is easy to see that* $\mathbf{curl} \operatorname{curl} \mathbf{b} = \mathbf{curl}\, \mathbf{curl}\, \widetilde{\mathbf{b}}$. *Similarly,* $\mathbf{curl}(\mathbf{u} \times \mathbf{b}) = \mathbf{curl}(\widetilde{\mathbf{u}} \times \widetilde{\mathbf{b}})$ *and* $\mathbf{b} \times \operatorname{curl} \mathbf{b} = \widetilde{\mathbf{b}} \times \mathbf{curl}\, \widetilde{\mathbf{b}}$.

2.2 Weighted Spaces

To derive a weak formulation for (2.1)-(2.6) based on weighted regularization, we need to introduce the weighted Sobolev spaces from [7].

To this end, we denote by $\mathcal{C}$ the set of all corners of the polyhedron Ω. For $\mathbf{c} \in \mathcal{C}$, we set $r_{\mathbf{c}}(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{c})$. At each corner $\mathbf{c}$ there is a ball $B(\mathbf{c}, R_{\mathbf{c}})$ of radius $R_{\mathbf{c}}$ such that $\mathcal{V}_{\mathbf{c}} = \Omega \cap B(\mathbf{c}, R_{\mathbf{c}})$ is a cone which, in local spherical coordinates, is of the form $\mathcal{V}_{\mathbf{c}} = \{(r_{\mathbf{c}}, \theta_{\mathbf{c}}) \,|\, 0 < r_{\mathbf{c}} < R_{\mathbf{c}},\ \theta_{\mathbf{c}} \in G_{\mathbf{c}}\}$. Here, we use the function $r_{\mathbf{c}}$ as the radial coordinate while $\theta_{\mathbf{c}}$ is the angular coordinate with values in $G_{\mathbf{c}}$, a spherical polygonal domain in the unit sphere $\mathbb{S}^2$.

Furthermore, let $\mathcal{E}$ denote the set of all (open) edges of Ω. Then, for each point $\mathbf{x}$ of an edge e, there is a ball $B(\mathbf{x}, R_{\mathbf{x}})$ of radius $R_{\mathbf{x}}$ such that $\mathcal{V}_e(\mathbf{x}) = \Omega \cap B(\mathbf{x}, R_{\mathbf{x}})$ is diffeomorphic to a wedge $\Gamma_e(\mathbf{x}) \times \mathbb{R}$ with $\Gamma_e(\mathbf{x})$ being a plane sector with opening angle $\omega_e \in (0, 2\pi)$. This angle is intrinsic and is called the opening angle of Ω at the edge e. We set $r_e(\mathbf{y}) = \operatorname{dist}(\mathbf{y}, \overline{e})$; r_e is equivalent to the radial coordinate in $\Gamma_e(\mathbf{x})$. Let $\mathbf{c}$ and $\mathbf{c}'$ be the two endpoints of an edge e. Then we define ϱ_e by $r_e(\mathbf{x}) = r_{\mathbf{c}}(\mathbf{x}) r_{\mathbf{c}'}(\mathbf{x}) \varrho_e(\mathbf{x})$. Note that ϱ_e is equivalent to $r_e / r_{\mathbf{c}}$ in $\mathcal{V}_{\mathbf{c}}$, to $r_e / r_{\mathbf{c}'}$ in $\mathcal{V}_{\mathbf{c}'}$, and to r_e outside of $\mathcal{V}_{\mathbf{c}} \cup \mathcal{V}_{\mathbf{c}'}$.

Next, we introduce weight vectors $\underline{\gamma}$ which are collections of real numbers $\{\gamma_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}} \cup \{\gamma_e\}_{e \in \mathcal{E}}$. We set $|\underline{\gamma}| = \max\{\{\gamma_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}}, \{\gamma_e\}_{e \in \mathcal{E}}\}$. For two weight vectors $\underline{\beta}$ and $\underline{\gamma}$, we use the notation $\underline{\beta} \leq \underline{\gamma}$ to mean that $\beta_{\mathbf{c}} \leq \gamma_{\mathbf{c}}$ and $\beta_e \leq \gamma_e$

for all $\mathbf{c} \in \mathcal{C}$ and $e \in \mathcal{E}$. Similarly, $\underline{\beta} \pm \underline{\gamma}$ is the weight vector given by the components $\beta_{\mathbf{c}} \pm \gamma_{\mathbf{c}}$ and $\beta_e \pm \gamma_e$. For constants κ_1 and κ_2, we write $\kappa_1 \leq \underline{\gamma} \leq \kappa_2$ to mean that $\kappa_1 \leq \gamma_{\mathbf{c}} \leq \kappa_2$ and $\kappa_1 \leq \gamma_e \leq \kappa_2$ for all $\mathbf{c} \in \mathcal{C}$ and $e \in \mathcal{E}$.

With a weight vector $\underline{\gamma}$ we associate the weight function

$$\omega_{\underline{\gamma}}(\mathbf{x}) = \left(\prod_{\mathbf{c} \in \mathcal{C}} r_{\mathbf{c}}^{\gamma_{\mathbf{c}}} \right) \left(\prod_{e \in \mathcal{E}} r_e^{\gamma_e} \right). \tag{2.14}$$

Moreover, we need the distance function $d(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathcal{C} \cup \mathcal{E})$. For $s \in \mathbb{N}_0$, we define the weighted space

$$V_{\underline{\gamma}}^s(\Omega) = \{ v \in L^2_{\text{loc}}(\Omega) : \omega_{\underline{\gamma}} d^{|\alpha|-s} \partial_\alpha v \in L^2(\Omega),\ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq s \}, \tag{2.15}$$

equipped with the norm

$$\|\varphi\|_{s,\underline{\gamma}}^2 = \sum_{|\alpha| \leq s} \|\omega_{\underline{\gamma}} d^{|\alpha|-s} \partial_\alpha v\|_{L^2(\Omega)}^2.$$

A special role is played by the weight vector $\underline{\delta}^{\text{dir}}$ given by

$$\delta_{\mathbf{c}}^{\text{dir}} = \frac{1}{2} - \lambda_{\mathbf{c},1}^{\text{dir}}, \qquad \mathbf{c} \in \mathcal{C}, \tag{2.16}$$

$$\delta_e^{\text{dir}} = 1 - \frac{\pi}{\omega_e}, \qquad e \in \mathcal{E}. \tag{2.17}$$

Here, following [7, Section 4.3], we have set

$$\lambda_{\mathbf{c},1}^{\text{dir}} = -\frac{1}{2} + \sqrt{\mu_1^{\text{dir}} + \frac{1}{4}},$$

with $\mu_1^{\text{dir}} > 0$ denoting the smallest Dirichlet eigenvalue of the Laplace-Beltrami operator in the cone $G_{\mathbf{c}}$. Note that we always have $\lambda_{\mathbf{c},1}^{\text{dir}} > 0$. Thus, there holds

$$|\underline{\delta}^{\text{dir}}| < \frac{1}{2} - \delta_\star^{\text{dir}}, \tag{2.18}$$

for a parameter $\delta_\star^{\text{dir}} > 0$. We remark that $\delta_\star^{\text{dir}}$ approaches zero if one of the opening angles of Ω approaches 2π.

The following result holds; see [7, Theorem 4.1].

Theorem 2.2 *Let $\underline{\delta}^{\text{dir}}$ be the weight vector in (2.16)–(2.17). Then for any weight vector $\underline{\gamma}$ with*

$$\underline{\delta}^{\text{dir}} < \underline{\gamma} \qquad \textit{and} \qquad 0 \leq \underline{\gamma} \leq 1, \tag{2.19}$$

the Laplacian with Dirichlet boundary conditions is an isomorphism from $V_{\underline{\gamma}}^2(\Omega) \cap H_0^1(\Omega)$ onto $V_{\underline{\gamma}}^0(\Omega)$.

Remark 2.3 *In a Lipschitz polygon $\Omega \subset \mathbb{R}^2$ the definitions of $\omega_{\underline{\gamma}}$ and the corresponding weighted spaces are easier as the edges do not need to be taken into account. To define $\omega_{\underline{\gamma}}$, let $\mathcal{C}$ be the set of all corners of Ω. The opening angle at the corner $\mathbf{c} \in \mathcal{C}$ is denoted by $\omega_{\mathbf{c}}$, with $\omega_{\mathbf{c}} \in (0, 2\pi)$. Then, for a weight vector $\underline{\gamma} = \{\gamma_{\mathbf{c}}\}_{\mathbf{c}\in\mathcal{C}}$, the weight function $\omega_{\underline{\gamma}}$ is given by*

$$\omega_{\underline{\gamma}}(\mathbf{x}) \;=\; \prod_{\mathbf{c}\in\mathcal{C}} r_{\mathbf{c}}(\mathbf{x})^{\gamma_{\mathbf{c}}}, \qquad r_{\mathbf{c}}(\mathbf{x}) \;=\; \mathrm{dist}(\mathbf{x}, \mathbf{c}). \tag{2.20}$$

Introducing the distance function $d(\mathbf{x})$ by $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \mathcal{C})$, the spaces $V^s_{\underline{\gamma}}(\Omega)$ are defined as in (2.15). In two dimensions, the critical weight vector $\underline{\delta}^{\mathrm{dir}}$ is

$$\delta^{\mathrm{dir}}_{\mathbf{c}} = 1 - \frac{\pi}{\omega_{\mathbf{c}}}, \qquad \mathbf{c} \in \mathcal{C}. \tag{2.21}$$

The result of Theorem 2.2 then holds true, with $\underline{\delta}^{\mathrm{dir}}$ given in (2.21).

Next, we introduce the Sobolev spaces

$$H(\mathrm{curl};\Omega) \;=\; \{\mathbf{b} \in L^2(\Omega)^3 \mid \mathbf{curl}\,\mathbf{b} \in L^2(\Omega)^3\},$$

as well as

$$H_0(\mathrm{curl};\Omega) \;=\; \{\mathbf{b} \in H(\mathrm{curl};\Omega) \mid \mathbf{n} \times \mathbf{b} = \mathbf{0} \text{ on } \partial\Omega\},$$

and endow them with the norm

$$\|\mathbf{b}\|^2_{\mathrm{curl}} \;=\; \|\mathbf{b}\|^2_{L^2(\Omega)} \;+\; \|\,\mathbf{curl}\,\mathbf{b}\|^2_{L^2(\Omega)}.$$

For a weight vector $\underline{\gamma}$, we further define the space

$$X_{\underline{\gamma}}(\Omega) = \;\{\mathbf{b} \in H_0(\mathrm{curl};\Omega) \mid \mathrm{div}\,\mathbf{b} \in V^0_{\underline{\gamma}}(\Omega)\}, \tag{2.22}$$

and equip it with the norm

$$\|\mathbf{b}\|^2_{X_{\underline{\gamma}}} = \;\|\,\mathbf{curl}\,\mathbf{b}\|^2_{L^2(\Omega)} \;+\; \|\,\mathrm{div}\,\mathbf{b}\|^2_{0,\underline{\gamma}} + \|\mathbf{b}\|^2_{L^2(\Omega)}. \tag{2.23}$$

Our finite element discretization will be based on the subspace $H_{\underline{\gamma}}(\Omega) \subset X_{\underline{\gamma}}(\Omega)$ given by

$$H_{\underline{\gamma}}(\Omega) = \{\mathbf{b} \in H^1(\Omega)^3 \mid \mathbf{n} \times \mathbf{b} = \mathbf{0} \text{ on } \partial\Omega\}, \tag{2.24}$$

equipped with the norm $\|\cdot\|_{X_{\underline{\gamma}}}$.

The following result is crucial; see [7, Theorem 5.1].

Theorem 2.4 *Let $\underline{\gamma}$ be a weight vector satisfying (2.19). Then the space $H_{\underline{\gamma}}(\Omega)$ is dense in $X_{\underline{\gamma}}(\Omega)$.*

Remark 2.5 *The same result holds true in polygons* $\Omega \subset \mathbb{R}^2$*, with* $\omega_{\underline{\gamma}}$ *and* $\underline{\delta}^{\mathrm{dir}}$ *defined as in (2.20) and (2.21), respectively.*

Finally, let us show a Poincaré-type inequality that will be needed in our analysis. We set

$$|\mathbf{b}|^2_{X_{\underline{\gamma}}} = \|\,\mathbf{curl}\,\mathbf{b}\|^2_{L^2(\Omega)} + \|\operatorname{div}\mathbf{b}\|^2_{0,\underline{\gamma}},$$

and have the following result.

Proposition 2.6 *Let* $\underline{\gamma}$ *be a weight vector satisfying (2.19). There holds:*

(i) $|\cdot|_{X_{\underline{\gamma}}}$ *is a norm on* $X_{\underline{\gamma}}(\Omega)$.
(ii) There exists a constant $C > 0$ *only depending on* Ω *and* $\underline{\gamma}$ *such that*

$$|\mathbf{b}|_{X_{\underline{\gamma}}} \geq C\|\mathbf{b}\|_{X_{\underline{\gamma}}} \qquad \forall \mathbf{b} \in X_{\underline{\gamma}}(\Omega).$$

Proof: To prove the first assertion, it is sufficient to show that $|\mathbf{b}|_{X_{\underline{\gamma}}} = 0$ implies $\mathbf{b} = \mathbf{0}$. Indeed, if $|\mathbf{b}|_{X_{\underline{\gamma}}} = 0$, we conclude that $\mathbf{curl}\,\mathbf{b} = \mathbf{0}$ and $\omega_{\underline{\gamma}} \operatorname{div}\mathbf{b} = 0$. Since $\omega_{\underline{\gamma}} > 0$ in Ω, we also have $\operatorname{div}\mathbf{b} = 0$. By using that $\mathbf{n} \times \mathbf{b} = \mathbf{0}$ on $\partial\Omega$ and the Poincaré-type inequality from [12, Proposition 7.4], we obtain

$$\|\mathbf{b}\|_{L^2(\Omega)} \leq C\|\,\mathbf{curl}\,\mathbf{b}\|_{L^2(\Omega)},$$

with a constant $C > 0$ only depending on Ω. Thus, $\mathbf{b} = \mathbf{0}$, which shows the first assertion.

For the second assertion let $\varphi \in H^1_0(\Omega)$ be the solution of $\Delta\varphi = \operatorname{div}\mathbf{b}$ in Ω, $\varphi = 0$ on $\partial\Omega$. Since $\operatorname{div}\mathbf{b} \in V^0_{\underline{\gamma}}(\Omega)$ and the Laplacian is an isomorphism from $V^2_{\underline{\gamma}}(\Omega) \cap H^1_0(\Omega)$ onto $V^0_{\underline{\gamma}}(\Omega)$, see Theorem 2.2, we have $\varphi \in V^2_{\underline{\gamma}}(\Omega)$ and

$$\|\varphi\|_{2,\underline{\gamma}} \leq C\|\operatorname{div}\mathbf{b}\|_{0,\underline{\gamma}} \leq C|\mathbf{b}|_{X_{\underline{\gamma}}}, \tag{2.25}$$

for a constant $C > 0$ solely depending on Ω and $\underline{\gamma}$.

By setting $\mathbf{b}_0 = \mathbf{b} - \nabla\varphi$, we have $\operatorname{div}\mathbf{b}_0 = 0$ and $\mathbf{curl}\,\mathbf{b}_0 = \mathbf{curl}\,\mathbf{b}$ in Ω, as well as $\mathbf{n} \times \mathbf{b}_0 = \mathbf{0}$ on $\partial\Omega$. As before, the inequality in [12, Proposition 7.4] yields

$$\|\mathbf{b}_0\|_{L^2(\Omega)} \leq C\|\,\mathbf{curl}\,\mathbf{b}_0\|_{L^2(\Omega)} = C\|\,\mathbf{curl}\,\mathbf{b}\|_{L^2(\Omega)} \leq C|\mathbf{b}|_{X_{\underline{\gamma}}}, \tag{2.26}$$

for a constant $C > 0$ only depending on Ω.

Referring to (2.25) and (2.26) gives

$$\|\mathbf{b}\|_{L^2(\Omega)} \leq \|\mathbf{b}_0\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leq \|\mathbf{b}_0\|_{L^2(\Omega)} + C\|\varphi\|_{2,\underline{\gamma}} \leq C|\mathbf{b}|_{X_{\underline{\gamma}}}, \tag{2.27}$$

for any $\mathbf{b} \in X_{\underline{\gamma}}(\Omega)$, with a constant $C > 0$ solely depending on Ω and $\underline{\gamma}$. Equation (2.27) implies the second assertion. □

We point out that the results in Proposition 2.6 can be easily adapted to the two-dimensional case.

2.3 Weak Formulation

For a weight vector $0 \leq \underline{\gamma} \leq 1$, we define the following weak form for the MHD problem (2.1)–(2.6): Find $\mathbf{u} \in H_0^1(\Omega)^3$, $\mathbf{b} \in X_{\underline{\gamma}}(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} a_s(\mathbf{u},\mathbf{v}) + o_s(\mathbf{u};\mathbf{u},\mathbf{v}) + b_s(p,\mathbf{v}) + c_1(\mathbf{b};\mathbf{b},\mathbf{v}) &= (\mathbf{f},\mathbf{v}), \\ a_m(\mathbf{b},\mathbf{c}) - c_2(\mathbf{b};\mathbf{u},\mathbf{c}) &= (\mathbf{g},\mathbf{c}), \\ b_s(q,\mathbf{u}) &= 0 \end{aligned} \tag{2.28}$$

for all $\mathbf{v} \in H_0^1(\Omega)^3$, $\mathbf{c} \in X_{\underline{\gamma}}(\Omega)$ and $q \in L_0^2(\Omega)$. Here, we use the following forms:

$$a_s(\mathbf{u},\mathbf{v}) = R_e^{-1} \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x}, \tag{2.29}$$

$$a_m(\mathbf{b},\mathbf{c}) = R_m^{-1} S \int_\Omega \mathbf{curl}\,\mathbf{b} \cdot \mathbf{curl}\,\mathbf{c}\, d\mathbf{x} + D \int_\Omega \omega_{\underline{\gamma}}(\mathbf{x})^2 \operatorname{div}\mathbf{b} \operatorname{div}\mathbf{c}\, d\mathbf{x}, \tag{2.30}$$

$$b_s(q,\mathbf{v}) = -\int_\Omega \operatorname{div}\mathbf{v}\, q\, d\mathbf{x}, \tag{2.31}$$

$$o_s(\mathbf{w};\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \Big[(\mathbf{w}\cdot\nabla)\mathbf{u}\Big] \cdot \mathbf{v}\, d\mathbf{x} - \frac{1}{2}\int_\Omega \Big[(\mathbf{w}\cdot\nabla)\mathbf{v}\Big] \cdot \mathbf{u}\, d\mathbf{x}, \tag{2.32}$$

$$c_1(\mathbf{d};\mathbf{b},\mathbf{v}) = S \int_\Omega (\mathbf{d} \times \mathbf{curl}\,\mathbf{b}) \cdot \mathbf{v}\, d\mathbf{x}, \tag{2.33}$$

$$c_2(\mathbf{d};\mathbf{u},\mathbf{c}) = S \int_\Omega (\mathbf{u} \times \mathbf{d})\, \mathbf{curl}\,\mathbf{c}\, d\mathbf{x}. \tag{2.34}$$

Here, we have incorporated the regularization term $\int_\Omega \omega_{\underline{\gamma}}^2 \operatorname{div}\mathbf{b} \operatorname{div}\mathbf{c}\, d\mathbf{x}$ into the magnetic form a_m, following [7]. The parameter D is a positive constant that can be used to dimensionalize the regularization term and to balance it with the curl-curl term. Furthermore, we use the standard anti-symmetric trilinear form o_s for the discretization of the non-linear convection term in the Navier-Stokes operator; see, e.g., [27, Chapter II] for details. The trilinear forms c_1 and c_2 arise due to the coupling terms in (2.1) and (2.2); we show in Section 2.4 below that these forms are well-defined for suitable choices of $\underline{\gamma}$.

The regularization term ensures that the magnetic field is solenoidal.

Proposition 2.7 *Assume (2.19) and let* $(\mathbf{u}, \mathbf{b}, p)$ *be a solution of (2.28). Then we have* $\operatorname{div} \mathbf{b} = 0$.

Proof: Since $\operatorname{div} \mathbf{b} \in V^0_{\underline{\gamma}}(\Omega)$, the problem $\Delta\varphi = \operatorname{div} \mathbf{b}$ in Ω, $\varphi = 0$ on $\partial\Omega$, is well-posed and, as before, has a unique weak solution $\varphi \in V^2_{\underline{\gamma}}(\Omega) \cap H^1_0(\Omega)$. By construction, $\nabla\varphi \in X_{\underline{\gamma}}(\Omega)$. Choosing $\mathbf{c} = \nabla\varphi$ as a test function in the second equation of formulation (2.28) yields

$$
\begin{aligned}
\int_\Omega \mathbf{g} \cdot \mathbf{c}\, d\mathbf{x} &= D \int_\Omega \omega_{\underline{\gamma}}(\mathbf{x})^2 \operatorname{div} \mathbf{b} \operatorname{div} \nabla\varphi\, d\mathbf{x} \\
&= D \int_\Omega \omega_{\underline{\gamma}}(\mathbf{x})^2 \operatorname{div} \mathbf{b} \operatorname{div} \mathbf{b}\, d\mathbf{x} = D \| \operatorname{div} \mathbf{b} \|^2_{0,\underline{\gamma}}.
\end{aligned}
$$

Here, we have used that $\mathbf{curl}\, \mathbf{c} = \mathbf{0}$. Furthermore, by integration by parts,

$$
\int_\Omega \mathbf{g} \cdot \mathbf{c}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \nabla\varphi\, d\mathbf{x} = - \int_\Omega \varphi \operatorname{div} \mathbf{g}\, d\mathbf{x} = 0,
$$

since $\varphi = 0$ on $\partial\Omega$ and $\operatorname{div} \mathbf{g} = 0$, as assumed in (2.7). Thus, since $\omega_{\underline{\gamma}}(\mathbf{x}) > 0$ in Ω and $D > 0$, we have $\operatorname{div} \mathbf{b} = 0$. $\square$

For the purpose of our analysis, we rewrite the formulation (2.28) in the compact form: Find $(\mathbf{u}, \mathbf{b}, p) \in H^1_0(\Omega)^3 \times X_{\underline{\gamma}}(\Omega) \times L^2_0(\Omega)$ such that

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + O(\mathbf{u}, \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + B(p; \mathbf{v}, \mathbf{c}) &= (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{c}), \\
B(q; \mathbf{u}, \mathbf{b}) &= 0
\end{aligned}
\tag{2.35}
$$

for all $(\mathbf{v}, \mathbf{c}, q) \in H^1_0(\Omega)^3 \times X_{\underline{\gamma}}(\Omega) \times L^2_0(\Omega)$. Here, we use the forms

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) &= a_s(\mathbf{u}, \mathbf{v}) + a_m(\mathbf{b}, \mathbf{c}), \\
B(q; \mathbf{v}, \mathbf{c}) &= b_s(q, \mathbf{v}), \\
O(\mathbf{w}, \mathbf{d}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) &= o_s(\mathbf{w}; \mathbf{u}, \mathbf{v}) + c_1(\mathbf{d}; \mathbf{b}, \mathbf{v}) - c_2(\mathbf{d}; \mathbf{u}, \mathbf{c}).
\end{aligned}
$$

The adaptation of the forms in (2.29)–(2.34) and the weak formulation (2.28) to two-dimensional MHD problems of the form (2.8)–(2.13) is straightforward.

2.4 Well-Posedness

We show that the variational formulation (2.28) is well-posed and uniquely solvable for small data. We begin by establishing the continuity of the forms.

Lemma 2.8 *For any weight vector* $\underline{\gamma}$*, the forms* a_s*,* a_m*, and* b_s *satisfy the* *following continuity properties:*

$$
\begin{aligned}
|a_s(\mathbf{u},\mathbf{v})| &\leq R_e^{-1}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}, && \mathbf{u},\mathbf{v}\in H_0^1(\Omega)^3,\\
|a_m(\mathbf{b},\mathbf{c})| &\leq \max\{R_m^{-1}S, D\}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\mathbf{c}\|_{X_{\underline{\gamma}}}, && \mathbf{b},\mathbf{c}\in X_{\underline{\gamma}}(\Omega),\\
|b_s(q,\mathbf{v})| &\leq \sqrt{3}\|q\|_{L^2(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}, && q\in L_0^2(\Omega),\ \mathbf{v}\in H_0^1(\Omega)^3.
\end{aligned}
$$

Furthermore, there exists a constant C_o only depending on Ω such that

$$
|o_s(\mathbf{w};\mathbf{u},\mathbf{v})| \leq C_o\|\mathbf{w}\|_{H^1(\Omega)}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}, \qquad \mathbf{w},\mathbf{u},\mathbf{v}\in H_0^1(\Omega)^3.
$$

Proof: The continuity properties of the forms a_s, a_m and b_s follow straightforwardly using Cauchy-Schwarz inequalities and the fact that $\|\operatorname{div}\mathbf{v}\|_{L^2(\Omega)} \leq \sqrt{3}\|\mathbf{v}\|_{H^1(\Omega)}$. The continuity property for o_s follows from the continuous embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$ and Hölder's inequality; see, e.g., [14, Chapter IV].$\Box$

To show the continuity of the forms c_1 and c_2, we make use of the following embedding result, which is valid for suitable values of $\underline{\gamma}$ (see also Remark 2.11 below). For each corner $\mathbf{c}\in\mathcal{C}$, we write $\mathcal{E}_{\mathbf{c}}$ for the set of all edges that contain the corner $\mathbf{c}$.

Lemma 2.9 *Let the weight vector $\underline{\gamma}$ satisfy*

$$
0\leq\underline{\gamma}<1/2 \qquad \textit{and} \qquad \gamma_e\geq\gamma_{\mathbf{c}} \quad \forall e\in\mathcal{E}_{\mathbf{c}}. \tag{2.36}
$$

Then we have $H^{|\underline{\gamma}|}(\Omega)\subset V^0_{-\underline{\gamma}}(\Omega)$ and $\|v\|_{0,-\underline{\gamma}}\leq C\|v\|_{H^{|\underline{\gamma}|}(\Omega)}$ for a constant only depending on Ω and $\underline{\gamma}$.

Proof: As described in [7, Section 4.1], we can decompose Ω into

$$
\Omega=\mathcal{V}^0\cup\Big(\bigcup_{e\in\mathcal{E}}\mathcal{V}_e^0\Big)\cup\Big(\bigcup_{\mathbf{c}\in\mathcal{C}}\mathcal{V}_{\mathbf{c}}^0\cup\Big(\bigcup_{e\in\mathcal{E}_{\mathbf{c}}}\mathcal{V}_e(\mathbf{c})\Big)\Big). \tag{2.37}
$$

Here, $\mathcal{V}^0$ is a subregion of Ω away from corners and edges, and $\mathcal{V}_e^0$ is a subregion of Ω such that $\overline{\mathcal{V}}_e^0$ does not contain any corners or parts of any other edge than e. The subregion $\mathcal{V}_{\mathbf{c}}^0$ is such that $\mathbf{c}\in\overline{\mathcal{V}}_{\mathbf{c}}^0$ and $e\cap\overline{\mathcal{V}}_{\mathbf{c}}^0=\emptyset$ for any edge $e\in\mathcal{E}$. Finally, for any edge $e\in\mathcal{E}_{\mathbf{c}}$ the subregion $\mathcal{V}_e(\mathbf{c})$ is such that $\overline{\mathcal{V}}_e(\mathbf{c})$ only contains $\mathbf{c}$ and parts of e. Note that this decomposition is not unique and that the different subregions may be overlapping.

As in [7, Equation (4.9)], we define the distance functions

$$
d_{\mathcal{C}}(\mathbf{x})=\operatorname{dist}(\mathbf{x},\mathcal{C}), \qquad d_{\mathcal{E}}(\mathbf{x})=\operatorname{dist}(\mathbf{x},\mathcal{E}).
$$

Furthermore, we can choose exponents $\gamma_{\mathcal{C}}$ and $\gamma_{\mathcal{E}}$ such that $|\underline{\gamma}| \geq \gamma_{\mathcal{C}} \geq 0$, $|\underline{\gamma}| \geq \gamma_{\mathcal{E}} \geq 0$ and

$$\begin{aligned} \gamma_{\mathcal{C}}(\mathbf{x}) &= \gamma_{\mathbf{c}}, \qquad \mathbf{x} \in \mathcal{V}_{\mathbf{c}}, \\ \gamma_{\mathcal{E}}(\mathbf{x}) &= \gamma_{e}, \qquad \mathbf{x} \in \mathcal{V}_e^0 \cup \Big(\bigcup_{\mathbf{c} \in \overline{e}} \mathcal{V}_e(\mathbf{c}) \Big). \end{aligned}$$

The weight $w_{-\underline{\gamma}}$ is then equivalent to

$$w_{-\underline{\gamma}} \approx d_{\mathcal{C}}^{-\gamma_{\mathcal{C}}+\gamma_{\mathcal{E}}} d_{\mathcal{E}}^{-\gamma_{\mathcal{E}}}. \tag{2.38}$$

Let $\underline{\gamma}$ be a weight vector satisfying (2.36) and let v be in $H^{|\underline{\gamma}|}(\Omega)$. We may assume that $1/2 > |\underline{\gamma}| > 0$, the case $|\underline{\gamma}| = 0$ being trivial. We obtain

$$\begin{aligned} \int_\Omega \omega_{-\underline{\gamma}}^2 v^2 \, d\mathbf{x} &\leq C \int_\Omega d_{\mathcal{C}}^{-2\gamma_{\mathcal{C}}+2\gamma_{\mathcal{E}}} d_{\mathcal{E}}^{-2\gamma_{\mathcal{E}}} \, v^2 \, d\mathbf{x} \\ &\leq C \int_\Omega d_{\mathcal{E}}^{-2\gamma_{\mathcal{E}}} \, v^2 \, d\mathbf{x} \\ &\leq C \int_\Omega \operatorname{dist}(\mathbf{x}, \partial\Omega)^{-2\gamma_{\mathcal{E}}} \, v^2 \, d\mathbf{x} \leq C \int_\Omega \operatorname{dist}(\mathbf{x}, \partial\Omega)^{-2|\underline{\gamma}|} \, v^2 \, d\mathbf{x}. \end{aligned}$$

Here, we have used that $\gamma_e \geq \gamma_{\mathbf{c}}$ for all $\mathbf{c} \in \mathcal{E}_{\mathbf{c}}$, $\operatorname{dist}(\mathbf{x}, \partial\Omega) \leq d_{\mathcal{E}}$ and $\gamma_{\mathcal{E}} \leq |\underline{\gamma}|$. Since $0 < |\underline{\gamma}| < \frac{1}{2}$, the continuous embedding in [15, Theorem 1.4.4.3] ensures that

$$\int_\Omega \operatorname{dist}(\mathbf{x}, \partial\Omega)^{-2|\underline{\gamma}|} \, v^2 \, d\mathbf{x} \leq C \|v\|^2_{H^{|\underline{\gamma}|}(\Omega)},$$

which completes the proof. □

Lemma 2.10 *Let the weight vector* $\underline{\gamma}$ *satisfy (2.36). Then there is a constant* C_c *depending on* Ω *and* $\underline{\gamma}$ *such that*

$$\begin{aligned} |c_1(\mathbf{d}; \mathbf{b}, \mathbf{v})| &\leq S C_c \|\mathbf{d}\|_{X_{\underline{\gamma}}} \|\mathbf{b}\|_{X_{\underline{\gamma}}} \|\mathbf{v}\|_{H^1(\Omega)}, \qquad \mathbf{d}, \mathbf{b} \in X_{\underline{\gamma}}(\Omega), \ \mathbf{v} \in H_0^1(\Omega)^3, \\ |c_2(\mathbf{d}; \mathbf{u}, \mathbf{c})| &\leq S C_c \|\mathbf{d}\|_{X_{\underline{\gamma}}} \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{c}\|_{X_{\underline{\gamma}}}, \qquad \mathbf{d}, \mathbf{c} \in X_{\underline{\gamma}}(\Omega), \ \mathbf{u} \in H_0^1(\Omega)^3. \end{aligned}$$

Proof: We start by noting that, from [7, Theorem 2.2] any field $\mathbf{d} \in X_{\underline{\gamma}}(\Omega)$ can be decomposed as

$$\mathbf{d} = \mathbf{d}_0 + \nabla\varphi,$$

with $\mathbf{d}_0 \in H_{\underline{\gamma}}(\Omega)$ and $\varphi \in V_{\underline{\gamma}}^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, there exists a constant $C > 0$ only depending on Ω and $\underline{\gamma}$, such that

$$\|\mathbf{d}_0\|_{H^1(\Omega)} + \|\Delta\varphi\|_{0,\underline{\gamma}} \leq C \|\mathbf{d}\|_{X_{\underline{\gamma}}}. \tag{2.39}$$

Let us now establish the assertion for c_1; the proof for c_2 is completely analogous. We write $c_1(\mathbf{d}; \mathbf{b}, \mathbf{v})$ as

$$c_1(\mathbf{d}; \mathbf{b}, \mathbf{v}) = c_1(\mathbf{d}_0; \mathbf{b}, \mathbf{v}) + c_1(\nabla\varphi; \mathbf{b}, \mathbf{v}). \tag{2.40}$$

We first bound $c_1(\mathbf{d}_0; \mathbf{b}, \mathbf{v})$. By Hölder's inequality, the continuous embedding of $H^1(\Omega)$ into $L^4(\Omega)$, and the estimate (2.39), we have

$$
\begin{aligned}
|c_1(\mathbf{d}_0; \mathbf{b}, \mathbf{v})| \leq S| \int_\Omega (\mathbf{d}_0 \times \mathbf{curl}\,\mathbf{b}) \cdot \mathbf{v}\, d\mathbf{x}| &\leq S\|\mathbf{d}_0\|_{L^4(\Omega)} \|\mathbf{curl}\,\mathbf{b}\|_{L^2(\Omega)} \|\mathbf{v}\|_{L^4(\Omega)} \\
&\leq CS\|\mathbf{d}_0\|_{H^1(\Omega)} \|\mathbf{b}\|_{X_{\underline{\gamma}}} \|\mathbf{v}\|_{H^1(\Omega)} \leq CS\|\mathbf{d}\|_{X_{\underline{\gamma}}} \|\mathbf{b}\|_{X_{\underline{\gamma}}} \|\mathbf{v}\|_{H^1(\Omega)},
\end{aligned}
\tag{2.41}
$$

with a constant $C > 0$ only depending on Ω and $\underline{\gamma}$.

Next, we bound $c_1(\nabla\varphi; \mathbf{b}, \mathbf{v})$. To do so, we first note that, since $|\underline{\gamma}| < 1/2$, we have $H^{|\underline{\gamma}|}(\Omega) = H_0^{|\underline{\gamma}|}(\Omega)$; see [15, Corollary 1.4.4.5]. Thus, from Lemma 2.9 and duality we have that

$$
V_{\underline{\gamma}}^0(\Omega) \subset H^{-|\underline{\gamma}|}(\Omega) \qquad \text{and} \qquad \|v\|_{H^{-|\underline{\gamma}|}(\Omega)} \leq C\|v\|_{0,\underline{\gamma}} \quad \forall v \in V_{\underline{\gamma}}^0(\Omega),
$$

with a constant only depending on Ω and $\underline{\gamma}$. Hence, since $\Delta\varphi \in V_{\underline{\gamma}}^0(\Omega)$,

$$
\Delta\varphi \in H^{-|\underline{\gamma}|}(\Omega) \qquad \text{and} \qquad \|\Delta\varphi\|_{H^{-|\underline{\gamma}|}(\Omega)} \leq C\|\Delta\varphi\|_{0,\underline{\gamma}}.
$$

In view of $|\underline{\gamma}| < 1/2$, the elliptic shift theorem for polyhedral domains implies that

$$
\nabla\varphi \in H^{1/2+\varepsilon}(\Omega) \qquad \text{and} \qquad \|\nabla\varphi\|_{H^{1/2+\varepsilon}(\Omega)} \leq C\|\Delta\varphi\|_{H^{-|\underline{\gamma}|}(\Omega)} \leq C\|\Delta\varphi\|_{0,\underline{\gamma}},
$$

for a parameter $\varepsilon > 0$; see [8]. Further, we have that $H^{1/2+\varepsilon}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for an exponent $q > 3$; see [14, Theorem I.1.3 and Definition I.1.2]. We can then find a second exponent $p < 6$ such that $1/2 = p^{-1} + q^{-1}$. Using Hölder's inequality with these exponents and the continuous embedding of $H^1(\Omega)$ into $L^p(\Omega)$, we obtain

$$
\begin{aligned}
|c_1(\nabla\varphi; \mathbf{b}, \mathbf{v})| &\leq S| \int_\Omega (\nabla\varphi \times \mathbf{curl}\,\mathbf{b}) \cdot \mathbf{v}\, d\mathbf{x}| \\
&\leq S\|\nabla\varphi\|_{L^q(\Omega)} \|\mathbf{curl}\,\mathbf{b}\|_{L^2(\Omega)} \|\mathbf{v}\|_{L^p(\Omega)} \\
&\leq CS\|\nabla\varphi\|_{H^{1/2+\varepsilon}(\Omega)} \|\mathbf{b}\|_{X_{\underline{\gamma}}} \|\mathbf{v}\|_{H^1(\Omega)} \\
&\leq CS\|\Delta\varphi\|_{0,\underline{\gamma}} \|\mathbf{b}\|_{X_{\underline{\gamma}}} \|\mathbf{v}\|_{H^1(\Omega)}.
\end{aligned}
$$

This, together with (2.39)–(2.41), proves the assertion for c_1. □

Remark 2.11 *The assumptions in (2.36) restrict the choice of $\underline{\gamma}$ to quite a small range. In view of (2.18), this is particularly evident when we simultaneously seek to fulfill (2.36) and (2.19), that is,*

$$
0 \leq \underline{\gamma}, \qquad \underline{\delta}^{\mathrm{dir}} < \underline{\gamma} < 1/2 \qquad \textit{and} \qquad \gamma_e \geq \gamma_{\mathbf{c}} \quad \forall e \in \mathcal{E}_{\mathbf{c}};
$$

see also Theorem 2.17 below. The upper bound $1/2$ shows up because of the use of the embedding result in Lemma 2.9. Whether or not this upper bound can *be improved with a different analysis technique remains an open question. We also point out that for linear MHD problems no restrictions on $\underline{\gamma}$ are necessary.*

Next, let us show that for the two-dimensional analogues of the forms c_1 and c_2 in (2.33)–(2.34) it is possible to obtain a result with less restrictions on $\underline{\gamma}$.

Lemma 2.12 *Let Ω be a polygon in $\mathbb{R}^2$ and let the two-dimensional weight vector $\underline{\gamma}$ in (2.20) satisfy $0 \leq \underline{\gamma} < 1$. Then there is a constant C_c depending on Ω and $\underline{\gamma}$ such that the two-dimensional analogues of the forms c_1 and c_2 satisfy*

$$
\begin{aligned}
|c_1(\mathbf{d};\mathbf{b},\mathbf{v})| &\leq SC_c\|\mathbf{d}\|_{X_{\underline{\gamma}}}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\mathbf{v}\|_{H^1(\Omega)}, \qquad && \mathbf{d},\mathbf{b} \in X_{\underline{\gamma}}(\Omega),\ \mathbf{v} \in H_0^1(\Omega)^2,\\
|c_2(\mathbf{d};\mathbf{u},\mathbf{c})| &\leq SC_c\|\mathbf{d}\|_{X_{\underline{\gamma}}}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{c}\|_{X_{\underline{\gamma}}}, \qquad && \mathbf{d},\mathbf{c} \in X_{\underline{\gamma}}(\Omega),\ \mathbf{u} \in H_0^1(\Omega)^2.
\end{aligned}
$$

Proof: As in the proof of Lemma 2.10, we can write $\mathbf{d} = \mathbf{d}_0 + \nabla\varphi$, with $\mathbf{d}_0 \in H_{\underline{\gamma}}(\Omega)$ and $\varphi \in V_{\underline{\gamma}}^2(\Omega) \cap H_0^1(\Omega)$. The contribution $c_1(\mathbf{d}_0;\mathbf{b},\mathbf{v})$ is bounded as in Lemma 2.10. To bound $c_1(\nabla\varphi;\mathbf{b},\mathbf{v})$, we proceed as follows. First note that $\nabla\varphi \in V_{\underline{\gamma}}^1(\Omega)^2$ and $\|\nabla\varphi\|_{1,\underline{\gamma}} \leq C\|\varphi\|_{2,\underline{\gamma}} < \infty$. From [26, Proposition 25], we obtain

$$
\omega_{\underline{\gamma}}\nabla\varphi \in H^1(\Omega)^2 \qquad \text{and} \qquad \|\omega_{\underline{\gamma}}\nabla\varphi\|_{H^1(\Omega)} \leq C\|\nabla\varphi\|_{1,\underline{\gamma}} \leq C\|\varphi\|_{2,\underline{\gamma}},
$$

for a constant $C > 0$ solely depending on Ω and $\underline{\gamma}$. Let p and q be parameters with $2 < q \leq p < \infty$ and $q^{-1} + p^{-1} = 1/2$. By Hölder's inequality, Rellich's embedding theorem and the above estimate, we obtain

$$
\begin{aligned}
\Big|\int_\Omega (\nabla\varphi \times \operatorname{curl}\mathbf{b}) \cdot \mathbf{v}\,d\mathbf{x}\Big| &= \Big|\int_\Omega (\omega_{\underline{\gamma}}\nabla\varphi \times \operatorname{curl}\mathbf{b}) \cdot (\omega_{\underline{\gamma}}^{-1}\mathbf{v})\,d\mathbf{x}\Big| \\
&\leq \|\omega_{\underline{\gamma}}\nabla\varphi\|_{L^p(\Omega)}\|\operatorname{curl}\mathbf{b}\|_{L^2(\Omega)}\|\omega_{\underline{\gamma}}^{-1}\mathbf{v}\|_{L^q(\Omega)} \\
&\leq C\|\omega_{\underline{\gamma}}\nabla\varphi\|_{H^1(\Omega)}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\omega_{\underline{\gamma}}^{-1}\mathbf{v}\|_{L^q(\Omega)} \\
&\leq C\|\varphi\|_{2,\underline{\gamma}}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\omega_{\underline{\gamma}}^{-1}\mathbf{v}\|_{L^q(\Omega)},
\end{aligned}
\tag{2.42}
$$

with a constant $C > 0$ solely depending on Ω, $\underline{\gamma}$, and the exponents p and q. It remains to show that q can be chosen so that $\|\omega_{\underline{\gamma}}^{-1}\mathbf{v}\|_{L^q(\Omega)} \leq C\|\mathbf{v}\|_{H^1(\Omega)}$. To this end, let s and s' be two other parameters with $1 < s' \leq s < \infty$ and $s'^{-1} + s^{-1} = 1$. We have

$$
\|\omega_{\underline{\gamma}}^{-1}\mathbf{v}\|_{L^q(\Omega)}^q \leq \Big(\int_\Omega |\mathbf{v}|^{qs}\,d\mathbf{x}\Big)^{1/s}\Big(\int_\Omega |\omega_{\underline{\gamma}}^{-1}|^{qs'}\,d\mathbf{x}\Big)^{1/s'} = \|\mathbf{v}\|_{L^{qs}(\Omega)}^q\|\omega_{\underline{\gamma}}^{-1}\|_{L^{qs'}(\Omega)}^q.
$$

Let $\mathcal{V}_\mathbf{c}$ be a small neighborhood of the corner $\mathbf{c} \in \mathcal{C}$. In local polar coordinates $(r_\mathbf{c},\phi)$ at the point $\mathbf{c}$, there holds

$$
\int_{\mathcal{V}_\mathbf{c}} |\omega_{\underline{\gamma}}|^{-qs'}\,d\mathbf{x} \leq C\int_{\mathcal{V}_\mathbf{c}} r_\mathbf{c}^{-qs'\gamma_\mathbf{c}+1}\,dr_\mathbf{c}\,d\phi < \infty,
$$

provided that $qs'\gamma_{\mathbf{c}} < 2$. The constant C only depends on Ω. Since we have $\max_{\mathbf{c}\in\mathcal{C}} \gamma_{\mathbf{c}} < 1$, there is a parameter $\varepsilon > 0$ (depending on $\underline{\gamma}$) such that the condition $qs'\gamma_{\mathbf{c}} < 2$ is fulfilled for $q = 2 + \varepsilon$ and $s' = 1 + \varepsilon$. With this choice, we obtain

$$\|\omega_{\underline{\gamma}}^{-1}\|^q_{L^{qs'}(\Omega)} \leq C < \infty, \tag{2.43}$$

with a constant $C > 0$ depending on Ω and $\underline{\gamma}$. Combining (2.42) and (2.43), using Rellich's embedding theorem and the estimate in (2.39), results in

$$\begin{aligned}|c_1(\nabla\varphi; \mathbf{b}, \mathbf{v})| &\leq SC\,\|\varphi\|_{2,\underline{\gamma}}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\mathbf{v}\|_{L^{qs}(\Omega)}\\ &\leq SC\,\|\varphi\|_{2,\underline{\gamma}}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\mathbf{v}\|_{H^1(\Omega)} \leq SC\|\mathbf{d}\|_{X_{\underline{\gamma}}}\|\mathbf{b}\|_{X_{\underline{\gamma}}}\|\mathbf{v}\|_{H^1(\Omega)},\end{aligned}$$

for a constant $C > 0$ depending on Ω and $\underline{\gamma}$.

This yields the result for c_1, the proof for c_2 is analogous. $\square$

Remark 2.13 *We point out that the proof in Lemma 2.12 is based on the continuous embedding of $H^1(\Omega)$ into $L^q(\Omega)$ for all $q \geq 1$. Since in the three-dimensional $H^1(\Omega)$ is continuously embedded into $L^q(\Omega)$ only for $q \in [1, 6]$, a similar argument in three dimensions shows the continuity of c_1 and c_2 only for polyhedral domains whose maximal opening angle is smaller than $2\pi/3$.*

Next, we address the coercivity of the forms a_s and a_m.

Lemma 2.14 *Let $\underline{\gamma}$ be a weight vector satisfying (2.19). Then:*

$$\begin{aligned} a_s(\mathbf{u}, \mathbf{u}) &\geq C_1 R_e^{-1}\,\|\mathbf{u}\|^2_{H^1(\Omega)}, && \mathbf{u} \in H^1_0(\Omega)^3,\\ a_m(\mathbf{b}, \mathbf{b}) &\geq C_2 \min\{R_m^{-1}\,S, D\}\|\mathbf{b}\|^2_{X_{\underline{\gamma}}}, && \mathbf{b} \in X_{\underline{\gamma}}(\Omega),\end{aligned}$$

with a constant $C_1 > 0$ only depending on Ω, and a constant $C_2 > 0$ only depending on Ω and $\underline{\gamma}$.

Proof: The coercivity of a_s is a standard property and the coercivity of a_m follows from the inequality in Proposition 2.6. $\square$

Finally, we recall the following inf-sup condition for the form b_s; see, e.g., [14, Section I.5.1].

Lemma 2.15 *There is a constant $\beta > 0$, only depending on Ω, such that*

$$\inf_{q\in L^2_0(\Omega)} \sup_{\mathbf{v}\in H^1_0(\Omega)^3} \frac{b_s(q, \mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta.$$

For notational convenience, we introduce the space

$$W_{\underline{\gamma}}(\Omega) \;=\; H^1_0(\Omega)^3 \,\times\, X_{\underline{\gamma}}(\Omega),$$

and endow it with the norm

$$\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}}^2 = \|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\mathbf{c}\|_{X_{\underline{\gamma}}}^2. \tag{2.44}$$

We then have the following stability results for the forms in (2.35).

Proposition 2.16 *Let $\underline{\gamma}$ be a weight vector satisfying (2.19). There holds:*

(i) There are continuity constants C_A, C_B solely depending on the data and on D such that

$$\begin{aligned} |A(\mathbf{u},\mathbf{b};\mathbf{v},\mathbf{c})| &\leq C_A\|(\mathbf{u},\mathbf{b})\|_{W_{\underline{\gamma}}}\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}}, \qquad && (\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c}) \in W_{\underline{\gamma}}(\Omega), \\ |B(q;\mathbf{v},\mathbf{c})| &\leq C_B\|q\|_{L^2(\Omega)}\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}}, && q \in L_0^2(\Omega),\ (\mathbf{v},\mathbf{c}) \in W_{\underline{\gamma}}(\Omega). \end{aligned}$$

If we additionally assume (2.36) to hold, then there is a constant C_C depending on the data, the domain, and the weight $\underline{\gamma}$ such that

$$|O(\mathbf{w},\mathbf{d};\mathbf{u},\mathbf{b};\mathbf{v},\mathbf{c})| \leq C_C\|(\mathbf{w},\mathbf{d})\|_{W_{\underline{\gamma}}}\|(\mathbf{u},\mathbf{b})\|_{W_{\underline{\gamma}}}\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}},$$

for any $(\mathbf{w},\mathbf{d}) \in W_{\underline{\gamma}}(\Omega)$, $(\mathbf{u},\mathbf{b}) \in W_{\underline{\gamma}}(\Omega)$, and $(\mathbf{v},\mathbf{c}) \in W_{\underline{\gamma}}(\Omega)$.

(ii) There is a coercivity constant $\alpha > 0$, depending on the data, the parameter D, the domain, and $\underline{\gamma}$, such that

$$A(\mathbf{u},\mathbf{b};\mathbf{u},\mathbf{b}) \geq \alpha\|(\mathbf{u},\mathbf{b})\|_{W_{\underline{\gamma}}}^2, \qquad (\mathbf{u},\mathbf{b}) \in W_{\underline{\gamma}}(\Omega).$$

(iii) Let $L := (\|\mathbf{f}\|_{L^2(\Omega)}^2 + \|\mathbf{g}\|_{L^2(\Omega)}^2)^{1/2}$. We have

$$|(\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{c})| \leq L\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}}, \qquad (\mathbf{v},\mathbf{c}) \in W_{\underline{\gamma}}(\Omega).$$

(iv) We have the skew-symmetry property

$$O(\mathbf{w},\mathbf{d};\mathbf{u},\mathbf{b};\mathbf{u},\mathbf{b}) = 0, \qquad (\mathbf{w},\mathbf{d}),(\mathbf{u},\mathbf{b}) \in W_{\underline{\gamma}}(\Omega).$$

(v) There holds, for the same constant β as in Lemma 2.15 and independently of $\underline{\gamma}$,

$$\inf_{q\in L_0^2(\Omega)} \sup_{(\mathbf{v},\mathbf{c})\in W_{\underline{\gamma}}(\Omega)} \frac{B(q;\mathbf{v},\mathbf{c})}{\|(\mathbf{v},\mathbf{c})\|_{W_{\underline{\gamma}}}\|q\|_{L^2(\Omega)}} \geq \beta.$$

Proof: The continuity and coercivity in (i) and (ii) are immediate consequences of Lemma 2.8, Lemma 2.10 and Lemma 2.14, respectively. The continuity property in (iii) holds since

$$\begin{aligned} |(\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{c})| &\leq \|\mathbf{f}\|_{L^2(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}\|\mathbf{c}\|_{X_{\underline{\gamma}}} \\ &\leq (\|\mathbf{f}\|_{L^2(\Omega)}^2 + \|\mathbf{g}\|_{L^2(\Omega)}^2)^{1/2}\,(\|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\mathbf{c}\|_{X_{\underline{\gamma}}}^2)^{1/2}. \end{aligned}$$

To see the skew-symmetry property of O in (iv), it is enough to note that $o_s(\mathbf{w};\mathbf{u},\mathbf{u}) = 0$ and $c_1(\mathbf{d};\mathbf{b},\mathbf{u}) = c_2(\mathbf{d};\mathbf{u},\mathbf{b})$. The latter identity follows since $(\mathbf{d} \times \mathbf{curl}\,\mathbf{b}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{d})\ \mathbf{curl}\,\mathbf{b}$.

It only remains to establish the inf-sup condition in (v) for the form B. To see this, fix $q \in L^2_0(\Omega)$. From Lemma 2.15, there is an element $\mathbf{v} \in H^1_0(\Omega)^3$ such that

$$b_s(q, \mathbf{v}) \,\geq\, \beta \|q\|^2_{L^2(\Omega)}, \qquad \|\mathbf{v}\|_{H^1(\Omega)} \,\leq\, \|q\|_{L^2(\Omega)}.$$

We obtain

$$B(q; \mathbf{v}, \mathbf{0}) \,\geq\, \beta \|q\|^2_{L^2(\Omega)}, \qquad \|(\mathbf{v}, \mathbf{0})\|_{X_{\underline{\gamma}}} \,\leq\, \|q\|_{L^2(\Omega)},$$

and the inf-sup condition for B follows. □

Proceeding as in the proof of [23, Theorem 10.1.1], we obtain from Proposition 2.16 the following existence and uniqueness result for small data.

Theorem 2.17 *Let $\underline{\gamma}$ be a weight vector satisfying (2.19) and (2.36). Assume further that*

$$\frac{C_C L}{\alpha^2} < 1. \tag{2.45}$$

Then the weak formulation in (2.28) has a unique solution $(\mathbf{u}, \mathbf{b}, p) \in H^1_0(\Omega)^3 \times X_{\underline{\gamma}}(\Omega) \times L^2_0(\Omega)$ and we have the stability bounds

$$\begin{aligned} \|(\mathbf{u}, \mathbf{b})\|_{W_{\underline{\gamma}}} \,&\leq\, \alpha^{-1} L, \\ \|p\|_{L^2(\Omega)} \,&\leq\, \beta^{-1} L\,(1 + \alpha^{-1} C_A + \alpha^{-2} C_C L), \end{aligned}$$

with α, β, C_A, C_B, C_C, and L denoting the stability constants from Proposition 2.16.

Remark 2.18 *As has been pointed out in Remark 2.11, the restrictions on $\underline{\gamma}$ in (2.36) are most likely suboptimal. For the two-dimensional MHD problem in (2.8)–(2.13), on the other hand, the continuity result in Lemma 2.12 can be invoked and the result of Theorem 2.17 is obtained for any weight $\underline{\gamma}$ satisfying $\underline{\delta}^{\mathrm{dir}} < \underline{\gamma} < 1$, with $\underline{\delta}^{\mathrm{dir}}$ defined in (2.21).*

Remark 2.19 *The extension of the result in Theorem 2.17 to MHD problems with inhomogeneous boundary conditions is not straightforward. While it is easily possible to lift inhomogeneous boundary data in a divergence-free fashion into the domain, see [14, Lemma IV.2.3] for the velocity field and [22, Proposition A.1] for the magnetic field, these liftings affect the size of the data for which existence and uniqueness of solutions can be proved; see [14, Section IV.2.1]. For velocity boundary data, this effect can be minimized and* *controlled by using the so-called Hopf construction which yields a divergence-free lifting $\mathbf{u}_0 \in H^1(\Omega)^3$ such that $|\int_\Omega [(\mathbf{v}\cdot\nabla)\mathbf{u}_0]\cdot\mathbf{v}\,d\mathbf{x}|$ is arbitrarily small relative to $\|\nabla\mathbf{v}\|^2_{L^2(\Omega)}$, for all $\mathbf{v} \in H^1_0(\Omega)^3$; see [14, Lemma IV.2.3]. However, analogous Hopf-type liftings for magnetic boundary data seem not to be available in the literature and remain to be constructed.*

3 Finite Element Approximation

In this section, we introduce and analyze the finite element approximation of the mixed formulation in (2.28). We derive quasi-optimal error bounds in the energy norm and show that the weighted regularization technique ensures convergence of the approximation in possibly non-convex domains.

3.1 Galerkin Approximation

We choose conforming finite element spaces $V^h \subset H^1_0(\Omega)^3$, $X^h_{\underline{\gamma}} \subset X_{\underline{\gamma}}(\Omega)$, and $L^h \subset L^2_0(\Omega)$, and endow them with the norms $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{X_{\underline{\gamma}}}$, and $\|\cdot\|_{L^2(\Omega)}$, respectively. Here, we use the index h to denote the discretization parameter. We generically refer to it as the mesh-size.

Throughout, we assume that the pair $V^h \times L^h$ gives rise to an inf-sup stable Stokes discretization, that is, we assume that there is a constant $\beta_h > 0$ independent of the mesh-size h, such that

$$\inf_{q\in L^h}\sup_{\mathbf{v}\in V^h}\frac{b_s(q,\mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta_h. \tag{3.1}$$

A wide variety of spaces V^h and L^h fulfilling (3.1) have been proposed in the literature; we refer to [5, Chapter IV], [14, Chapter II] and the references cited therein. A specific choice of finite element spaces based on Hood-Taylor elements will be discussed in Section 3.2 below.

Given a weight vector $\underline{\gamma}$, the finite element approximation of (2.28) is: Find $(\mathbf{u}_h, \mathbf{b}_h, p_h) \in V^h \times X^h_{\underline{\gamma}} \times L^h$ such that

$$\begin{aligned} a_s(\mathbf{u}_h,\mathbf{v}) + o_s(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) + b_s(p_h,\mathbf{v}) + c_1(\mathbf{b}_h;\mathbf{b}_h,\mathbf{v}) &= (\mathbf{f},\mathbf{v}),\\ a_m(\mathbf{b}_h,\mathbf{c}) - c_2(\mathbf{b}_h;\mathbf{u}_h,\mathbf{c}) &= (\mathbf{g},\mathbf{c}),\\ b_s(q,\mathbf{u}_h) &= 0 \end{aligned} \tag{3.2}$$

for all $\mathbf{v} \in V^h$, $\mathbf{c} \in X^h_{\underline{\gamma}}$ and $q \in L^h$. As before, problem (3.2) is equivalent to:

Find $(\mathbf{u}_h, \mathbf{b}_h, p_h) \in V^h \times X^h_{\underline{\gamma}} \times L^h$ such that

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + O(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + B(p_h; \mathbf{v}, \mathbf{c}) &= (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{c}), \\ B(q; \mathbf{u}_h, \mathbf{b}_h) &= 0 \end{aligned} \tag{3.3}$$

for all $(\mathbf{v}, \mathbf{c}, q) \in V^h \times X^h_{\underline{\gamma}} \times L^h$.

Introducing the space $W^h_{\underline{\gamma}} = V^h \times X^h_{\underline{\gamma}}$, endowed with the norm $\| \cdot \|_{W_{\underline{\gamma}}}$, we have the following discrete inf-sup condition for the form B

$$\inf_{q \in L^h} \sup_{(\mathbf{v},\mathbf{c}) \in W^h_{\underline{\gamma}}} \frac{B(q; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}} \|q\|_{L^2(\Omega)}} \geq \beta_h, \tag{3.4}$$

with the same inf-sup constant $\beta_h > 0$ as in (3.1). Condition (3.4) can be proved by using arguments that are completely analogous to those on the continuous level.

The discrete version of Theorem 2.17 is then an immediate consequence.

Corollary 3.1 *Let $\underline{\gamma}$ be a weight vector satisfying (2.19) and (2.36). Let the smallness assumption (2.45) be satisfied. Then the finite element formulation in (3.2) has a unique solution $(\mathbf{u}_h, \mathbf{b}_h, p_h) \in V^h \times X^h_{\underline{\gamma}} \times L^h$ and we have the stability bounds*

$$\begin{aligned} \|(\mathbf{u}, \mathbf{b})\|_{W_{\underline{\gamma}}} &\leq \alpha^{-1} L, \\ \|p\|_{L^2(\Omega)} &\leq \beta_h^{-1} L \,(1 + \alpha^{-1} C_A + \alpha^{-2} C_C L), \end{aligned}$$

with α, C_A, C_B, C_C, and L denoting the stability constants from Proposition 2.16, and with β_h denoting the discrete inf-sup constant from (3.1).

As before, in the two-dimensional case, this result holds for weight vectors $\underline{\gamma}$ with $\underline{\delta}^{\mathrm{dir}} < \underline{\gamma} < 1$.

Remark 3.2 *The solution $(\mathbf{u}_h, \mathbf{b}_h, p_h) \in X^h_{\underline{\gamma}} \times V^h \times L^h$ of the finite element formulation (3.2) can be found by the following Picard iteration: Given $(\mathbf{u}^n_h, \mathbf{b}^n_h, p^n_h) \in V^h \times X^h_{\underline{\gamma}} \times L^h$, let $(\mathbf{u}^{n+1}_h, \mathbf{b}^{n+1}_h, p^{n+1}_h) \in V^h \times X^h_{\underline{\gamma}} \times L^h$ be the solution of the linearized problem*

$$\begin{aligned} a_s(\mathbf{u}^{n+1}_h, \mathbf{v}) + o_s(\mathbf{u}^n_h; \mathbf{u}^{n+1}_h, \mathbf{v}) + b_s(p^{n+1}_h, \mathbf{v}) + c_1(\mathbf{b}^n_h; \mathbf{b}^{n+1}_h, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \\ a_m(\mathbf{b}^{n+1}_h, \mathbf{c}) - c_2(\mathbf{b}^n_h; \mathbf{u}^{n+1}_h, \mathbf{c}) &= (\mathbf{g}, \mathbf{c}), \\ b_s(q, \mathbf{u}^{n+1}_h) &= 0 \end{aligned}$$

for all $\mathbf{v} \in V^h$, $\mathbf{c} \in X^h_{\underline{\gamma}}$ and $q \in L^h$. Under the smallness assumption in (2.45), the sequence $\{(\mathbf{u}^n_h, \mathbf{b}^n_h, p^n_h)\}_{n \geq 1}$ converges to the solution $(\mathbf{u}_h, \mathbf{b}_h, p_h)$ of (3.2).

Other procedures based on Newton's method are possible as well; cf. [18]. We point out that, if the linearized problems above are strongly convection-dominated, it might be necessary for their efficient solution to include additional stabilization terms along the lines of [13]. As our analysis is mainly concerned with the incorporation of the divergence constraint $\operatorname{div} \mathbf{b} = 0$ *via the weighted regularization approach, this point is not further investigated in this paper.*

We derive quasi-optimal error bounds for the proposed finite element approximation. To this end, we introduce on $W_{\underline{\gamma}}(\Omega) \times L_0^2(\Omega)$ the norm $\|\!|\!| (\cdot,\cdot,\cdot) |\!|\!|_{\underline{\gamma}}$ given by

$$|\!|\!| (\mathbf{v}, \mathbf{c}, q) |\!|\!|_{\underline{\gamma}}^2 = \|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}^2 + \|q\|_{L^2(\Omega)}^2.$$

The following theorem holds.

Theorem 3.3 *Let* $\underline{\gamma}$ *be a weight vector satisfying (2.19) and (2.36). Assume further that*

$$\frac{C_C L}{\alpha^2} \leq \frac{1}{2}. \tag{3.5}$$

Let $(\mathbf{u}, \mathbf{b}, p)$ *be the (unique) solution of (2.28), and let* $(\mathbf{u}_h, \mathbf{b}_h, p_h)$ *its finite element approximition obtained by (3.2). Then we have the quasi-optimal error bound*

$$|\!|\!| (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h, p - p_h) |\!|\!|_{\underline{\gamma}} \leq C \inf_{(\mathbf{v},\mathbf{c},q) \in V^h \times X^h_{\underline{\gamma}} \times L^h} |\!|\!| (\mathbf{u} - \mathbf{v}, \mathbf{b} - \mathbf{c}, p - q) |\!|\!|_{\underline{\gamma}},$$

with a constant $C > 0$ *independent of the mesh-size* h*.*

Proof: We proceed in several steps.

Step 1: We first note that we have the error equation

$$\begin{aligned} &A(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + O(\mathbf{u} - \mathbf{u}_h; \mathbf{b} - \mathbf{b}_h; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) \\ &\qquad + O(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + B(p - p_h; \mathbf{v}, \mathbf{c}) = 0, \end{aligned}$$

for any $(\mathbf{v}, \mathbf{c}) \in W^h_{\underline{\gamma}}$.

Step 2: Set $\ker B_h = \{(\mathbf{v}, \mathbf{c}) \in W^h_{\underline{\gamma}} \mid B(q; \mathbf{v}, \mathbf{c}) = 0 \ \forall q \in L^h\}$. We claim that

$$\|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{W_{\underline{\gamma}}} \leq C \Big[\|(\mathbf{u} - \mathbf{v}, \mathbf{b} - \mathbf{c})\|_{W_{\underline{\gamma}}} + \|p - q\|_{L^2(\Omega)} \Big], \tag{3.6}$$

for any $(\mathbf{v}, \mathbf{c}) \in \ker B_h$ and $q \in L^h$, with a constant $C > 0$ that is independent of the mesh-size.

To see (3.6), fix $(\mathbf{v}, \mathbf{c}) \in \ker B_h$ and $q \in L^h$. Clearly, $\mathbf{v} - \mathbf{u}_h \in \ker B_h$. Using the error equation in Step 1, it can be easily seen that

$$\begin{aligned}
&A(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h;\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)+O(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h;\mathbf{u},\mathbf{b};\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\\
&= \quad A(\mathbf{v}-\mathbf{u},\mathbf{c}-\mathbf{b};\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)+O(\mathbf{v}-\mathbf{u},\mathbf{c}-\mathbf{b};\mathbf{u},\mathbf{b};\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\\
&\qquad +O(\mathbf{u}_h,\mathbf{b}_h;\mathbf{v}-\mathbf{u},\mathbf{c}-\mathbf{b};\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)-B(p-p_h;\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h).
\end{aligned} \tag{3.7}$$

We first estimate the left-hand side of (3.7) from below. To this end, we use the coercivity and continuity properties in Proposition 2.16, the stability bound in Theorem 2.17, and the smallness assumption in (3.5), and obtain

$$\begin{aligned}
\text{l.h.s. of (3.7)} &= A(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h;\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\\
&\qquad +O(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h;\mathbf{u},\mathbf{b};\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\\
&\geq \left[\alpha - C_C\|(\mathbf{u},\mathbf{b})\|_{W_{\underline{\gamma}}}\right]\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|^2_{W_{\underline{\gamma}}}\\
&\geq \left[1-\frac{C_C L}{\alpha^2}\right]\alpha\,\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|^2_{W_{\underline{\gamma}}}\\
&\geq \frac{1}{2}\alpha\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|^2_{W_{\underline{\gamma}}}.
\end{aligned} \tag{3.8}$$

To bound the right-hand side of (3.7) from above, we first note that, because $\mathbf{v}-\mathbf{u}_h \in \ker B_h$, we have

$$B(p-p_h;\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)=B(p-q;\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h).$$

Using the continuity properties in Proposition 2.16 and the bounds in Theorem 2.17 and Corollary 3.1 we get

$$\begin{aligned}
\text{r.h.s. of (3.7)} &\leq \left(C_A + C_C\|(\mathbf{u},\mathbf{b})\|_{W_{\underline{\gamma}}} + C_C\|(\mathbf{u}_h,\mathbf{b}_h)\|_{W_{\underline{\gamma}}}\right)\\
&\qquad \times \|(\mathbf{v}-\mathbf{u},\mathbf{c}-\mathbf{b})\|_{W_{\underline{\gamma}}}\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}}\\
&\qquad + C_B\|p-q\|_{L^2(\Omega)}\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}}\\
&\leq (C_A + 2C_C L\alpha^{-1})\|(\mathbf{v}-\mathbf{u},\mathbf{c}-\mathbf{b})\|_{W_{\underline{\gamma}}}\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}}\\
&\qquad + C_B\|p-q\|_{L^2(\Omega)}\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}}.
\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) results in

$$\|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}} \leq C\|(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\|_{W_{\underline{\gamma}}} + C\|p-q\|_{L^2(\Omega)}.$$

Since

$$\|(\mathbf{u}-\mathbf{u}_h,\mathbf{b}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}} \leq \|(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\|_{W_{\underline{\gamma}}} + \|(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}},$$

the assertion (3.6) follows.

Step 3: It is well-known that we can use the discrete inf-sup condition in (3.4) in order to establish the approximation result (3.6) for any $(\mathbf{v}, \mathbf{c}) \in W^h_{\underline{\gamma}}$; see, e.g., [5,14]. This proves the quasi-optimality of the error $\|(\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h)\|_{W_{\underline{\gamma}}}$.

Step 4: It remains to bound the error in the pressure. To do so, let $q \in L^h$. The inf-sup condition (3.4) yields

$$
\begin{aligned}
\beta_h \|q - p_h\|_{L^2(\Omega)} &\leq \sup_{(\mathbf{v},\mathbf{c}) \in W^h_{\underline{\gamma}}} \frac{B(q - p_h; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} \\
&= \sup_{(\mathbf{v},\mathbf{c}) \in W^h_{\underline{\gamma}}} \frac{B(q - p; \mathbf{v}, \mathbf{c}) + B(p - p_h; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} \\
&\leq \sup_{(\mathbf{v},\mathbf{c}) \in W^h_{\underline{\gamma}}} \frac{B(q - p; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} + \sup_{(\mathbf{v},\mathbf{c}) \in W^h_{\underline{\gamma}}} \frac{B(p - p_h; \mathbf{v}, \mathbf{c})}{\|\mathbf{v}, \mathbf{c}\|_{W_{\underline{\gamma}}}}.
\end{aligned}
$$

Using the error equation from Step 1, the continuity properties in Property 2.16, and the stability bounds in Theorem 2.17 and Corollary 3.1, we have

$$
\begin{aligned}
\frac{B(p - p_h; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} &\leq \Big(C_A + C_C \|(\mathbf{u}, \mathbf{b})\|_{W_{\underline{\gamma}}} + C_C \|(\mathbf{u}_h, \mathbf{b}_h)\|_{W_{\underline{\gamma}}} \Big) \\
&\qquad\qquad\qquad\qquad \times \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{W_{\underline{\gamma}}} \\
&\leq \Big(C_A + 2 C_C L \alpha^{-1} \Big) \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{W_{\underline{\gamma}}}.
\end{aligned}
$$

Thus,

$$
\frac{B(p - p_h; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} \leq C \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{W_{\underline{\gamma}}}.
$$

Moreover, employing the continuity of B,

$$
\frac{B(q - p; \mathbf{v}, \mathbf{c})}{\|(\mathbf{v}, \mathbf{c})\|_{W_{\underline{\gamma}}}} \leq C_B \|p - q\|_{L^2(\Omega)}.
$$

We obtain

$$
\begin{aligned}
\|p - p_h\|_{L^2(\Omega)} &\leq \|p - q\|_{L^2(\Omega)} + \|q - p_h\|_{L^2(\Omega)} \\
&\leq C \Big[\|p - q\|_{L^2(\Omega)} + \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{W_{\underline{\gamma}}} \Big].
\end{aligned}
$$

The estimate and the assertion follows then with the bounds of Step 3. $\square$

Let now $\{V^h\}_h$, $\{X^h_{\underline{\gamma}}\}_h$, and $\{L^h\}_h$ be sequences of finite element spaces, chosen such that

$$\begin{aligned}
&\forall\, \mathbf{u} \in H_0^1(\Omega)^3 : \qquad && \inf_{\mathbf{v}\in V^h} \|\mathbf{u}-\mathbf{v}\|_{H^1(\Omega)} \to 0, \qquad && h \to 0,\\
&\forall\, \mathbf{b} \in X_{\underline{\gamma}}(\Omega) : && \inf_{\mathbf{c}\in X^h_{\underline{\gamma}}} \|\mathbf{b}-\mathbf{c}\|_{X_{\underline{\gamma}}} \to 0, && h \to 0,\\
&\forall\, p \in L_0^2(\Omega) : && \inf_{q\in L^h} \|p-q\|_{L^2(\Omega)} \to 0, && h \to 0.
\end{aligned}$$

Note that due to Theorem 2.4 and the density of $C^\infty(\overline{\Omega})$ functions with vanishing trace on $\partial\Omega$ in $V^2_{\underline{\gamma}}(\Omega)\cap H^1_0(\Omega)$, see [7, Proposition 3.2 and Theorem 4.1], the density assumption for $\mathbf{b}$ is justified, provided that the weight vector $\underline{\gamma}$ satisfies (2.19).

Corollary 3.4 *Assume (3.5) and that the weight vector $\underline{\gamma}$ satisfies (2.19) and (2.36). Then, we have for the above sequence of spaces*

$$\lim_{h\to 0} ||| \, (\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h, p-p_h) \, |||_{\underline{\gamma}} = 0.$$

In two dimensions, the same result holds for weights $\underline{\gamma}$ satisfying $\underline{\delta}^{\mathrm{dir}} < \underline{\gamma} < 1$, with $\underline{\delta}^{\mathrm{dir}}$ given in (2.21).

Theorem 3.3 ensures convergence of the finite element approximation in non-convex polyhedra as $h \to 0$, provided that the weight vector is properly chosen. The choice $\gamma_{\mathbf{c}} = 0$ and $\gamma_e = 0$ for all $\mathbf{c} \in \mathcal{C}$ and $e \in \mathcal{E}$ (no weighted regularization), for example, does not lead to convergent FEM solutions in non-convex polygons. This is due to the fact that, without weighted regularization, the space $H_{\underline{\gamma}}(\Omega)$ is known to be a closed subspace of $X_{\underline{\gamma}}(\Omega)$ and the strongest magnetic singularities lie in the complement $X_{\underline{\gamma}}(\Omega) \setminus H_{\underline{\gamma}}(\Omega)$; see [7]. Hence, it is impossible to correctly capture the magnetic fields. This behavior is clearly confirmed in our numerical results in Section 4, demonstrating that weighted regularization is indispensable in non-convex domains.

3.2 Convergence Rates for Hood-Taylor Elements

In this section, we present a specific finite element family based on Hood-Taylor elements for the unknowns $\mathbf{u}$ and p, and discuss the corresponding convergence rates.

To this end, let $\mathcal{T}_h = \{K\}$ be a regular and shape-regular partition of Ω into hexahedral elements $\{K\}$. We assume that each element K is affinely equivalent to the reference cube $\hat{K} = (0,1)^3$. We denote by h_K the diameter of element K and set $h = \max_{K\in\mathcal{T}_h}\{h_K\}$. For an approximation order $k \geq 2$, we introduce the following finite element spaces

$$\begin{aligned} V^h &= \{\mathbf{v} \in H_0^1(\Omega)^3 \mid \mathbf{v}_{|K} \in Q_k^3(K),\ \forall K \in \mathcal{T}_h\}, \\ X_{\underline{\gamma}}^h &= \{\mathbf{c} \in H_{\underline{\gamma}}(\Omega) \mid \mathbf{c}_{|K} \in Q_k^3(K),\ \forall K \in \mathcal{T}_h\}, \\ L^h &= \{q \in H^1(\Omega) \cap L_0^2(\Omega) \mid q_{|K} \in Q_{k-1}(K),\ \forall K \in \mathcal{T}_h\}. \end{aligned} \tag{3.10}$$

Here, $Q_k(K)$ denotes the space of polynomials of degree $\leq k$ in each variable on K. The velocity-pressure pair $V^h \times L^h$ is referred to as "Hood-Taylor" elements. It is well-known that the spaces V^h and L^h satisfy the discrete inf-sup condition in (3.1); see [4].

For this family, let us discuss the convergence rates that can be expected from Theorem 3.3. We first consider the case of a smooth solution $(\mathbf{u}, \mathbf{b}, p)$. Standard approximation properties then give straightforwardly the following optimal convergence rates.

Corollary 3.5 *Let the exact solution* $(\mathbf{u}, \mathbf{b}, p)$ *of (2.1)–(2.6) satisfy*

$$(\mathbf{u}, \mathbf{b}, p) \in H^{k+1}(\Omega)^3 \times H^{k+1}(\Omega)^3 \times H^k(\Omega).$$

Under the assumptions of Theorem 3.3, there holds

$$\| (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h, p - p_h) \|_{\underline{\gamma}} \leq Ch^k (\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \|\mathbf{b}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}),$$

with a constant $C > 0$ *independent of the mesh-size* h.

Next, we show that positive convergence rates are still possible for solutions that exhibit singularities at the corners of the domain Ω. To this end, we consider a model situation where we assume that the exact solution $(\mathbf{u}, \mathbf{b}, p)$ can be decomposed into a regular and a singular part, according to

$$(\mathbf{u}, \mathbf{b}, p) = (\mathbf{u}^{\text{reg}}, \mathbf{b}^{\text{reg}}, p^{\text{reg}}) + (\mathbf{u}^{\text{sing}}, \mathbf{b}^{\text{sing}}, p^{\text{sing}}).$$

Such decompositions can be found in, e.g., [9,21,19,6,7] in the context of the Navier-Stokes equations, linearizations thereof, and Maxwell's equations. Analogous results then hold for linear MHD problems, as the one considered in Section 4 below. However, for the nonlinear MHD problems under consideration, decompositions of the above type do not seem to be available in detail. Thus, here we assume that the regular part is smooth and satisfies

$$\mathbf{u}^{\text{reg}} \in H^2(\Omega)^3, \qquad \mathbf{b}^{\text{reg}} \in H^2(\Omega)^3, \qquad p^{\text{reg}} \in H^1(\Omega).$$

It is then clear that there is an interpolant $(\mathbf{v}^{\text{reg}}, \mathbf{c}^{\text{reg}}, q^{\text{reg}}) \in V^h \times X_{\underline{\gamma}}^h \times L^h$ so that

$$\| (\mathbf{u}^{\text{reg}} - \mathbf{v}^{\text{reg}}, \mathbf{b}^{\text{reg}} - \mathbf{c}^{\text{reg}}, p^{\text{reg}} - q^{\text{reg}}) \|_{\underline{\gamma}} \leq Ch.$$

The part $(\mathbf{u}^{\text{sing}}, \mathbf{b}^{\text{sing}}, p^{\text{sing}})$ consists of the singular functions. These functions have a low global regularity, but are typically smooth in the interior of the domain. This behavior can be described best in terms of the following limits of weighted spaces

$$K^{\infty}_{\underline{\beta}}(\Omega) = \bigcap_{\underline{\gamma}>\underline{\beta}} \left(\bigcap_{m\in\mathbb{N}} V^m_{\underline{\gamma}+m}(\Omega) \right).$$

We then assume that the velocity-pressure singularities belong to

$$\mathbf{u}^{\mathrm{sing}} \in K^{\infty}_{\underline{\delta}^{\mathrm{dir}}-2}(\Omega)^3 \cap H^1_0(\Omega)^3, \qquad p^{\mathrm{sing}} \in K^{\infty}_{\underline{\delta}^{\mathrm{dir}}-1}(\Omega).$$

Note that $\mathbf{u}^{\mathrm{sing}} \notin H^2(\Omega)^3$ and $p^{\mathrm{sing}} \notin H^1(\Omega)$. With arguments similar to those in, e. g., [26] and the references therein, an interpolant $(\mathbf{v}^{\mathrm{sing}}, q^{\mathrm{sing}}) \in V^h \times L^h$ can be constructed such that

$$\|\mathbf{u}^{\mathrm{sing}} - \mathbf{v}^{\mathrm{sing}}\|_{H^1(\Omega)} + \|p^{\mathrm{sing}} - q^{\mathrm{sing}}\|_{L^2(\Omega)} \leq Ch^{\tau_1},$$

for an exponent $\tau_1 > 0$ depending on $\underline{\delta}^{\mathrm{dir}}$.

Concerning the magnetic field $\mathbf{b}^{\mathrm{sing}}$, we assume, in agreement with [6,7], that it consists of Neumann singularities of the Laplace operator and singularities that are gradients of Dirichlet singularities of the Laplacian. That is,

$$\mathbf{b}^{\mathrm{sing}} = \mathbf{b}_N + \nabla\varphi, \qquad \mathbf{b}_N \in K^{\infty}_{\underline{\delta}^{\mathrm{neu}}-2}(\Omega)^2 \cap H^1(\Omega)^3, \qquad \varphi \in K^{\infty}_{\underline{\delta}^{\mathrm{dir}}-2}(\Omega) \cap H^1_0(\Omega). \tag{3.11}$$

Here, similarly to (2.16)–(2.17), $\underline{\delta}^{\mathrm{neu}}$ are the minimal singularity exponents for the Laplacian with Neumann boundary conditions; see [7, Section 6]. Note that $\mathbf{b}^{\mathrm{sing}} \notin H^1(\Omega)^3$.

As in [7, Section 7 and Section 8], it is possible to construct an approximation $\mathbf{c}^{\mathrm{sing}} \in X^h_{\underline{\gamma}}$ such that

$$\|\mathbf{b}^{\mathrm{sing}} - \mathbf{c}^{\mathrm{sing}}\|_{X_{\underline{\gamma}}} \leq Ch^{\tau_2},$$

for a parameter $\tau_2 \in (0,1)$, provided that we have $k \geq k_0$ for a sufficiently large threshold value k_0. The restrictions on the polynomial degree k are due to the proof in [7, Section 7] where it is necessary to construct H^2-conforming interpolants of the gradient components of the magnetic singularities. However, our numerical results in two dimensions indicate that positive convergence rates are already achieved for $k \geq 2$ on rectangular grids.

In the model situation described above, we have the following result.

Corollary 3.6 *Under the above assumptions and those in Theorem 3.3, there holds*

$$\|\!|\!| (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h, p - p_h) |\!|\!|_{\underline{\gamma}} \leq Ch^{\min\{\tau_1,\tau_2\}}.$$

The constant $C > 0$ is independent of the mesh-size h.

Similar convergence results are obtained in the two-dimensional case.

4 Numerical Results

In this section, we present several numerical experiments for the linear two-dimensional Oseen-type MHD problem

$$-R_e^{-1}\,\Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p + S\,\mathbf{d}\times\operatorname{curl}\mathbf{b} = \mathbf{f} \qquad \text{in } \Omega\subset\mathbb{R}^2, \tag{4.1}$$

$$R_m^{-1}\,S\,\mathbf{curl}(\operatorname{curl}\mathbf{b}) - S\,\mathbf{curl}(\mathbf{u}\times\mathbf{d}) = \mathbf{g} \qquad \text{in } \Omega, \tag{4.2}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{4.3}$$

$$\operatorname{div}\mathbf{b} = 0 \qquad \text{in } \Omega, \tag{4.4}$$

where $\mathbf{d}$ is a given smooth magnetic field and $\mathbf{w}$ a smooth flow field. Problems of this type arise in each step of the Picard iteration in Remark 3.2. On non-convex domains, problem (4.1)–(4.4) already exhibits magnetic singularities with regularity below $H^1(\Omega)^2$. Hence, it is well suited to test the performance of the proposed finite element method.

We approximate (4.1)–(4.4) using the two-dimensional analogue of the Taylor-Hood family (3.10) on square meshes. Our implementation is based on the finite element library `deal.II`; see [3,2]. It provides powerful C^{++} classes for handling meshes and degrees of freedom, and for solving the resulting linear systems of equations. In our experiments, we have solved these systems by BICGSTAB, using a simple Jacobi preconditioner. While this worked well in our examples, we point out that the systematic design and analysis of efficient solvers for the weighted regularization approach proposed in this paper remain open issues. For comprehensive discussions of efficient preconditioning and solution techniques for incompressible Navier-Stokes discretizations in the absence of electro-magnetic effects, we refer the reader to, e.g., [11,10,28] and the references therein.

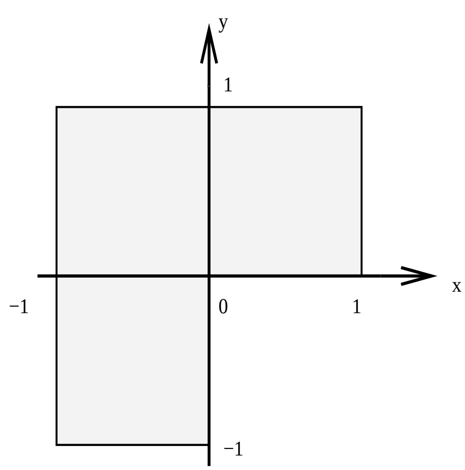

Fig. 1. L-shaped domain Ω.

Throughout, we consider the L-shaped domain Ω with opening angle $3\pi/2$ shown in Figure 1. We always set $R_e = R_m = S = 1$ in (4.1)–(4.4), and prescribe the right-hand sides $\mathbf{f}$, $\mathbf{g}$, as well as the field $\mathbf{d}$ and $\mathbf{w}$. Furthermore, we allow for non-homogeneous Dirichlet boundary conditions for $\mathbf{u}$ and $\mathbf{b}\cdot\mathbf{t}$ on the boundary $\partial\Omega$ of Ω. These conditions are taken into account in the usual fashion by interpolating the boundary data at the corresponding nodal degrees of freedom. We always take $D = 1$ and choose the weight function $\omega_{\underline{\gamma}}$ in the bilinear form a_m as $\omega_\gamma(\mathbf{x}) = |\mathbf{x}|^\gamma$, with a parameter $0 \leq \gamma \leq 1$ that we are varying in our experiments. We point out that, for the linear MHD problem in (4.1)–(4.4), the theoretical results of the previous sections hold without any restrictions on γ.

4.1 Smooth Solution

In our first experiment, we validate the a priori error bounds in Corollary 3.5 for a smooth solution. We solve the problem (4.1)–(4.4) with $\mathbf{w} = \mathbf{d} = (1, 1)$, and with $\mathbf{f}$, $\mathbf{g}$ and the boundary data chosen so that the exact solution $(\mathbf{u}, \mathbf{b}, p) = (u_1, u_2, b_1, b_2, p)$ is given by

$$\begin{aligned} u_1(x_1, x_2) &= -e^{x_1}(x_2 \cos(x_2) + \sin(x_2)), \\ u_2(x_1, x_2) &= e^{x_1} x_2 \sin(x_2), \\ b_1(x_1, x_2) &= -e^{x_1}(x_2 \cos(x_2) + \sin(x_2)), \\ b_2(x_1, x_2) &= e^{x_1} x_2 \sin(x_2), \\ p(x_1, x_2) &= 2e^{x_1} \sin(x_2). \end{aligned} \tag{4.5}$$

Note that $\mathbf{u}$, $\mathbf{b}$, and the corresponding right-hand side $\mathbf{g}$ are solenoidal.

We compute finite element approximations to this MHD solution using two-dimensional $Q_2^2 - Q_2^2 - Q_1$ Hood-Taylor elements on a sequence of successively refined square meshes $\{\mathcal{T}_i\}_{i\geq 1}$, referring to the index i as cycle i. The number of elements in the mesh $\mathcal{T}_i$ is proportional to 2^{2i}; the mesh-size h_i of $\mathcal{T}_i$ is thus proportional to 2^{-i}. If e_i denotes the error in some component of the approximation on cycle i (in a suitable norm), the corresponding numerical rate of convergence is given by

$$r_i = \frac{\log(e_i/e_{i-1})}{\log(h_i/h_{i-1})}.$$

In Table 1 and Table 2, we show the errors in the indicated norms for the hydrodynamic variables $(\mathbf{u}, p)$ and the magnetic field $\mathbf{b}$, respectively, obtained with exponents $\gamma = 0$, $\gamma = 0.5$, and $\gamma = 1$. We also list the number of degrees of freedom (dofs) for each of the solution components. For all choices of γ, the rates in the H^1-error in $\mathbf{u}$, the L^2-error in p, and in the $X_{\underline{\gamma}}$-error in $\mathbf{b}$ are of order two, in full agreement with the results of Corollary 3.5. As can be expected, the convergence rates in the L^2-errors of $\mathbf{u}$ and $\mathbf{b}$ are of one order higher and of optimal third order. The difference in the results with respect to the different values of γ is minimal and almost negligible, indicating that the weighted regularization term has no influence on the performance of the proposed method if the solution is smooth.

γ	cycle	dofs in $\mathbf{u}/p$	L^2-error in $\mathbf{u}$		H^1-error in $\mathbf{u}$		L^2-error in p	
0	1	130/21	5.77e-03	-	7.33e-02	-	3.06e-02	-
	2	450/65	7.07e-04	3.03	1.82e-02	2.01	8.02e-03	1.93
	3	1666/225	8.70e-05	3.01	4.55e-03	2.00	2.13e-03	1.91
	4	6402/833	1.10e-05	3.00	1.15e-03	2.00	5.50e-04	1.96
	5	25090/3201	1.37e-06	3.00	2.84e-04	2.00	1.40e-04	1.98
0.5	1	130/21	5.79e-03	-	7.32e-02	-	3.34e-02	-
	2	450/65	7.08e-04	3.03	1.82e-02	2.01	8.49e-03	1.98
	3	1666/225	8.79e-05	3.01	4.55e-03	2.00	2.19e-03	1.95
	4	6402/833	1.10e-05	3.00	1.14e-03	2.00	5.55e-04	1.98
	5	25090/3201	1.37e-06	3.00	2.84e-04	2.00	1.40e-04	1.99
1	1	130/21	5.77e-03	-	7.33e-02	-	3.05e-02	-
	2	450/65	7.07e-04	3.03	1.82e-02	2.01	8.02e-03	1.93
	3	1666/225	8.79e-05	3.01	4.55e-03	2.00	2.14e-03	1.91
	4	6402/833	1.10e-05	3.00	1.14e-03	2.00	5.50e-04	1.96
	5	25090/3201	1.37e-06	3.00	2.85e-04	2.00	1.39e-04	2.00

Table 1
Smooth solution: Errors and convergence rates in $(\mathbf{u}, p)$.

4.2 Non-Smooth Solution

Next, we consider the MHD problem (4.1)–(4.4) where all the solution components have corner singularities at the origin. We set again $\mathbf{w} = \mathbf{d} = (1, 1)$, and choose the data so that the magnetic field $\mathbf{b}$ is given by the strongest singularity of the curl-curl operator for the L-shaped domain in Figure 1, namely

$$\mathbf{b}(\mathbf{x}) = \nabla(r^{2/3} \sin(2\phi/3)), \tag{4.6}$$

with (r, ϕ) denoting the standard polar coordinates. Evidently, $\operatorname{curl} \mathbf{b} = 0$ and $\operatorname{div} \mathbf{b} = 0$. We also point out that $\mathbf{b} \notin H^1(\Omega)^2$. The hydrodynamic pair $(\mathbf{u}, p)$ is taken to be the strongest corner singularity of the Stokes operator for the

γ	cycle	dofs in $\mathbf{b}$	L^2-error in $\mathbf{b}$		$X_{\underline{\gamma}}$-error in $\mathbf{b}$	
0	1	130	5.69e-03	-	7.10e-02	-
	2	450	7.04e-04	3.01	1.81e-02	1.99
	3	1666	8.79e-05	3.00	4.53e-03	2.00
	4	6402	1.10e-05	3.00	1.13e-03	2.00
	5	25090	1.49e-06	2.89	2.84e-04	2.00
0.5	1	130	5.89e-03	-	7.14e-02	-
	2	450	7.23e-04	3.03	1.79e-02	1.99
	3	1666	8.92e-05	3.02	4.51e-03	1.99
	4	6402	1.11e-05	3.01	1.13e-03	2.00
	5	25090	1.38e-06	3.01	2.83e-04	2.00
1	1	130	5.89e-03	-	7.14e-02	-
	2	450	7.23e-04	3.03	1.79e-02	1.99
	3	1666	8.92e-05	3.02	4.51e-03	1.99
	4	6402	1.11e-05	3.01	1.13e-03	2.00
	5	25090	1.89e-06	2.55	2.83e-04	2.00

Table 2
Smooth solution: Errors and convergence rates in $\mathbf{b}$.

L-shaped domain in Figure 1. This singularity is given by

$$\begin{aligned} u_1(x_1,x_2) &= r^{\lambda}((1+\lambda)\sin(\phi)\psi(\phi)+\cos(\phi)\psi'(\phi)), \\ u_2(x_1,x_2) &= r^{\lambda}(-(1+\lambda)\cos(\phi)\psi(\phi)+\sin(\phi)\psi'(\phi)), \\ p(x_1,x_2) &= -r^{\lambda-1}((1+\lambda)^2\psi'(\phi)+\psi'''(\phi))/(1-\lambda), \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} \psi(\phi) &= \sin((1+\lambda)\phi)\cos(\lambda w)/(1+\lambda) - \cos((1+\lambda)\phi) \\ &\quad - \sin((1-\lambda)\phi)\cos(\lambda w)/(1-\lambda) + \cos((1-\lambda)\phi). \end{aligned}$$

The exponent λ is the smallest positive solution of $\sin(\lambda 3\pi/2)+\lambda\sin(3\pi/2) = 0$, which is $\lambda \approx 0.54448373678246$. Note that $\mathbf{u}$ is solenoidal, and that $(\mathbf{u},p) \in H^{1+\lambda}(\Omega)^2 \times H^{\lambda}(\Omega)$. We consider the same sequence of meshes as in Section 4.1.

The performance of the proposed method is shown in Table 3 and Table 4; it now strongly depends on the choice of the exponent γ. The choice $\gamma = 0$ (no regularization) does not lead to convergent FEM solutions and the errors in the field $\mathbf{b}$ do not decrease at all. This is due to the fact that for $\gamma = 0$ the space $H_{\underline{\gamma}}(\Omega)$ is known to be a closed subspace of $X_{\underline{\gamma}}(\Omega)$ and that the singular solution (4.6) lies in the complement $X_{\underline{\gamma}}(\Omega) \setminus H_{\underline{\gamma}}(\Omega)$; see [7]. Hence, for $\gamma = 0$ it is impossible to correctly capture the singular magnetic field. The tables clearly show convergence of the method for $\gamma = 0.5$ and $\gamma = 1$ when the weighted regularization is switched on, hereby confirming the results of Corollary 3.4 and Corollary 3.6. The convergence rates in the H^1-norm for $\mathbf{u}$ and the L^2-norm for p are of the expected order λ. The L^2-norm in $\mathbf{u}$ converges with twice that order. In contrast to the restrictions on the polynomial degree in the theoretical results, good convergence rates for $\mathbf{b}$ are already obtained for quadratic elements. For γ close to one, this order is close to the order 2/3; cf. the discussion in [7].

We point out that the errors are only slightly better with increased approximation order k, and that the rates remain comparable. In all our tests we started to observe convergence as soon as $\gamma > 1/3$, the lower critical bound in (2.21) that is required for convergence. However, the best results in the $X_{\underline{\gamma}}$-norm and L^2-norm were always obtained with the upper bound $\gamma = 1$.

It is evident that the weighted regularization term is needed to correctly capture the singular behavior of the solution.

γ	cycle	dofs in $\mathbf{u}/p$	L^2-error in $\mathbf{u}$		H^1-error in $\mathbf{u}$		L^2-error in p	
0	1	130/21	1.54e-01	-	1.55e+00	-	2.71e+00	-
	2	450/65	6.55e-02	1.23	1.11e+00	0.49	1.50e+00	0.85
	3	1666/225	2.89e-02	1.18	7.65e-01	0.53	1.12e+00	0.43
	4	6402/833	1.52e-02	0.92	5.27e-01	0.54	9.25e-01	0.27
	5	25090/3201	1.02e-02	0.57	3.62e-01	0.54	8.34e-01	0.15
0.5	1	130/21	1.51e-01	-	1.55e+00	-	2.84e+00	-
	2	450/65	6.31e-02	1.26	1.09e+00	0.49	1.51e+00	0.90
	3	1666/225	2.60e-02	1.28	7.58e-01	0.53	1.01e+00	0.58
	4	6402/833	1.15e-02	1.18	5.21e-01	0.54	6.97e-01	0.54
	5	25090/3201	5.63e-03	1.03	3.58e-01	0.54	4.90e-01	0.51
1	1	130/21	1.49e-01	-	1.53e+00	-	2.95e+00	-
	2	450/65	6.15e-02	1.28	1.00e+00	0.49	1.58e+00	0.90
	3	1666/225	2.46e-02	1.32	7.53e-01	0.53	1.04e+00	0.61
	4	6402/833	1.01e-02	1.28	5.19e-01	0.54	6.96e-01	0.57
	5	25090/3201	4.34e-03	1.22	3.56e-01	0.54	4.73e-01	0.56

Table 3
Non-smooth solution: Errors and convergence rates in $(\mathbf{u}, p)$.

γ	cycle	dofs in $\mathbf{b}$	L^2-error in $\mathbf{b}$		$X_{\underline{\gamma}}$-error in $\mathbf{b}$	
0	1	130	7.31e-01	-	1.06e+00	-
	2	450	7.07e-01	0.05	1.04e+00	0.03
	3	1666	6.96e-01	0.02	1.04e+00	0.01
	4	6402	6.90e-01	0.01	1.03e+00	0.01
	5	25090	6.86e-01	0.01	1.03e+00	0.00
0.5	1	130	5.00e-01	-	8.34e-01	-
	2	450	4.08e-01	0.29	7.37e-01	0.18
	3	1666	3.37e-01	0.28	6.61e-01	0.16
	4	6402	2.79e-01	0.27	5.95e-01	0.15
	5	25090	2.30e-01	0.28	5.36e-01	0.15
1	1	130	4.30e-01	-	5.88e-01	-
	2	450	3.35e-01	0.36	4.16e-01	0.50
	3	1666	2.56e-01	0.39	2.93e-01	0.51
	4	6402	1.85e-01	0.47	1.97e-01	0.57
	5	25090	1.26e-01	0.56	1.28e-01	0.62

Table 4
Non-smooth solution: Errors and convergence rates in $\mathbf{b}$.

5 Conclusions

In this paper, we have introduced and analyzed a new finite element method for incompressible MHD problems in polygonal and polyhedral domains. The method employs nodal elements for the discretization of the magnetic fields and standard inf-sup elements for the hydrodynamic variables. In order to account for singular solution behavior, the magnetic bilinear form has been modified using the weighted regularization technique recently developed in [7]. The analysis in this paper shows that this approach leads to convergent schemes in non-convex domains. Our two-dimensional numerical results on an L-shaped domain confirm that the weighted regularization approach is indispensable for the numerical resolution of singular solution components in the magnetic fields.

We finally point out that this paper is mainly concerned with the discretization of the elliptic operator underlying the MHD problems under consideration. For strongly convection-dominated problems, additional stabilization techniques might be necessary to ensure the robustness of the schemes; see, e.g., [13] and the references therein.

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