Pacific Institute for the Mathematical Sciences

Anti-selfdual Hamiltonians: Variational resolutions for Navier-Stokes and other nonlinear evolutions

Nassif Ghoussoub
Department of Mathematics
University of British Columbia
Vancouver, BC, Canada

Preprint number: PIMS-05-5
Received on March 24, 2005
Revised on April 1, 2005

Anti-selfdual Hamiltonians: Variational resolutions for Navier-Stokes and other nonlinear evolutions

Nassif Ghoussoub*
Department of Mathematics, University of British Columbia,
Vancouver BC Canada V6T 1Z2
nassif@math.ubc.ca

April 5, 2005

Abstract

The theory of anti-selfdual (ASD) Lagrangians developed in [8] allows a variational resolution for equations of the form $\Lambda u + Au + \partial\varphi(u) + f = 0$ where φ is a convex lower-semi-continuous function on a reflexive Banach space X, $f \in X^*$, $A : D(A) \subset X \to X^*$ is a positive linear operator and where $\Lambda : D(\Lambda) \subset X \to X^*$ is a non-linear operator that satisfies suitable continuity and anti-symmetry properties. ASD Lagrangians on path spaces also yield variational resolutions for nonlinear evolution equations of the form $\dot{u}(t) + \Lambda u(t) + Au(t) + f \in -\partial\varphi(u(t))$ starting at $u(0) = u_0$. In both stationary and dynamic cases, the equations associated to the proposed variational principles are not derived from the fact they are critical points of the action functional, but because they are also zeroes of the Lagrangian itself. For that we establish a general –and remarkably encompassing– nonlinear variational principle which has many applications, in particular to Navier-Stokes type equations. More applications, especially to the differential systems of magnetohydrodynamics and thermohydraulics will be given in a forthcoming paper [9].

Contents

*Research partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The author gratefully acknowledges the hospitality and support of the Centre de Recherches Mathématiques in Montréal and the Université de Nice where this work was initiated.

1 Introduction

A new variational framework was developed in [8] where solutions of various equations, not normally of Euler-Lagrange type, can still be obtained as minima of functionals of the form

$$I(u) = L(u, Au) + \ell(b_1(x), b_2(x)) \quad \text{or} \quad I(u) = \int_0^T L(t, u(t), \dot{u}(t) + Au(t))dt + \ell(u(0), u(T)).$$

The Lagrangians L (resp., ℓ) must obey certain anti-selfdual (resp., selfdual) conditions, while the operators A are essentially skew-adjoint modulo boundary terms represented by a pair of operators (b_1, b_2). For such "anti-selfdual" (ASD) Lagrangians, the minimal value will always be zero and –just like the self (and antiself) dual equations of quantum field theory (e.g. Yang-Mills and others)– the equations associated to such minima are not derived from the fact they are critical points of the functional I, but because they are also zeroes of the Lagrangian L itself. In other words, the solutions will satisfy

$$L(u, Au) + \langle u, Au \rangle = 0 \quad \text{and} \quad L(t, u(t), \dot{u}(t) + A_t u(t)) + \langle u(t), \dot{u}(t) \rangle = 0.$$

It is also shown in [8] that ASD Lagrangians possess remarkable permanence properties making them more prevalent than expected and quite easy to construct and/or identify. The variational game changes from the analytical proofs of existence of extremals for general action functionals, to a more algebraic search of an appropriate ASD Lagrangian for which the minimization problem is remarkably simple with value always equal to zero. This makes them efficient new tools for proving –variationally– existence and uniqueness results for a large array of differential equations.

In this paper, we tackle boundary value problems of the form:

$$\begin{cases} -\Lambda u - Au + f & \in & \partial\varphi(u) \\ b_1(u) & = & 0 \end{cases} \tag{1}$$

as well as parabolic evolution equations of the form:

$$\begin{cases} -\dot{u}(t) - \Lambda u(t) - Au(t) & \in & \partial\varphi(t, u(t)) & \text{a.e. } t \in [0, T] \\ b_1(u(t)) & = & b_1(u_0) & \text{a.e } t \in [0, T] \\ u(0) & = & u_0 \end{cases} \tag{2}$$

where u_0 is a given initial value. Here φ is a convex lower semicontinuous functional, Λ is a non-linear "conservative" operator, A is a linear not necessarily bounded but essentially skew-adjoint operator modulo the operators (b_1, b_2), a notion to be defined below.

As applications to the method, we provide a variational resolution to equations involving nonlinear operators such as the Navier-Stokes equation for a fluid driven by its boundary:

$$\begin{cases} (u \cdot \nabla)u + f & = & \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u & = & 0 \quad \text{on } \Omega \\ u & = & u^0 \quad \text{on } \partial\Omega \end{cases}$$

where $u^0 \in H^{3/2}(\partial\Omega)$ is such that $\int_{\partial\Omega} u^0 \cdot \mathbf{n}\, d\sigma = 0$, $\nu > 0$ and $f \in L^p(\Omega; \mathbb{R}^3)$.

We can also deal with the superposition of such non-linear operators with non self-adjoint first order operators such as linear transport maps:

$$\begin{cases} (u \cdot \nabla)u + \vec{a} \cdot \nabla u + a_0 u + |u|^{m-2}u + f & = & \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u & = & 0 \quad \text{on } \Omega \\ u & = & 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\vec{a} \in C^\infty(\bar{\Omega})$ is a smooth vector field and $a_0 \in L^\infty$ are such that $a_0 - \frac{1}{2}\text{div}(a) \geq 0$.

The methods extend to the dynamic case where typically we give a variational resolution to the Navier-Stokes evolution

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - f & = & \nu\Delta u - \nabla p \quad \text{on } [0, T] \times \Omega \\ \text{div} u & = & 0 \quad \text{on } [0, T] \times \Omega \\ u(t, x) & = & 0 \quad \text{on } [0, T] \times \partial\Omega \\ u(0, x) & = & u_0(x) \quad \text{on } \Omega. \end{cases}$$

The key to our approach is again a minimization principle for functionals of the form $I(x) = L(x, \Lambda x) + \langle x, \Lambda x\rangle$ where L is an anti-selfdual Lagrangian, however Λ is now a fairly general non-linear operator with suitable continuity and symmetry properties.
The paper, though sufficiently self-contained, is better read in conjunction with [8]. It is organized as follows: In section 2, we introduce the concept of anti-selfdual Hamiltonian which is the appropriate dual notion to anti-selfdual Lagrangians. In section 3, we give the main non-linear variational principle, which is applied in section 4 to obtain variational proofs for the existence of stationary solutions for various nonlinear equations. In section 5, we deal with the dynamic case where we provide a variational resolution to several nonlinear parabolic initial-value problems, including those appearing in the basic models of hydrodynamics. Further applications will follow in the forthcoming paper [9].

2 Basic properties of Anti-selfdual Hamiltonians

Definition 2.1 *Let X be a reflexive Banach space. Say that a functional $H : X \times X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is an anti-selfdual Hamiltonian if for each $y \in X$, the function $x \to -H(x, y)$ from X to $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is convex and the function $x \to H(-y, -x)$ is its convex lower semi-continuous envelope.*

It readily follows that for an ASD Hamiltonian H, the function $y \to H(x, y)$ is convex and lower semi-continuous for each $x \in X$, and that the following inequality holds for every $(x, y) \in X \times X$,

$$H(-y, -x) \le -H(x, y), \tag{3}$$

In particular, we have for every $x \in X$,

$$H(x, -x) \le 0. \tag{4}$$

The class of anti-selfdual Hamiltonian on a space X, will be denoted by $\mathcal{H}_{\rm AD}(X)$. The most basic ASD Hamiltonian is $H(x, y) = \|y\|^2 - \|x\|^2$ (Maxwell's Hamiltonian) or more generally $H(x, y) = \varphi(-y) - \varphi(x)$ where φ is any finite convex lower semi-continuous function on X. More generally, if $B : X \to X^*$ is a skew-adjoint bounded linear operator, $f \in X^*$, and if $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is proper convex and lower semi-continuous, then

$$H(x, y) = \begin{cases} \varphi(-y) - \varphi(x) - \langle Bx, y\rangle + \langle f, x + y\rangle & \text{if } x \in \mathrm{Dom}(\varphi) \\ -\infty & \text{if } x \notin \mathrm{Dom}(\varphi) \end{cases} \tag{5}$$

is also an anti-selfdual Hamiltonian. We define the (partial) domain of H to be

$$\mathrm{Dom}_1(H) = \{x \in X; H(x, y) > -\infty \text{ for all } y \in X\}. \tag{6}$$

Note that if φ is a convex lower semi-continuous function that is bounded below on X, and if $H(x, y) = \varphi(-y) - \varphi(x)$ is the anti-selfdual Hamiltonian associated to φ, then $\mathrm{Dom}_1 H = \mathrm{Dom}\varphi$. Note also that for any $z \in \mathrm{Dom}_1(H)$, we have that the function $\varphi_z : x \to -H(x, -z)$ is convex and valued in $\mathbb{R} \cup \{+\infty\}$. Moreover $\mathrm{Dom}_1(H) \subset \mathrm{Dom}(\varphi_z)$, hence for any $z, y \in \mathrm{Dom}_1(H)$,

$$H(z, -y) = -H(y, -z) \text{ if and only if the function } x \to H(x, -z) \text{ is upper semi-continuous at } y.$$

We can now introduce the following

Definition 2.2 *Say that an Anti-selfdual Hamiltonian $H : X \times X \to \mathbb{R}$ is tempered if for every $y \in \mathrm{Dom}_1(H)$, the function $x \to H(x, -y)$ is concave and upper semi-continuous from X to $\mathbb{R} \cup \{-\infty\}$.*

It then follows that

$$H(y, x) = -H(-x, -y) \text{ for all } (x, y) \in X \times \mathrm{Dom}_1(H). \tag{7}$$

and therefore

$$H(x, -x) = 0 \quad \text{for all } x \in \mathrm{Dom}_1(H). \tag{8}$$

The class of tempered anti-selfdual Hamiltonian on a space X, will be denoted by $\mathcal{H}_{\rm TAD}(X)$.
The most basic tempered ASD Hamiltonian is $H(x, y) = \varphi(y) - \varphi(-x) + \langle x, By\rangle + \langle f, x + y\rangle$ where φ is any finite convex lower semi-continuous function on X, $f \in X^*$, and where $B : X \to X^*$ is a skew-adjoint bounded linear operator. Tempered ASD Hamiltonians satisfy some obvious permanence properties that we summarize in the following proposition.

Proposition 2.1 *Let X be a reflexive Banach space, then the following holds:*

1. *If H and K are in $\mathcal{H}_{\rm TAD}(X)$ and $\lambda > 0$, then the Hamiltonians $H + K$ (defined as $-\infty$ if the first variable is not in $\mathrm{Dom}_1(H) \cap \mathrm{Dom}_1(K)$), and $\lambda{\cdot}H$ also belong to $\mathcal{H}_{\rm TAD}(X)$.*

2. *If $H_i \in \mathcal{H}_{\rm TAD}(X_i)$ where X_i is a reflexive Banach space for each $i \in I$, then the Hamiltonian $H := \Sigma_{i\in I} H_i$ defined by $H((x_i)_i, (y_i)_i) = \Sigma_{i\in I} H_i(x_i, y_i)$ is in $\mathcal{H}_{\rm TAD}(\Pi_{i\in I} X_i)$.*

3. *If $H \in \mathcal{H}_{\rm TAD}(X)$ and $B : X \to X^*$ is a skew-adjoint bounded linear operator then the Hamiltonian H_B defined by $H_B(x, y) = H(x, y) + \langle Bx, y\rangle$ is also in $\mathcal{H}_{\rm TAD}(X)$.*

4. *If $H \in \mathcal{H}_{\rm TAD}(X)$ and $K \in \mathcal{H}_{\rm TAD}(Y)$, then for any bounded linear operator $A : X \to Y^*$, the Hamiltonian $H +_A K$ defined by*

$$(H +_A K)((x, y), (z, w)) = H(x, z) + K(y, w) + \langle A^*y, z\rangle - \langle Ax, w\rangle$$

belongs to $\mathcal{H}_{\rm TAD}(X \times Y)$.

5. *If φ is a proper convex lower semi-continuous function on $X \times Y$ and A is any bounded linear operator $A : X \to Y^*$, then the Hamiltonian $H_{\varphi,A}$ defined by*

$$H_{\varphi,A}((x, y), (z, w)) = \varphi(-z, -w) - \varphi(x, y) + \langle A^*y, z\rangle - \langle Ax, w\rangle$$

also belongs to $\mathcal{H}_{\rm TAD}(X \times Y)$

This notion is in a certain sense dual to the notion of anti-selfdual Lagrangian introduced and developed in [8]. Indeed, let $\mathcal{L}(X)$ be the class of convex Lagrangians on a reflexive Banach space X. These are all functions $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ which are convex and lower semi-continuous (in both variables) and which are not identically $+\infty$. The (partial) domain of the Lagrangian L is defined as

$$\mathrm{Dom}_1(L) = \{x \in X; L(x, p) < +\infty \text{ for some } p \in X^*\}. \tag{9}$$

To each Lagrangian L on $X \times X^*$, we can associate its *Hamiltonian* on $X \times X$ defined as the Legendre transform in the second variable, i.e.,

$$H_L(x, y) := \sup\{\langle p, y\rangle - L(x, p); p \in X^*\}.$$

It is clear that $\mathrm{Dom}_1(L) = \mathrm{Dom}_1(H_L)$.
The Legendre-Fenchel dual (in both variables) of L is defined at any pair $(q, y) \in X^* \times X$ by:

$$L^*(q, y) = \sup\{\langle q, x\rangle + \langle y, p\rangle - L(x, p);\ x \in X, p \in X^*\}.$$

We recall the notion of anti-selfdual Lagrangians developed in [8].

Definition 2.3 Let L be a Lagrangian in $\mathcal{L}(X)$. We say that
(1) L is an *anti-self dual Lagrangian* on $X \times X^*$, if

$$L^*(p, x) = L(-x, -p) \quad \text{for all } (p, x) \in X^* \times X. \tag{10}$$

(2) L is *anti-self dual on the graph of* Λ, the latter being a map from a subset $D \subset X$ into X^*, if

$$L^*(\Lambda x, x) = L(-x, -\Lambda x) \quad \text{for all } x \in D. \tag{11}$$

We denote by $\mathcal{L}_{\rm AD}(X)$ the class of ASD-Lagrangians. We now procced to identify the class of Hamiltonians associated to ASD-Lagrangians. We denote by K_2^* (resp., K_1^*) the Legendre dual of a functional $K(x, y)$ with respect to the second variable (resp., the first variable), we have the following

Proposition 2.2 *Let L be an ASD Lagrangian on a reflexive Banach space X, then its corresponding Hamiltonian $H = H_L$ is anti-selfdual.*

Proof: Since a Lagrangian $L \in \mathcal{L}(X)$ is convex in both variables, its corresponding Hamiltoninan H_L is always concave in the first variable. Also note that the Legendre transform of $-H_L(\cdot, y)$ with respect to the first variable is related to the Legendre transform in both variables of its Lagrangian in the following way.

$$\begin{aligned}(-H_L)_1^*(p,y) &= \sup\{\langle p,x\rangle + H_L(x,y); x\in X\}\\ &= \sup\{\langle p,x\rangle + \sup\{\langle y,q\rangle - L(x,q); q\in X^*\}; x\in X\}\\ &= L^*(p,y).\end{aligned}$$

If now L is an ASD Lagrangian, then the convex lower semi-continuous envelope of the function $x \to -H_L(x,y)$ (i.e., the largest convex lower semi-continuous function below the function $x \to -H_L(x,y)$) is

$$\begin{aligned}(-H_L)_1^{**}(x,y) &= \sup\{\langle p,x\rangle - (-H_L)_1^*(p,y); p\in X^*\}\\ &= \sup\{\langle p,x\rangle - L^*(p,y); p\in X^*\}\\ &= \sup\{\langle p,x\rangle - L(-y,-p); p\in X^*\}\\ &= H_L(-y,-x).\end{aligned}$$

Note that a characterization of anti-selfdual Hamiltonian that correspond to an ASD Lagrangian (i.e., $H = H_L$ for some $L \in \mathcal{L}_{\rm AD}(X)$) is that

$$H_2^*(-x,-p) = (-H)_1^*(p,x),$$

for each $(x,p) \in X \times X^*$. In this case, the corresponding ASD Lagrangian is nothing else but $L(x,p) := H_2^*(x,p) = (H)_1^*(-p,-x)$.

As mentioned above since a Lagrangian $L \in \mathcal{L}(X)$ is convex in both variables, then its corresponding Hamiltoninan H_L is always concave in the first variable. However, H_L is not necessarily upper semi-continuous in the first variable, even if L is an anti-selfdual Lagrangian. This leads to the following notion.

Definition 2.4 *A Lagrangian $L \in \mathcal{L}(X)$ will be called tempered if for each $y \in \mathrm{Dom}_1(H)$, the map $x \to H(x,-y)$ from X to $\mathbb{R} \cup \{-\infty\}$ is upper semi-continuous.*

A typical tempered Lagrangian (resp., tempered ASD-Lagrangian) is $L(x,p) = \varphi(x) + \psi^*(p)$ (resp., $L(x,p) = \varphi(x) + \varphi^*(-p)$) where φ and ψ are convex and lower semi-continuous on X. We let $\mathcal{L}_T(X)$ denote the class of tempered Lagrangians and $\mathcal{L}_{TAD}(X)$ the class of tempered ASD Lagrangians on X.
We now recall from [8] a few of the operations defined on the class of Lagrangians $\mathcal{L}(X)$ and study the permanence properties of the class $\mathcal{L}_{\rm TAD}(X)$ of tempered ASD Lagrangians.

- *Addition:* If $L, M \in \mathcal{L}(X)$, define the Lagrangian $L \oplus M$ on $X \times X^*$ by:

$$(L \oplus M)(x,p) = \inf\{L(x,r) + M(x,p-r); r \in X^*\}$$

- *Convolution:* If $L, M \in \mathcal{L}(X)$, define the Lagrangian $L \star M$ on $X \times X^*$ by:

$$(L \star M)(x,p) = \inf\{L(z,p) + M(x-z,p); z \in X\}$$

- *Right operator shift:* If $L \in \mathcal{L}(X)$ and $B : X \to X^*$ is a bounded linear operator, define the Lagrangian L_B on $X \times X^*$ by

$$L_B(x,p) := L(x, Bx+p).$$

Lemma 2.5 *Let X be a reflexive Banach space, then the following hold:*

1. *If L and M are two Lagrangians in $\mathcal{L}(X)$, then $H_{L\oplus M}(x,y) = H_L(x,y) + H_M(x,y)$, where H_L and H_M denote the corresponding Hamiltonians.*

2. *If L and M are in $\mathcal{L}_{\rm AD}(X)$, then $L^* \oplus M^*(q,y) = L \star M(-y,-q)$ for every $(y,q) \in X \times X^*$.*

3. *If L is an ASD Lagrangian and M is of the form $M(x,p) = \varphi(x) + \varphi^*(-p)$ for some convex l.s.c. function φ, then $(L \oplus M)^* = L^* \star M^*$ and $(L \star M)^* = L^* \oplus M^*$.*

Proof: (1) and (2) are straightforward, while (3) was established in [8]. It follows that the λ-regularization of an ASD Lagrangian L, that is $L_\lambda := L \star T_\lambda$ where $T_\lambda(x,p) = \frac{\lambda^2\|x\|^2}{2} + \frac{\|p\|^2}{2\lambda^2}$, is also an ASD Lagrangian.

We shall see later that not all ASD Lagrangians are automatically tempered. This lemma shows that it is the case under certain coercivity conditions.

Proposition 2.3 *Let L be an ASD Lagrangian on a reflexive Banach space X. If for some $p_0 \in X$ and $\alpha > 1$, we have that $L(x,p_0) \leq C(1+\|x\|^\alpha)$ for all $x \in X$, then L belongs to $\mathcal{L}_{TAD}(X)$.*

Proof: Note that in this case, we readily have that $\mathrm{Dom}_1(L) = \mathrm{Dom}_1(H_L) = X$.
Assume first that $\lim\limits_{\|x\|+\|p\|\to+\infty} \frac{L(x,p)}{\|x\|+\|p\|} = \infty$, and write

$$\begin{aligned} H_L(x,y) &= \sup\{\langle p,y\rangle - L(x,p); p \in X^*\} \\ &= \sup\{\langle p,y\rangle - L^*(-p,-x); p \in X^*\} \\ &= \sup\{\langle p,y\rangle - \sup\{\langle -p,z\rangle + \langle -x,q\rangle - L(z,q);\ z\in X, q\in X^*\};\ p\in X^*\} \\ &= \sup\{\langle p,y\rangle + \inf\{\langle p,z\rangle + \langle x,q\rangle + L(z,q);\ z\in X, q\in X^*\};\ p\in X^*\} \\ &= \sup_{p\in X^*}\inf_{(z,q)\in X\times X^*}\{\langle p,y\rangle + \langle p,z\rangle + \langle x,q\rangle + L(z,q)\}. \end{aligned}$$

The function S defined on the product space $(X\times X^*)\times X^*)$ as

$$S((z,q),p) = \langle p, y+z\rangle + \langle x,q\rangle + L(z,q)$$

is convex and lower semi-continuous in the first variable (z,q) and concave and upper semi-continuous in the second variable p, hence in view of the coercivity condition, Von-Neuman's min-max theorem applies and we get:

$$\begin{aligned} H(x,y) &= \sup_{p\in X^*}\inf_{(z,q)\in X\times X^*}\{\langle p,y+z\rangle + \langle x,q\rangle + L(z,q)\} \\ &= \inf_{(z,q)\in X\times X^*}\sup_{p\in X^*}\{\langle p,y+z\rangle + \langle x,q\rangle + L(z,q)\} \\ &= \inf_{q\in X^*}\{\langle x,q\rangle + L(-y,q)\} \\ &= -H_L(-y,-x). \end{aligned}$$

It follows that L is tempered under the coercivity assumption.
Suppose now $L(x,p_0) \leq C(1+\|x\|^\alpha)$, and consider the λ-regularization of its conjugate L^*, that is $M_\lambda = L^* \star T^*_\lambda$ where $T^*_\lambda(x,p) = \frac{\|x\|^2}{2\lambda^2} + \frac{\lambda^2\|p\|^2}{2}$. Since obviously L^* is ASD on X^*, we get from Lemma 2.5.3 that M_λ is ASD on X^*. Moreover,

$$M_\lambda(p,x) = \inf\{L^*(q,x) + \frac{\|p-q\|^2}{2\lambda^2} + \frac{\lambda^2\|x\|^2}{2}; q\in X^*\} \leq L(x,p_0) + \frac{1}{2\lambda^2}\|p-p_0\|^2 + \frac{\lambda^2\|x\|^2}{2} \leq C_1 + C_2\|x\|^\beta + C_3\|p\|^2$$

which means that its dual M^*_λ is an ASD Lagrangian on X that is coercive in both variables. By the first part of the proof, $H_{M^*_\lambda}$ is a tempered ASD Hamiltonian on X. But in view of Lemma 2.5.1, we have $M^*_\lambda = L \oplus T_\lambda$ and therefore $H_{M^*_\lambda} = H_L + H_{T_\lambda}$. Consequently $x \to H_L(x,y)$ is upper semi-continuous and L itself is a tempered ASD Lagrangian.

By exploiting the duality between tempered ASD Lagrangians and ASD Hamiltonians, we get the following

Proposition 2.4 *The class $\mathcal{L}_{TAD}(X)$ possesses the following permanence properties.*

1. *If L and M are in $\mathcal{L}_{TAD}(X)$ and $\lambda > 0$, then the Lagrangians $L\oplus M$, and $\lambda{\cdot}L$ also belong to $\mathcal{L}_{TAD}(X)$.*
2. *If L is an ASD Lagrangian, then its λ-regularization $L_\lambda \in \mathcal{L}_{TAD}(X)$.*
3. *If $L \in \mathcal{L}_{TAD}(X)$ and $B : X \to X^*$ is a skew-adjoint operator, then L_B is also in $\mathcal{L}_{TAD}(X)$.*

Proof: They all follow from Proposition 2.1, Lemma 2.5 and Proposition 2.3. Note also that

$$H_{L_B}(x,y) = H_L(x,y) - \langle Bx, y\rangle.$$

Let now B be a linear –not necessarily bounded- map from its domain $D(B) \subset X$ into X^* such that $D(B)$ is dense in X, we consider the domain of its adjoint B^* which is defined as:

$$D(B^*) = \{x \in X; \sup\{\langle x, By\rangle; y \in D(B), \|y\|_X \le 1\} < +\infty\}.$$

Definition 2.6 *Say that*

1. *B is antisymmetric if $D(B) \subset D(B^*)$ and if $B^* = -B$ on $D(B)$.*
2. *B is skew-adjoint if it is antisymmetric and if $D(B) = D(B^*)$.*

We then have the following easy lemma (See also [10]).

Lemma 2.7 *Let $L : X \times X^* \to \mathbb{R}$ be an ASD Lagrangian on a reflexive Banach space X and let B be a linear skew-adjoint map from its domain $D(B) \subset X$ into X^* such that the function $x \to L(x,0)$ is bounded on the unit ball of X. The Lagrangian L_B defined by*

$$L_B(x,p) = \begin{cases} L(x, Bx + p) & \text{if } x \in D(B) \\ +\infty & \text{if } x \notin D(B) \end{cases}$$

is then itself anti-selfdual on X. Moreover, if L is tempered then so is L_B whose Hamiltonian is given by

$$H_B(x,y) = \begin{cases} H_L(x,y) - \langle Bx, y\rangle & \text{if } x \in D(B) \\ -\infty & \text{if } x \notin D(B) \end{cases}$$

We shall also deal with situations where operators are skew-adjoint provided one takes into account certain boundary terms. We consider the following notion introduced in [10].

Definition 2.8 Let B be a linear map from its domain $D(B)$ in a reflexive Banach space X into X^* and consider (b_1, b_2) to be a pair of linear maps from its domain $D(b_1, b_2)$ in X into the product of two Hilbert spaces $H_1 \times H_2$. Associate the set

$$D^*(B, b_1, b_2) = \left\{ y \in X; \sup\{\langle y, Bx\rangle - \frac{1}{2}(\|b_1(x)\|^2_{H_1} + \|b_2(x)\|^2_{H_2}); x \in S, \|x\|_X < 1\} < \infty \right\}.$$

- Say that B is *anti-symmetric modulo the boundary operators* (b_1, b_2) if the following properties are satisfied:
  1. The set $S = D(B) \cap D(b_1, b_2)$ is dense in X.
  2. The space $X_0 := Ker(b_1, b_2) \cap D(B)$ is dense in X.
  3. The image of S by (b_1, b_2) is dense in $H_1 \times H_2$.
  4. For every $x, y \in S$, we have $\langle y, Bx\rangle = -\langle By, x\rangle + \langle b_2(x), b_2(y)\rangle_{H_2} - \langle b_1(x), b_1(y)\rangle_{H_1}$.
- Say that B is *skew-adjoint modulo the boundary operators* (b_1, b_2) if it is *anti-symmetric modulo the boundary operators* (b_1, b_2) and if in addition $D^*(B, b_1, b_2) = D(B) \cap D(b_1, b_2)$.

It is clear that if b_1, b_2 are identically zero, then our definition coincides with the notions in Definition 2.5. For problems involving boundaries, we may start with an ASD Lagrangian L, but if the linear operator B is skew-adjoint modulo a term involving the boundary, the Lagrangian L_B is not ASD but we may recover anti-selfduality by adding a correcting term via a "Boundary Lagrangian" ℓ.

Definition 2.9 *We say that $\ell : H_1 \times H_2 \to \mathbb{R} \cup \{+\infty\}$ is a self-dual boundary Lagrangian if*

$$\ell^*(-h_1, h_2) = \ell(h_1, h_2) \quad \text{for all } (h_1, h_2) \in H_1 \times H_2. \tag{12}$$

It is easy to see that such a boundary Lagrangian will always satisfy the inequality

$$\ell(r,s) \geq \tfrac{1}{2}(\|s\|^2 - \|r\|^2) \text{ for all } (r,s) \in H_1 \times H_2. \tag{13}$$

The basic example of a self dual boundary Lagrangian is given by a function ℓ on $H_1 \times H_2$, of the form $\ell(x,p) = \psi_1(x) + \psi_2(p)$, with $\psi_1^*(x) = \psi_1(-x)$ and $\psi_2^*(p) = \psi_2(p)$. Here the choices for ψ_1 and ψ_2 are rather limited and the typical sample is:

$$\psi_1(x) = \frac{1}{2}\|x\|^2 - 2\langle a, x\rangle + \|a\|^2, \quad \text{and} \quad \psi_2(p) = \tfrac{1}{2}\|p\|^2.$$

where a is given in H_1. Boundary operators and Lagrangians allow us to build new ASD Lagrangians. Here is the situation when the skew-adjoint operators are not necessarily bounded. Most of it was established in [10], but we include here a proof for completness.

Proposition 2.5 *Let $\ell : H_1 \times H_2 \to \mathbb{R} \cup \{+\infty\}$ be a self-dual boundary Lagrangian on the product of two Hilbert spaces $H_1 \times H_2$, and let $L : X \times X^* \to \mathbb{R}$ be an ASD Lagrangian on a reflexive Banach space X such that for every $p \in X^*$, the function $x \to L(x,p)$ is bounded on the bounded sets of X. Let B be a linear map from its domain $D(B) \subset X$ into X^*, and let $(b_1, b_2) : D(b_1, b_2) \subset X \to H_1 \times H_2$ be linear boundary operators. Assume one of the following two conditions:*

1. *B is antisymmetric modulo the boundary operators (b_1, b_2), and $0 \in \mathrm{Dom}_1(L) \subset D(B) \cap D(b_1, b_2)$.*
2. *B is skew-adjoint modulo the boundary operators (b_1, b_2) and $\ell(r,s) \leq C(1 + \|r\|^2 + \|s\|^2)$ for all $(r,s) \in H_1 \times H_2$.*

Then the Lagrangian defined by

$$L_{B,\ell}(x,p) = \begin{cases} L(x, Bx + p) + \ell(b_1(x), b_2(x)) & \text{if } x \in D(B) \cap D(b_1, b_2) \\ +\infty & \text{if } x \notin D(B) \cap D(b_1, b_2) \end{cases}$$

is anti-self dual on X. Its Hamiltonian is then given by

$$H_{B,\ell}(x,y) = \begin{cases} H_L(x,y) - \langle Bx, y\rangle - \ell(b_1(x), b_2(x)) & \text{if } x \in D(B) \cap D(b_1, b_2) \\ -\infty & \text{if } x \notin D(B) \cap D(b_1, b_2) \end{cases}$$

Proof: Before we proceed with the proof, we note that while $H_{B,\ell}(-x,-y) \leq -H_{B,\ell}(y,x)$ for every $(x,y) \in X \times X$ and consequently $H_{B,\ell}(x,-x) \leq 0$, we almost never have equality. In other words $L_{B,\ell}$ is never tempered even when L is. This is due to the fact that with the above assumptions on the density of their kernel, the operators b_1, b_2 can never be continuous.

Assume now that B is antisymmetric modulo the boundary operators (b_1, b_2), and that for every $p \in X^*$, the function $x \to L(x,p)$ is continuous on X. We shall prove that $L^*_{B,\ell}(\tilde{p}, \tilde{x}) = L_{B,\ell}(-\tilde{x}, -\tilde{p})$ if $\tilde{x} \in D(B) \cap D(b_1, b_2)$. Indeed, fix $\tilde{x} \in S := D(B) \cap D(b_1, b_2)$, and write

$$L^*_{B,\ell}(\tilde{p}, \tilde{x}) = \sup\{\langle \tilde{x}, p\rangle + \langle x, \tilde{p}\rangle - L(x, Bx + p) - \ell(b_1(x), b_2(x)); x \in S, p \in X^*\}$$

Substituting $q = Bx + p$, and since for $\tilde{x} \in S$, we have $\langle \tilde{x}, Bx\rangle = -\langle x, B\tilde{x}\rangle - \langle b_1(x), b_1(\tilde{x})\rangle + \langle b_2(x), b_2(\tilde{x})\rangle$, we obtain

$$\begin{aligned}
L^*_{B,\ell}(\tilde{p}, \tilde{x}) &= \sup_{\substack{x \in S \\ q \in X^*}} \{\langle \tilde{x}, q - Bx\rangle + \langle x, \tilde{p}\rangle - L(x,q) - \ell(b_1(x), b_2(x))\} \\
&= \sup_{\substack{x \in S \\ q \in X^*}} \Big\{\langle x, B\tilde{x}\rangle + \langle b_1(x), b_1(\tilde{x})\rangle - \langle b_2(x), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle + \langle x, \tilde{p}\rangle - L(x,q) - \ell(b_1(x), b_2(x))\Big\} \\
&= \sup\Big\{\langle x, B\tilde{x} + \tilde{p}\rangle + \langle b_1(x + x_0), b_1(\tilde{x})\rangle - \langle b_2(x + x_0), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle - L(x,q) \\
&\qquad -\ell(b_1(x + x_0), b_2(x + x_0)); x \in S, q \in X^*, x_0 \in \mathrm{Ker}(b_1, b_2) \cap D(B)\Big\}
\end{aligned}$$

Since S is a linear space, we may set $w = x + x_0$ and write

$$L^*_{B,\ell}(\tilde{p}, \tilde{x}) = \sup\Big\{ \langle w - x_0, B\tilde{x} + \tilde{p}\rangle + \langle b_1(w), b_1(\tilde{x})\rangle - \langle b_2(w), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle - L(w - x_0, q)$$
$$-\ell(b_1(w), b_2(w)); w \in S, q \in X^*, x_0 \in \text{Ker } (b_1, b_2) \cap D(B)\Big\}$$

Now, for each fixed $w \in S$ and $q \in X^*$, the supremum over $x_0 \in \text{Ker}(b_1, b_2) \cap D(B)$ can be taken as a supremum over $x_0 \in X$ since $\text{Ker}(b_1, b_2) \cap D(B)$ is dense in X and all terms involving x_0 are continuous in that variable. Furthermore, for each fixed $w \in S$ and $q \in X^*$, the supremum over $x_0 \in X$ of the terms $w - x_0$ can be written as supremum over $v \in X$ where $v = w - x_0$. So setting $v = w - x_0$ we get

$$\begin{aligned} L^*_{B,\ell}(\tilde{p}, \tilde{x}) &= \sup\Big\{ \langle v, B\tilde{x} + \tilde{p}\rangle + \langle b_1(w), b_1(\tilde{x})\rangle - \langle b_2(w), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle - L(v, q) \\ &\qquad - \ell(b_1(w), b_2(w)); v \in X, q \in X^*, w \in S\} \\ &= \sup_{v\in X} \sup_{q\in X^*} \{\langle v, B\tilde{x} + \tilde{p}\rangle + \langle \tilde{x}, q\rangle - L(v, q)\} \\ &\qquad + \sup_{w\in S} \{\langle b_1(w), b_1(\tilde{x})\rangle + \langle b_2(w), -b_2(\tilde{x})\rangle - \ell(b_1(w), b_2(w))\} \end{aligned}$$

Since the range of $(b_1, b_2) : S \to H_1 \times H_2$ is dense in the $H_1 \times H_2$ topology, the boundary term can be written as

$$\sup_{a\in H_1} \sup_{b\in H_2} \{\langle a, b_1(\tilde{x})\rangle + \langle b, -b_2(\tilde{x})\rangle - \ell(a, b)\} = \ell^*(b_1(\tilde{x}), -b_2(\tilde{x})) = \ell(-b_1(\tilde{x}), -b_2(\tilde{x})).$$

while the main term is clearly equal to $L^*(B\tilde{x} + \tilde{p}, \tilde{x}) = L(-\tilde{x}, -B\tilde{x} - \tilde{p})$ in such a way that $L^*_{B,\ell}(p, \tilde{x}) = L_{B,\ell}(-\tilde{x}, -\tilde{p})$ if $\tilde{x} \in D(B) \cap D(b_1, b_2)$.

Now assume $\tilde{x} \notin S = D(B) \cap D(b_1, b_2)$, then $-\tilde{x} \notin S$. and we distinguish the two cases:
Case 1: Under condition 1, we have that $-\tilde{x} \notin \text{Dom}_1(L)$, hence $-H_L(-\tilde{x}, 0) = +\infty$. Now the boundedness condition on L implies by Proposition 2.3 that it is a tempered ASD Lagrangian, which means since $0 \in \text{Dom}_1(L)$, that $H_L(0, \tilde{x}) = -H_L(-\tilde{x}, 0)$. It follows that

$$\begin{aligned} L^*_{B,\ell}(\tilde{p}, \tilde{x}) &= \sup_{\substack{x\in \text{Dom}_1(L)\\ p\in X^*}} \left\{ \langle \tilde{x}, p - Bx\rangle + \langle x, \tilde{p}\rangle - L(x, p) - \frac{\|b_1(x)\|^2_{H_1}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2} \right\} \\ &\geq \sup_{p\in X^*} \{\langle \tilde{x}, p\rangle - L(0, p)\} \\ &= H_L(0, \tilde{x}) = -H_L(-\tilde{x}, 0) = +\infty = L_{B,\ell}(-\tilde{x}, -\tilde{p}) \end{aligned}$$

Case 2: Under condition 2, write

$$\begin{aligned} L^*_{B,\ell}(\tilde{p}, \tilde{x}) &= \sup_{\substack{x\in S\\ q\in X^*}} \left\{ \langle \tilde{x}, q - Bx\rangle + \langle x, \tilde{p}\rangle - L(x, q) - \frac{\|b_1(x)\|^2_{H_1}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2} \right\} \\ &\geq \sup_{\substack{x\in S\\ \|x\|_X<1}} \left\{ \langle -\tilde{x}, Bx\rangle + \langle x, \tilde{p}\rangle - L(x, 0) - \frac{\|b_1(x)\|^2_{H_1}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2}. \right\} \end{aligned}$$

Since by assumption $L(x, 0) < K$ whenever $\|x\|_X < 1$, and $\ell(r, s) \leq C(1+\|r\|^2+\|s\|^2)$ for all $(r, s) \in H_1\times H_2$, we obtain that

$$\begin{aligned} L^*_{B,\ell}(\tilde{p}, \tilde{x}) &\geq \sup_{\substack{x\in S\\ \|x\|_X<1}} \left\{ \langle -\tilde{x}, Bx\rangle + \langle x, \tilde{p}\rangle - C - K - \frac{\|b(x)\|^2_{H_2}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2} \right\} \\ &= +\infty = L_{B,\ell}(-\tilde{x}, -\tilde{p}) \end{aligned}$$

since $\tilde{x} \notin S$ as soon as $-\tilde{x} \notin S$. Therefore $L^*_{B,\ell}(\tilde{p}, \tilde{x}) = L_{B,\ell}(-\tilde{x}, -\tilde{p})$ for all $(\tilde{x}, \tilde{p}) \in X \times X^*$ and $L_{B,\ell}$ is an anti-selfdual Lagrangian.

3 A nonlinear variational principle for ASD Lagrangians

Definition 3.1 (A) Say that a –non necessarily linear– map $\Lambda : D(\Lambda) \subset X \to X^*$ is a *regular map* if

$$u \to \Lambda u \text{ is weak-to-weak continuous,} \tag{14}$$

and

$$u \to \langle \Lambda u, u\rangle \text{ is weakly lower semi-continuous on } D(\Lambda). \tag{15}$$

(B) Say that $\Lambda : D(\Lambda) \subset X \to X^*$ is a *regular conservative map* if it satisfies (14) and

$$\langle \Lambda u, u\rangle = 0 \text{ for all } u \text{ in its domain } D(\Lambda). \tag{16}$$

It is clear that positive bounded linear operators are necessarily *regular maps* and that regular conservative maps (which include skew-symmetric bounded linear operators) are also *regular* maps. However, there are also plenty of nonlinear regular maps many of them appearing in the basic equations of hydrodynamics and magnetohydrodynamics (see below, [9] and [11]).
Now recall that ASD Lagrangians readily satisfy $L(x,p) \geq -\langle x,p\rangle$ for every $(x,p) \in X \times X^*$, which means that we always have the following inequality:

$$L(x, \Lambda x) + \langle x, \Lambda x\rangle \geq 0 \quad \text{for all } x \in D(\Lambda). \tag{17}$$

What is remarkable is that, just like in the case of linear skew-adjoint operators [8], the infimum will often be zero as long as Λ is a regular map, a fact that will allow us to derive variationally several nonlinear PDEs without using Euler-Lagrange theory. Here is our basic result.

Theorem 3.2 *Let L be an anti-self dual Lagrangian on a reflexive Banach space X and let $\Lambda : D(\Lambda) \subset X \to X^*$ be a regular map such that $\mathrm{Dom}_1(L) \subset D(\Lambda)$. Assume that*

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle \Lambda x, x\rangle = +\infty$$

where H_L is the Hamiltonian associated to L. Then there exists $\bar{x} \in \mathrm{Dom}_1(L)$ such that:

$$\begin{aligned} L(\bar{x}, \Lambda\bar{x}) + \langle \Lambda\bar{x}, \bar{x}\rangle &= \inf_{x\in X} L(x, \Lambda x) + \langle \Lambda x, x\rangle = 0 & (18)\\ (-\Lambda\bar{x}, -\bar{x}) &\in \partial L(\bar{x}, \Lambda\bar{x}). & (19)\end{aligned}$$

This is a corollary of the following much more general result.

Theorem 3.3 *Let $L : X \times X^* \to \mathbf{R} \cup \{+\infty\}$ be an anti-selfdual Lagrangian on a reflexive Banach space X and let $\ell : H_1 \times H_2 \to \mathbb{R} \cup \{+\infty\}$ be a self-dual boundary Lagrangian on the product of two Hilbert spaces $H_1 \times H_2$. Consider $B : D(B) \subset X \to X^*$ and $(b_1, b_2) : D(b_1, b_2) \subset X \to H_1 \times H_2$ to be linear operators such that*

$$\langle x, Bx\rangle = \tfrac{1}{2}(\|b_2 x\|^2 - \|b_1 x\|^2) \text{ for all } x \in D(B) \cap D(b_1, b_2), \tag{20}$$

and let $\Lambda : D(\Lambda) \subset X \to X^*$ be a regular operator such that the Lagrangian $L_{B,\ell}$ is anti-selfdual on the graph of Λ and $\mathrm{Dom}_1(L) \cap D(B) \cap D(b_1, b_2) \subset D(\Lambda)$. Assuming that

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle \Lambda x, x\rangle = +\infty,$$

then, there exists $\bar{x} \in \mathrm{Dom}_1(L) \cap D(B) \cap D(b_1, b_2)$ such that:

$$\begin{aligned} L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \Lambda\bar{x}, \bar{x}\rangle + \ell(b_1\bar{x}, b_2\bar{x}) &= \inf_{x\in X} \{L(x, Bx + \Lambda x) + \langle \Lambda x, x\rangle + \ell(b_1 x, b_2 x)\} = 0 & (21)\\ (-\Lambda\bar{x} - B\bar{x}, -\bar{x}) &\in \partial L(\bar{x}, B\bar{x} + \Lambda\bar{x}) & (22)\\ \ell(b_1(\bar{x}), b_2(\bar{x})) &= \frac{1}{2}(\|b_2(\bar{x})\|^2 - \|b_1(\bar{x})\|^2). & (23)\end{aligned}$$

We shall deduce Theorem 3.3 from the following Ky-Fan type min-max theorem due to Brezis-Nirenberg-Stampachia (see [6]).

Lemma 3.4 *Let D be a convex subset of a reflexive Banach space X and let $M(x,y)$ be a real valued function on $D \times D \subset X \times X$ that satisfies the following conditions:*

(1) $M(x,x) \leq 0$ for every $x \in D$.

(2) For each $x \in D$, the function $y \to M(x,y)$ is concave.

(3) For each $y \in D$, the function $x \to M(x,y)$ is weakly lower semi-continuous on X.

(4) There exists $K > 0$ and $y_0 \in X$ such that $\|y_0\| \leq K$ and $\inf\limits_{\|x\|>K} M(x,y_0) > 0$.

Then there exists $x_0 \in D$ such that $\sup\limits_{y\in D} M(x_0,y) \leq 0$.

Proof of Theorem 3.3: Since the Lagrangian $L_{B,\ell}$ defined above is anti-self dual on the graph of Λ, we can write for each $x \in D := \mathrm{Dom}(L) \cap D(B) \cap D(b_1,b_2) \subset D(\Lambda)$,

$$\begin{aligned}
I(x) &= L_{B,\ell}(x,\Lambda x) + \langle \Lambda x, x\rangle = L^*_{B,\ell}(-\Lambda x,-x) + \langle \Lambda x, x\rangle \\
&= \sup\{\langle y,-\Lambda x\rangle + \langle p,-x\rangle - L_{B,\ell}(y,p); y \in X, p \in X^*\} + \langle \Lambda x, x\rangle \\
&= \sup\{\langle y,-\Lambda x\rangle + \langle p,-x\rangle - L(y,By+p) - \ell(b_1(y),b_2(y)); y \in D, p \in X^*\} + \langle \Lambda x, x\rangle \\
&= \sup\{\langle y,-\Lambda x\rangle + \langle q-By,-x\rangle - L(y,q) - \ell(b_1(y),b_2(y)); y \in D, q \in X^*\} + \langle \Lambda x, x\rangle \\
&= \sup\{\langle y,-\Lambda x\rangle + \langle x,By\rangle - \ell(b_1(y),b_2(y)) + \sup\{\langle q,-x\rangle - L(y,q);, q \in X^*\}; y \in D\} + \langle \Lambda x, x\rangle \\
&= \sup\{\langle x-y,\Lambda x\rangle + \langle x,By\rangle - \ell(b_1(y),b_2(y)) + H_L(y,-x);\ y \in D\} \\
&= \sup_{y\in D} M(x,y)
\end{aligned}$$

where

$$M(x,y) = \langle x-y,\Lambda x\rangle + \langle x,By\rangle - \ell(b_1(y),b_2(y)) + H_L(y,-x),$$

and where H_L is the Hamiltonian associated to L.
We now claim that M satisfies all the properties of the Ky-Fan min-max lemma above. Indeed,
(1) For each $x \in D(B)\cap D(b_1,b_2)$, we have $y \to M(x,y)$ is concave since the first part $y \to \langle x-y,\Lambda x\rangle + \langle x,By\rangle$ is clearly linear, while $y \to -\ell(b_1(y),b_2(y))$ and $y \to H_L(y,x)$ are concave.
(2) For each $y \in D(A) \cap D(b_1,b_2)$, the function $x \to M(x,y)$ is weakly lower semi-continuous on $D(A) \cap D(b_1,b_2)$ since $x \to \langle x-y,\Lambda x\rangle + \langle x,By\rangle$ is weakly continuous by hypothesis while $x \to H_L(y,-x)$ is clearly the supremum of continuous affine functions.
(3) To show that $M(x,x) \leq 0$ for each $x \in D(A) \cap D(b_1,b_2)$, use the fact that H_L is an ASD Hamiltonian, hence $H_L(x,-x) \leq 0$ and Property (13) satisfied by ℓ to write

$$M(x,x) \leq \langle x,Bx\rangle - \ell(b_1(x),b_2(x)) = \frac{1}{2}(\|b_2(x)\|^2 - \|b_1(x)\|^2) - \ell(b_1(x),b_2(x)) \leq 0.$$

(4) The set $X_0 = \{x \in X; M(x,0) \leq 0\}$ is bounded in X since $M(x,0) = H_L(0,-x) + \langle \Lambda x, x\rangle - \ell(0,0)$ and the latter goes to infinity with $\|x\|$.
It follows from Lemma 3.4 that there exists $\bar{x} \in D$ such that $\sup\limits_{y\in D} M(\bar{x},y) \leq 0$. In other words

$$I(\bar{x}) = \sup_{y\in D} M(\bar{x},y) \leq 0.$$

On the other hand, for any $x \in X$, we have

$$\begin{aligned}
I(x) &= L(x,Bx+\Lambda x) + \langle \Lambda x, x\rangle + \ell(b_1x,b_2x) \\
&\geq -\langle x,Bx\rangle + \ell(b_1x,b_2x) \\
&= -\frac{1}{2}(\|b_2(x)\|^2 - \|b_1(x)\|^2) + \ell(b_1x,b_2x) \geq 0.
\end{aligned}$$

It follows that $I(\bar{x}) = 0 = \inf_{x\in X} I(x)$, which means

$$L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \Lambda\bar{x}, \bar{x}\rangle + \ell(b_1\bar{x}, b_2\bar{x}) = \inf_{x\in X} \{L(x, Bx + \Lambda x) + \langle \Lambda x, x\rangle + \ell(b_1 x, b_2 x)\} = 0. \tag{24}$$

To establish (21), write

$$\begin{aligned} 0 = L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \Lambda\bar{x}, \bar{x}\rangle + \ell(b_1\bar{x}, b_2\bar{x}) &= L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \bar{x}, B\bar{x} + \Lambda\bar{x}\rangle - \langle \bar{x}, B\bar{x}\rangle + \ell(b_1\bar{x}, b_2\bar{x}) \\ &= L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \bar{x}, B\bar{x} + \Lambda\bar{x}\rangle - \frac{1}{2}(\|b_2\bar{x}\|^2 - \|b_1\bar{x}\|^2) + \ell(b_1\bar{x}, b_2\bar{x}). \end{aligned}$$

Since $L(x,p) + \langle x, p\rangle \geq 0$ and $\ell(r,s) \geq \frac{1}{2}(\|s\|^2 - \|r\|^2)$, we get

$$\left\{ \begin{array}{rcl} L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \bar{x}, B\bar{x} + \Lambda\bar{x}\rangle &=& 0. \\ \ell(b_1\bar{x}, b_2\bar{x}) &=& \frac{1}{2}(\|b_2\bar{x}\|^2 - \|b_1\bar{x}\|^2). \end{array} \right. \tag{25}$$

To obtain the second claim, we use that L is anti-selfdual to write

$$\begin{aligned} \langle (\bar{x}, \Lambda\bar{x} + B\bar{x}), (-\Lambda\bar{x} - B\bar{x}, -\bar{x})\rangle &= -2\langle \bar{x}, \Lambda\bar{x} + B\bar{x}\rangle \\ &= 2L(\bar{x}, \Lambda\bar{x} + B\bar{x}) \\ &= L(\bar{x}, \Lambda\bar{x} + B\bar{x}) + L^*(-\Lambda\bar{x} - B\bar{x}, -\bar{x}). \end{aligned}$$

The last part of claim (21) now follows from the limiting case of the Legendre-Fenchel duality.

Theorem 3.2 is now immediate as it corresponds to the case where b_1, b_2 and ℓ are zero. Actually, we can add to Λ any positive linear operator $B : D(B) \subset X \to X^*$ with a large enough domain. We shall see in the next section that it is already sufficient to cover several nonlinear PDEs including Navier-Stokes equations and others.

Corollary 3.5 *Let L be an anti-self dual Lagrangian on a reflexive space X, $B : X \to X^*$ a positive bounded linear operator and let $\Lambda : D(\Lambda) \subset X \to X^*$ be a regular map such that $\mathrm{Dom}_1(L) \subset D(\Lambda)$ and*

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle x, Bx + \Lambda x\rangle = +\infty.$$

Then, there exists $\bar{x} \in \mathrm{Dom}_1(L)$ such that:

$$L(\bar{x}, B\bar{x} + \Lambda\bar{x}) + \langle \bar{x}, B\bar{x} + \Lambda\bar{x}\rangle = \inf_{x\in X} L(x, Bx + \Lambda x) + \langle x, Bx + \Lambda x\rangle = 0 \tag{26}$$

$$(-\Lambda\bar{x} - B\bar{x}, -\bar{x}) \in \partial L(\bar{x}, B\bar{x} + \Lambda\bar{x}). \tag{27}$$

Proof: It is sufficient to apply Theorem 3.3 to b_1, b_2 and ℓ being identically zero, while $\tilde{\Lambda} = \Lambda + B$ satisfies $\mathrm{Dom}_1(L) \subset D(\tilde{\Lambda})$.

If the domain of the linear operator B is not large enough, we can use Lemma 2.7 to obtain

Corollary 3.6 *Let L be an anti-self dual Lagrangian on a reflexive Banach space X such that $x \to L(x, 0)$ is bounded on the unit ball of X. Suppose $B : D(B) \subset X \to X^*$ is a linear skew-adjoint operator and $\Lambda : X \to X^*$ is a regular map such that $D(B) \cap \mathrm{Dom}_1 L \subset D(\Lambda)$ and*

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle x, \Lambda x\rangle = +\infty.$$

Then there exists $\bar{x} \in D(B) \cap \mathrm{Dom}_1 L$ satisfying (26) and (27).

Proof: By the above Lemma, L_B is an anti-selfdual Lagrangian, in particular it is so on the graph of Λ. The rest follows from Theorem 3.3 applied with b_1, b_2 and ℓ being identically zero.

In order to deal with situations where operators are skew-adjoint provided one takes into account certain boundary terms, we have the following

Corollary 3.7 *Let $\ell : H_1 \times H_2 \to \mathbb{R} \cup \{+\infty\}$ be a self-dual boundary Lagrangian on the product of two Hilbert spaces $H_1 \times H_2$, and let $L : X \times X^* \to \mathbf{R} \cup \{+\infty\}$ be an anti-selfdual Lagrangian on a reflexive Banach space X such that for every $p \in X^*$, $x \to L(x,p)$ is bounded on the bounded sets of X. Let $(B,(b_1,b_2)) : D(B) \times D(b_1,b_2) \to X^* \times (H_1 \times H_2)$ be linear operators such that one of the following two conditions hold:*

1. *B is antisymmetric modulo boundary operators (b_1,b_2), and $0 \in \mathrm{Dom}_1(L) \subset D(B) \cap D(b_1,b_2)$.*
2. *B is skew-adjoint modulo the boundary operators (b_1,b_2) and $\ell(r,s) \leq C(1+\|r\|^2+\|s\|^2)$ for all $(r,s) \in H_1 \times H_2$.*

Then, for any regular operator $\Lambda : D(\Lambda) \subset X \to X^*$ such that $\mathrm{Dom}(L) \cap D(B) \cap D(b_1,b_2) \subset D(\Lambda)$ and

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle \Lambda x, x\rangle = +\infty,$$

there exists $\bar{x} \in \mathrm{Dom}_1(L) \cap D(B) \cap D(b_1,b_2)$ such that:

$$L(\bar{x}, B\bar{x}+\Lambda\bar{x}) + \langle \bar{x}, \Lambda\bar{x}\rangle + \ell(b_1\bar{x}, b_2\bar{x}) = \inf_{x\in X}\{L(x,Bx+\Lambda x) + \langle x, \Lambda x\rangle + \ell(b_1x,b_2x)\} = 0 \quad (28)$$

$$(-\Lambda\bar{x} - B\bar{x}, -\bar{x}) \in \partial L(\bar{x}, B\bar{x}+\Lambda\bar{x}) \quad (29)$$

$$\ell(b_1(\bar{x}), b_2(\bar{x})) = \frac{1}{2}(\|b_2(\bar{x})\|^2 - \|b_1(\bar{x})\|^2). \quad (30)$$

Proof: This follows from Theorem 3.2 and Proposition 2.5, since under these conditions the Lagrangian $L_{B,\ell}$ is anti-selfdual.

4 A variational nonlinear Lax-Milgram theorem and applications

We now apply the above results to the most basic ASD Lagrangians of the form $L(x,p) = \varphi(x) + \varphi^*(Bx-p)$ where φ is a convex function and B is a linear anti-symmetric but not necessarily bounded operator. The applications differ as they will depend on the "position" of the domain of B. We start with the case where the linear operator component has a "large domain".

Theorem 4.1 *Let φ be a proper convex lower semi-continuous function on a reflexive Banach space X and let $B : D(B) \subset X \to X^*$ be an anti-symmetric linear operator such that $\mathrm{Dom}(\varphi) \subset D(B)$. Then, for any regular operator $\Lambda : D(\Lambda) \subset X \to X^*$ such that $\mathrm{Dom}(\varphi) \subset D(\Lambda)$ and $\lim\limits_{\|x\|\to+\infty} \frac{\varphi(x)+\langle \Lambda x, x\rangle}{\|x\|} = +\infty$, there exists for every $f \in X^*$, a solution $\bar{x} \in \mathrm{Dom}(\varphi)$ to the equation*

$$0 \in f + \Lambda x + Bx + \partial\varphi(x). \quad (31)$$

It is obtained as a minimizer of the problem:

$$\inf_{x\in X}\{\varphi(x) + \langle f, x\rangle + \varphi^*(-\Lambda x - Bx - f) + \langle x, \Lambda x\rangle\} = 0. \quad (32)$$

Proof: It is an immediate consequence of Corollary 3.5 applied to the Lagrangian $L(x,p) = \psi(x) + \psi^*(-p)$ where $\psi(x) = \varphi(x) + \langle f, x\rangle$. Note that its Hamiltonian is now $H(x,y) = \varphi(-y) - \varphi(x) - \langle f, x+y\rangle$ meaning that the coercivity hypothesis implies that $H(0,y) + \langle y, \Lambda y\rangle \to +\infty$ with $\|y\|$. Corollary 3.5 then applies with the Lagrangian L and the regular operator Λ to obtain that the minimum in (32) is attained at some $\bar{x} \in X$. We then get

$$\varphi(\bar{x}) + \varphi^*(-B\bar{x} - \Lambda\bar{x} - f) = \langle -B\bar{x} - \Lambda\bar{x} - f, \bar{x}\rangle$$

which yields, in view of Legendre-Fenchel duality that

$$-B\bar{x} - \Lambda\bar{x} - f \in \partial\varphi(\bar{x}).$$

An immediate application is the case where the linear operator component is bounded which already covers many interesting applications.

Corollary 4.2 *Let φ be a function on a reflexive Banach space X and let $B : X \to X^*$ be a bounded linear operator such that the function $\psi(x) := \varphi(x) + \frac{1}{2}\langle Bx, x\rangle$ is proper convex and lower semi-continuous on X. Assume*

$$\lim_{\|x\|\to\infty} \|x\|^{-1}(\varphi(x) + \frac{1}{2}\langle Bx, x\rangle) = +\infty. \tag{33}$$

Then, for any regular operator $\Lambda : X \to X^*$ and any $f \in X^*$, there exists a solution $\bar{x} \in X$ to the equation

$$0 \in f + \Lambda x + Bx + \partial\varphi(x). \tag{34}$$

It is obtained as a minimizer of the problem:

$$\inf_{x\in X} \{\psi(x) + \langle f, x\rangle + \psi^*(-\Lambda x - B^a x - f) + \langle x, \Lambda x\rangle\} = 0 \tag{35}$$

where B^a is the anti-symmetric part of B.

Proof: Apply the above theorem to $\psi(x) + \langle f, x\rangle$ and $B^a = \frac{1}{2}(B - B^*)$. We then get $\bar{x} \in X$ such that:

$$-B^a\bar{x} - \Lambda\bar{x} - f \in \partial\psi(\bar{x}) = B^s\bar{x} + \partial\varphi(\bar{x})$$

hence $\bar{x}$ satisfies (34).

Example 1: A variational resolution for the Stationary Navier-Stokes equation

Consider the incompressible stationary Navier-Stokes equation on a domain Ω of $\mathbb{R}^3$

$$\begin{cases} (u\cdot\nabla)u + f &= \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u &= 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{36}$$

where $\nu > 0$ and $f \in L^p(\Omega; \mathbb{R}^3)$. Let

$$\Phi(u) = \frac{\nu}{2}\int_\Omega \Sigma_{j,k=1}^3 (\frac{\partial u_j}{\partial x_k})^2\, dx \tag{37}$$

be the convex and coercive function on the Sobolev subspace $V = \{u \in H_0^1(\Omega; \mathbf{R}^3); \text{div} v = 0\}$. Its Legendre transform Φ^* on V can be characterized as $\Phi^*(v) = \langle Sv, v\rangle$ where $S : V^* \to V$ is the bounded linear operator that associates to $v \in V^*$ the solution $\hat{v} = Sv$ of the Stokes' problem

$$\begin{cases} \nu\Delta\hat{v} + \nabla p &= -\hat{v} \quad \text{on } \Omega \\ \text{div}\hat{v} &= 0 \quad \text{on } \Omega \\ \hat{v} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{38}$$

It is easy to see that (36) can be reformulated as

$$\begin{cases} (u\cdot\nabla)u + f &\in -\partial\Phi(u) = \nu\Delta u - \nabla p \\ u &\in V. \end{cases} \tag{39}$$

Consider now the nonlinear operator $\Lambda : V \to V^*$ defined as

$$\langle \Lambda u, v\rangle = \int_\Omega \Sigma_{j,k=1}^3 u_k \frac{\partial u_j}{\partial x_k} v_j\, dx = \langle (u\cdot\nabla)u, v\rangle.$$

We can deduce the following

Theorem 4.3 *Assume Ω is bounded domain in $\mathbb{R}^3$ and consider $f \in L^p(\Omega; \mathbb{R}^3)$ for $p > \frac{6}{5}$. Then the infimum of the functional*

$$I(u) = \Phi(u) + \Phi^*(-(u\cdot\nabla)u + f) - \int_\Omega \Sigma_{j=1}^3 f_j u_j$$

on V is equal to zero, and is attained at a solution of the Navier-Stokes equation (36).

Proof: To apply Theorem 4.1, it remains to show that Λ is a regular conservative operator. First note that $\langle \Lambda u, u\rangle = 0$ on V. For the weak-to weak continuity, assume that $u^n \to u$ weakly in $H^1(\Omega)$. We need to show that for a fixed $v \in V$, we have that

$$\langle \Lambda u^n, v\rangle = \int_\Omega \Sigma_{j,k=1}^3 u_k^n \frac{\partial u_j^n}{\partial x_k} v_j \, dx = -\int_\Omega \Sigma_{j,k=1}^3 u_k^n \frac{\partial v_j}{\partial x_k} u_j^n \, dx$$

converges to $\langle \Lambda u, v\rangle = \int_\Omega \Sigma_{j,k=1}^3 u_k \frac{\partial v_j}{\partial x_k} u_j \, dx$. But the Sobolev embedding in dimension 3 implies that (u^n) converges strongly in $L^p(\Omega;\mathbb{R}^3)$ for $1 \le p < 6$. On the other hand, $\frac{\partial u_j}{\partial x_k}$ is in $L^2(\Omega)$ and the result follows from an application of Hölder's inequality.

Example 2: Variational resolution for a fluid driven by its boundary

The full strength of Corollary 4.2 comes out when one deals with the Navier-Stokes equation with a boundary moving with a prescribed velocity:

$$\begin{cases} (u\cdot\nabla)u + f &= \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u &= 0 \quad \text{on } \Omega \\ u &= u^0 \quad \text{on } \partial\Omega \end{cases} \tag{40}$$

where $\int_{\partial\Omega} u^0 \cdot \mathbf{n} \, d\sigma = 0$, $\nu > 0$ and $f \in L^p(\Omega;\mathbb{R}^3)$. Assuming that $u^0 \in H^{3/2}(\partial\Omega)$ and that $\partial\Omega$ is connected, a classical result of Hopf then yields for each $\epsilon > 0$, the existence of $v^0 \in H^2(\Omega)$ such that

$$v^0 = u^0 \text{ on } \partial\Omega, \quad \text{div}\, v^0 = 0 \quad \text{and} \quad \int_\Omega \Sigma_{j,k=1}^3 u_k \frac{\partial v_j^0}{\partial x_k} u_j \, dx \le \epsilon \|u\|_V^2 \text{ for all } u \in V. \tag{41}$$

Setting $v = u + v^0$, then solving (40) reduces to finding a solution for

$$\begin{cases} (u\cdot\nabla)u + (v^0\cdot\nabla)u + (u\cdot\nabla)v^0 + f - \nu\Delta v^0 + (v^0\cdot\nabla)v^0 &= \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u &= 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

This can be reformulated as the following equation in the space V

$$(u\cdot\nabla)u + (v^0\cdot\nabla)u + (u\cdot\nabla)v^0 + g \in -\partial\Phi(u) \tag{42}$$

where Φ is again the convex functional $\Phi(u) = \frac{\nu}{2}\int_\Omega \Sigma_{j,k=1}^3 (\frac{\partial u_j}{\partial x_k})^2 \, dx$ as above and where

$$g := f - \nu\Delta v^0 + (v^0\cdot\nabla)v^0 \in V^*.$$

In other words, this is an equation of the form

$$\Lambda u + Bu + g \in -\partial\Phi(u) \tag{43}$$

with $\Lambda u = (u\cdot\nabla)u$ is a regular conservative operator, and $Bu = (v^0\cdot\nabla)u + (u\cdot\nabla)v^0$ is a bounded linear operator. Note that the component $B^1 u := (v^0\cdot\nabla)u$ is skew-symmetric which means that Hopf's result yields the required coercivity condition:

$$\Psi(u) := \Phi(u) + \frac{1}{2}\langle Bu, u\rangle \ge \frac{1}{2}(\nu - \epsilon)\|u\|^2 \quad \text{for all } u \in V.$$

In other words, Ψ is convex and coercive and therefore we can apply Theorem 4.1 to deduce

Theorem 4.4 *Under the above hypothesis, and letting A^a be the antisymmetric part of the operator $Au = (u\cdot\nabla)v^0$, the following functional*

$$I(u) = \Psi(u) + \Psi^*(-(u\cdot\nabla)u - (v^0\cdot\nabla)u - A^a u + g) - \int_\Omega \Sigma_{j=1}^3 g_j u_j$$

has zero for infimum on the Banach space V, which is attained at a solution $\bar{u}$ for (42).

The next application is a nonlinear Lax-Milgram type result with boundary constraints.

Theorem 4.5 *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semi-continuous on a reflexive Banach space such that for some constant $C > 0$ and $p_1, p_2 > 1$, we have*

$$\frac{1}{C}\left(\|x\|_X^{p_1} - 1\right) \le \varphi(x) \le C\left(\|x\|_X^{p_2} + 1\right) \text{ for every } x \in X, \tag{44}$$

Let $B : D(B) \subset X \to X^*$ be a skew-adjoint operator modulo the boundary $(b_1, b_2) : D(b_1, b_2) \to H_1 \times H_2$ where H_1, H_2 are two Hilbert spaces. Then for any regular operator $\Lambda : X \to X^*$ such that $D(B) \cap D(b_1, b_2) \subset D(\Lambda)$ and any $a \in X$, there exists a solution $\bar{x} \in X$ to the equation

$$\begin{array}{rcl} \Lambda x + Bx + f & \in & -\partial\varphi(x) \\ b_1(x) & = & a. \end{array} \tag{45}$$

It is obtained as a minimizer of the functional defined as:

$$I(x) = \varphi(x) + \langle f, x\rangle + \varphi^*(-\Lambda x - Bx - f) + \langle x, \Lambda x\rangle + \frac{1}{2}(\|b_1(x)\|^2 + \|b_2(x)\|^2) - 2\langle a, b_1(x)\rangle + \|a\|^2 \tag{46}$$

when $x \in D(A) \cap D(b_1, b_2)$ and $+\infty$ elsewhere. Moreover, $I(\bar{x}) = \inf_{x\in X} I(x) = 0$.

Proof: Let $\psi(x) = \varphi(x) + \langle f, x\rangle$ and apply Corollary 3.7 to the ASD Lagrangian $L(x, p) := \psi(x) + \psi^*(-p)$, to the boundary Lagrangian $\ell(r, s) = \frac{1}{2}(\|r\|^2 + \|s\|^2) - 2\langle a, r\rangle + \|a\|^2$, and to the skew-adjoint triplet (B, b_1, b_2). Note also that I can be rewritten as:

$$I(x) = \varphi(x) + \varphi^*(-\Lambda x - Bx - f) - \langle x, -\Lambda x - Bx - f\rangle + \frac{1}{2}(\|b_1(x) - a\|^2 \ge 0.$$

Example 3: Variational resolution for a fluid driven by a transport operator

Let $\vec{a} \in C^\infty(\bar{\Omega})$ be a smooth vector field on a neighborhood of a C^∞ bounded open set $\Omega \subset \mathbb{R}^3$, let $a_0 \in L^\infty(\Omega)$, and consider the space $X = \{u \in H_0^1(\Omega; \mathbb{R}^3); \text{div}(u) = 0\}$ and the transport operator $B : u \mapsto (a\cdot\nabla)u + \frac{1}{2}\text{div(a)}u$ from $D(B) = \{u \in X; a\cdot\nabla u + \frac{1}{2}\text{div}u \in X^*\}$ into X^*. It is easy to show using Green's formula that the operator B is skew-adjoint on the space X (See [10]). Consider now the following equation on the domain $\Omega \subset \mathbb{R}^3$

$$\left\{ \begin{array}{rcll} (u\cdot\nabla)u + (\vec{a}\cdot\nabla)u + a_0 u + |u|^{m-2}u + f & = & \nu\Delta u - \nabla p & \text{on } \Omega \\ \text{div}u & = & 0 & \text{on } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{47}$$

where $\nu > 0$, $6 \ge m \ge 1$ and $f \in L^q(\Omega; \mathbb{R}^3)$ for $q \ge \frac{6}{5}$. Suppose

$$\frac{1}{2}\text{div}(a) - a_0 \ge 0 \quad \text{on} \quad \Omega, \tag{48}$$

and consider the functional

$$\Psi(u) = \frac{\nu}{2}\int_\Omega \Sigma_{j,k=1}^3 (\frac{\partial u_j}{\partial x_k})^2\,dx + \frac{1}{4}\int_\Omega (\text{div}\vec{a} - 2a_0)|u|^2 dx + \frac{1}{m}\int_\Omega |u|^m dx + \int_\Omega uf\,dx \tag{49}$$

which is convex and coercive function on the space X. Theorem 4.5 then applies to yield

Theorem 4.6 *Under the above hypothesis,*
the functional

$$I(u) = \Psi(u) + \Psi^*(-(u\cdot\nabla)u - \vec{a}\cdot\nabla u - \frac{1}{2}\text{div}(\vec{a})u)$$

has zero for infimum and the latter is attained at a solution $\bar{u}$ for (47).

We can also give a variational resolution for nonlinear anti-Hamiltonian systems.

Theorem 4.7 *Let φ be a proper convex lower semi-continuous function on $X \times Y$, let $A : X \to Y^*$ be any bounded linear operator, let $B_1 : X \to X^*$ (resp., $B_2 : Y \to Y^*$) be two positive bounded linear operators, and assume $\Lambda := (\Lambda_1, \Lambda_2) : X \times Y \to X^* \times Y^*$ is a regular conservative operator. Assume that*

$$\lim_{\|x\|+\|y\|\to\infty} \frac{\varphi(x,y) + \frac{1}{2}\langle B_1 x, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle}{\|x\| + \|y\|} = +\infty,$$

then for any $(f,g) \in X^* \times Y^*$, there exists $(\bar{x},\bar{y}) \in X \times Y$ which solves the following system

$$\begin{cases} \Lambda_1(x,y) - A^*y - B_1 x + f & \in & \partial_1\varphi(x,y). \\ \Lambda_2(x,y) + Ax - B_2 y + g & \in & \partial_2\varphi(x,y). \end{cases} \tag{50}$$

The solution is obtained as a minimizer on $X \times Y$ of the functional

$$I(x,y) = \psi(x,y) + \psi^*(-A^*y - B_1^a x + \Lambda_1(x,y), Ax - B_2^a y + \Lambda_2(x,y)).$$

where

$$\psi(x,y) = \varphi(x,y) + \frac{1}{2}\langle B_1 x, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle - \langle f, x\rangle - \langle g, y\rangle.$$

and where B_1^a (resp., B_2^a) are the skew-symmetric parts of B_1 and B_2.

Proof: Consider the following ASD Lagrangian (see [8])

$$L((x,y),(p,q)) = \psi(x,y) + \psi^*(-A^*y - B_1^a x + p, Ax - B_2^a y + q).$$

Theorem 4.1 yields that $I(x,y) = L((x,y),\Lambda(x,y))$ attains its minimum at some point $(\bar{x},\bar{y}) \in X \times Y$ and that the minimum is actually 0. In other words,

$$\begin{aligned} 0 &= I(\bar{x},\bar{y}) = \psi(\bar{x},\bar{y}) + \psi^*(-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y})) \\ &= \psi(\bar{x},\bar{y}) + \psi^*(-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y})) \\ &\quad -\langle(\bar{x},\bar{y}), (-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y}))\rangle \end{aligned}$$

from which follows that

$$\begin{cases} -A^*y - B_1^a x + \Lambda_1(x,y) & \in & \partial_1\varphi(x,y) + B_1^s(x) - f \\ Ax - B_2^a y + \Lambda_1(x,y) & \in & \partial_2\varphi(x,y) + B_2^s(y) - g. \end{cases} \tag{51}$$

A typical example of such a system are the equations of magneto-hydrodynamics, but here is a simpler example communicated to us by A. Moameni.

Example 4: A variational resolution for doubly nonlinear coupled equations

Let $\mathbf{b_1} : \Omega \to \mathbf{R^n}$ and $\mathbf{b_2} : \Omega \to \mathbf{R^n}$ be two smooth vector fields on a bounded domain Ω of $\mathbf{R^n}$, verifying the conditions in example 3 and let $B_1 v = \mathbf{b_1} \cdot \nabla v$ and $B_2 v = \mathbf{b_2} \cdot \nabla v$ be the corresponding first order linear operators. Consider the Dirichlet problem:

$$\begin{cases} \Delta(v+u) + \mathbf{b_1} \cdot \nabla u &=& |u|^{p-2}u + u^{m-1}v^m + f \text{ on } \Omega \\ \Delta(v-u) + \mathbf{b_2} \cdot \nabla v &=& |v|^{q-2}q - u^m v^{m-1} + g \text{ on } \Omega \\ u = v &=& 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{52}$$

We can use the above to get

Theorem 4.8 *Assume* $\mathrm{div}(\mathbf{b_1}) \geq 0$ *and* $\mathrm{div}(\mathbf{b_2}) \geq 0$ *on* Ω, $2 < p, q \leq \frac{2n}{n-2}$ *and* $1 < m < \frac{n+2}{n-2}$ *and consider on* $H_0^1(\Omega) \times H_0^1(\Omega)$ *the functional*

$$I(u,v) = \Psi(u) + \Psi^*(\mathbf{b_1}.\nabla u + \frac{1}{2}\mathrm{div}(\mathbf{b_1})\,u + \Delta v - u^{m-1}v^m) + \Phi(v) + \Phi^*(\mathbf{b_2}.\nabla v + \frac{1}{2}\mathrm{div}(\mathbf{b_2})\,v - \Delta u + u^m v^{m-1})$$

where

$$\Psi(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx+\frac{1}{p}\int_\Omega|u|^pdx+\int_\Omega fudx+\frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b_1})\,|u|^2dx,$$
$$\Phi(v)=\frac{1}{2}\int_\Omega|\nabla v|^2dx+\frac{1}{q}\int_\Omega|v|^qdx+\int_\Omega gvdx+\frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b_2})\,|v|^2dx$$

and Ψ^* *and* Φ^* *are their Legendre transforms. Then there exists* $(\bar u,\bar v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *such that:*

$$I(\bar u,\bar v)=\inf\{I(u,v);(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\}=0,$$

and $(\bar u,\bar v)$ *is a solution of* (52).

Proof: Let $A=\Delta$ on H_0^1, $B_1=\operatorname{div}(\mathbf{b_1})$, $B_2=\operatorname{div}(\mathbf{b_2})$ and consider the ASD Lagrangian

$$L((u,v),(r,s)))=\Psi(u)+\Psi^*(\mathbf{b_1}.\nabla u+\frac{1}{2}\operatorname{div}(\mathbf{b_1})\,u+\Delta v+r)+\Phi(v)+\Phi^*(\mathbf{b_2}.\nabla v+\frac{1}{2}\operatorname{div}(\mathbf{b_2})\,v-\Delta u+s).$$

It is also easy to verify that the nonlinear operator $\Lambda: H_0^1\times H_0^1\to H^{-1}\times H^{-1}$ defined by

$$\Lambda(u,v)=(-u^{m-1}v^m,u^mv^{m-1})$$

is regular and conservative.

5 Nonlinear evolution equations

Consider now an evolution triple $X\subset H\subset X^*$, where H is a Hilbert space with $\langle,\rangle$ as scalar product, and where X is a dense vector subspace of H, that is a reflexive Banach space once equipped with its own norm $\|\cdot\|$. Assuming the canonical injection $X\to H$, continuous, we identify the Hilbert space H with its dual H^* and we "inject" H in X^* in such a way that

$$\langle h,u\rangle_{X^*,X}=\langle h,u\rangle_H\quad\text{for all } h\in H \text{ and all } u\in X$$

This injection is continuous, one-to-one, and H is also dense in X^*. In other words, the dual X^* of X is represented as the completion of H for the dual norm $\|h\|=\sup\{\langle h,u\rangle_H;\|u\|_X\le 1\}$.
Let $[0,T]$ be a fixed real interval and consider the following Banach spaces:

- The space L^2_X of Bochner integrable functions from $[0,T]$ into X with norm

$$\|u\|^2_{L_2(X)}=(\int_0^T\|u(t)\|^2_Xdt)^{\frac{1}{2}}.$$

- The space $\mathcal{X}_2$ of all functions in L^2_X such that $\dot u\in L^2_{X^*}$, equipped with the norm

$$\|u\|_{\mathcal{X}}=(\|u\|^2_{L_2(X)}+\|\dot u\|^2_{L_2(X^*)})^{1/2}.$$

Note that this last space is different from the Sobolev space

$$A^2_X=\{u:[0,T]\to X;\,\dot u\in L^2_X\}$$

and we actually have $A^2_X\subset\mathcal{X}_2\subset A^2_{X^*}$.

Definition 5.1 *A time dependent Lagrangian on* $[0,T]\times X\times X^*$ *is any function* $L:[0,T]\times X\times X^*\to\mathbb{R}\cup\{+\infty\}$ *that is measurable with respect to the* σ*-field generated by the products of Lebesgue sets in* $[0,T]$ *and Borel sets in* $H\times H$*. The Hamiltonian* H_L *of* L *is the function defined on* $[0,T]\times X\times X^*$ *by:*

$$H(t,x,y)=\sup\{\langle y,p\rangle-L(t,x,p);p\in X^*\}$$

We say that L *is an anti-self dual Lagrangian (ASD) on* $[0,T]\times X\times X^*$ *if for any* $t\in[0,T]$*, the map* $L_t:(x,p)\to L(t,x,p)$ *is in* $\mathcal{L}_{\rm AD}(X)$*: that is if*

$$L^*(t,p,x)=L(t,-x,-p)\quad\text{for all } (x,p)\in X\times X^*,$$

where here L^* *is the Legendre transform in the last two variables.*

The most basic time-dependent ASD-Lagrangians are again of the form

$$L(t,x,p) = \varphi(t,x) + \varphi^*(t,-p)$$

where for each t, the function $x \to \varphi(t,x)$ is convex and lower semi-continuous on X. We now show how this property naturally "lifts" to path space. For that we associate to each time-dependent Lagrangian L on $[0,T]\times X\times X^*$, the corresponding Lagrangian $\mathcal{L}$ on the path space $L^2_X \times L^2_{X^*}$ defined by

$$\mathcal{L}(u,p) := \int_0^T L(t,u(t),p(t))dt.$$

Define the dual of $\mathcal{L}$ in both variables as

$$\mathcal{L}^*(q,v) = \sup\left\{\int_0^T (\langle q(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),p(t)))dt\ ; (u,p)\in L^2_X\times L^2_{X^*}\right\}$$

and denote the associated Hamiltonian on path space by:

$$H_{\mathcal{L}}(u,v) = \sup\left\{\int_0^T (\langle p(t),v(t)\rangle - L(t,u(t),p(t)))dt\ ;\ p\in L^2_{X^*}\right\}$$

The following is standard (see [8]).

Proposition 5.1 *Suppose that L is a Lagrangian on $[0,T]\times X\times X^*$, and let $\mathcal{L}$ be the corresponding Lagrangian on the path space $L^2_X\times L^2_{X^*}$. Then*

1. $\mathcal{L}^*(p,u) = \int_0^T L^*(t,p(t),u(t))dt$.
2. $H_{\mathcal{L}}(u,v) = \int_0^T H_L(t,u(t),v(t))dt$.
3. *If L is an anti-self dual Lagrangian on $[0,T]\times X\times X^*$, then $\mathcal{L}$ is anti-selfdual on L^2_X.*

Proposition 5.2 *Suppose ℓ is a self-dual boundary Lagrangian on $H\times H$ and let L be an anti-self dual Lagrangian on $[0,T]\times X\times X^*$ such that*

$$\text{For each } p\in L^2_{X^*}, \text{ the map } u\to \int_0^T L(t,u(t),p(t))dt \text{ is continuous on } L^2_X \tag{53}$$

$$\text{The map } u\to \int_0^T L(t,u(t),0)dt \text{ is bounded on the bounded sets of } L^2_X \tag{54}$$

$$\ell(a,b)\le C(1+\|a\|^2_H+\|b\|^2_H) \text{ for all } (a,b)\in H\times H. \tag{55}$$

Then the Lagrangian

$$\mathcal{M}_L(u,p) = \begin{cases} \int_0^T L(t,u(t),p(t)+\dot u(t))dt + \ell(u(0),u(T)) & \text{if} \quad u\in\mathcal{X}_2 \\ +\infty & \text{otherwise}\end{cases}$$

is anti-self dual on L^2_X.

Proof: For $(q,v)\in L^2_X\times\mathcal{X}_2$, write:

$$\begin{aligned}\mathcal{M}^*_L(q,v) &= \sup_{u\in L^2_X}\sup_{p\in L^2_{X^*}}\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t),p(t)\rangle - L(t,u(t),p(t)+\dot u(t)))dt - \ell(u(0),u(T))\}\\ &= \sup_{u\in \mathcal{X}_2}\sup_{p\in L^2_{X^*}}\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t),p(t)\rangle - L(t,u(t),p(t)+\dot u(t)))dt - \ell(u(0),u(T))\}\end{aligned}$$

Make a substitution $p(t)+\dot u(t) = r(t)\in L^2_{X^*}$. Since u and v are both in $\mathcal{X}_2$, we have:

$$\int_0^T \langle v,\dot u\rangle = -\int_0^T \langle \dot v,u\rangle + \langle v(T),u(T)\rangle - \langle v(0),u(0)\rangle,$$

and since the subspace $\mathcal{X}_{2,0} = \{u \in \mathcal{X}_2;\, u(0) = u(T) = 0\}$ is dense in L^2_X, we obtain

$$\begin{aligned}
\mathcal{M}_L^*(q,v) &= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t), r(t)-\dot u(t)\rangle - L(t,u(t),r(t)))dt - \ell(u(0),u(T))\\
&= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\{\int_0^T (\langle u(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,u(t),r(t)))dt\\
&\qquad -\langle v(T),u(T)\rangle + \langle v(0),u(0)\rangle - \ell(u(0),u(T))\}\\
&= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{u_0\in\mathcal{X}_{2,0}}\{\int_0^T (\langle u(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,u(t),r(t)))dt\\
&\qquad -\langle v(T),(u+u_0)(T)\rangle + \langle v(0),(u+u_0)(0)\rangle - \ell((u+u_0)(0),(u+u_0)(T))\}\\
&= \sup_{w\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{u_0\in\mathcal{X}_{2,0}}\{\int_0^T (\langle w(t)-u_0(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,w(t)-u_0(t),r(t)))dt\\
&\qquad -\langle v(T),w(T)\rangle + \langle v(0),w(0)\rangle - \ell(w(0),w(T))\}\\
&= \sup_{w\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{x\in L^2_X}\{\int_0^T (\langle x(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x(t),r(t)))dt\\
&\qquad -\langle v(T),w(T)\rangle + \langle v(0),w(0)\rangle - \ell(w(0),w(T))\}
\end{aligned}$$

Here we have used the fact that $\mathcal{X}_{2,0}$ is dense in L^2_X and the continuity of $u \to \int_0^T L(t,u(t),p(t))dt$ on L^2_X for each p.
Now, for each $(a,b) \in X \times X$, there is $w \in \mathcal{X}_2$ such that $w(0) = a$ and $w(T) = b$, namely the linear path $w(t) = \frac{(T-t)}{T}a + \frac{t}{T}b$.
Since also X is dense in H and ℓ is continuous on H, we finally obtain that

$$\begin{aligned}
\mathcal{M}_L^*(q,v) &= \sup_{(a,b)\in X\times X}\sup_{r\in L^2_{X^*}}\sup_{x\in L^2_X}\{\int_0^T (\langle x(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x(t),r(t)))dt\\
&\qquad -\langle v(T),b\rangle + \langle v(0),a\rangle - \ell(a,b)\}\\
&= \sup_{x\in L^2_X}\sup_{r\in L^2_{X^*}}\{\int_0^T (\langle x(t),q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x(t),r(t)))dt\\
&\quad + \sup_{a\in H}\sup_{b\in H}\{-\langle v(T),b\rangle + \langle v(0),a\rangle - \ell(a,b)\}\\
&= \int_0^T L^*(t,q(t)+\dot v(t),v(t))dt + \ell^*(v(0),-v(T))\\
&= \int_0^T L(t,-v(t),-\dot v(t)-q(t))dt + \ell(-v(0),-v(T))\\
&= M(-v,-q).
\end{aligned}$$

If now $(q,v) \in L^2_{X^*} \times L^2_X \setminus \mathcal{X}_2$, then we use the fact that $u \to \int_0^T L(t,u(t),0)dt$ is bounded on the unit ball of $\mathcal{X}_2$ and the growth condition on ℓ to deduce

$$
\begin{aligned}
\mathcal{M}_L^*(q,v) &\geq \sup_{u\in\mathcal{X}_2}\sup_{r\in\mathcal{X}_2}\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t),r(t)\rangle - \langle v(t),\dot u(t)\rangle - L(t,u(t),r(t)))dt - \ell(u(0),u(T))\} \\
&\geq \sup_{u\in\mathcal{X}_2}\sup_{r\in\mathcal{X}_2}\{-\|u\|_{L^2_X}\|q\|_{L^2_{X^*}} - \|v\|_{L^2_X}\|r\|_{L^2_{X^*}} + \int_0^T (-\langle v(t),\dot u(t)\rangle - L(t,u(t),r(t)))dt - \ell(u(0),u(T))\} \\
&\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\{-\|q\|_2 + \int_0^T (\langle -v(t),\dot u(t)\rangle - L(t,u(t),0))dt - \ell(u(0),u(T))\} \\
&\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\{C + \int_0^T (\langle -v(t),\dot u(t)\rangle - L(t,u(t),0))dt - \frac{1}{2}(\|u(0)\|^2 + \|u(T)\|^2)\} \\
&\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\{D + \int_0^T \langle -v(t),\dot u(t)\rangle dt - \frac{1}{2}(\|u(0)\|_X^2 + \|u(T)\|_X^2)\}.
\end{aligned}
$$

Since now v does not belong to $\mathcal{X}_2$, we have that

$$\sup_{\|u\|_{\mathcal{X}_2}\leq 1}\int_0^T (\langle v(t),\dot u(t)\rangle dt + \frac{1}{2}(\|u(0)\|_X^2 + \|u(T)\|_X^2) = +\infty$$

which means that $M^*(q,v) = +\infty = M(-v,-q)$.

Now we can prove the following

Theorem 5.2 *Let $X \subset H \subset X^*$ be an evolution pair and consider an anti-self dual Lagrangian L on $[0,T]\times X\times X^*$ and a self-dual boundary Lagrangian ℓ on $H\times H$. Assume the following conditions:*

$$\text{For each } p \in L^2_{X^*}, \text{ the map } u \to \int_0^T L(t,u(t),p(t))dt \text{ is bounded on the bounded sets of } L^2_X \tag{56}$$

$$\lim_{\|v\|_{L^2(X)}\to+\infty}\int_0^T H_L(t,0,v(t))dt = +\infty, \tag{57}$$

and

$$\ell(a,b) \leq C(1+\|a\|_H^2 + \|b\|_H^2) \text{ for all } (a,b)\in H\times H. \tag{58}$$

(1) Then for any regular conservative operator $\Lambda : D(\Lambda)\subset L^2(X)\to L^2(X^*)$ such that $\mathcal{X}_2 \subset D(\Lambda)$, the following functional

$$I_{\ell,L,\Lambda}(u) = \int_0^T L(t,u(t),\Lambda u(t)+\dot u(t))dt + \ell(u(0),u(T))$$

has zero infimum. Moreover, there exists $v\in\mathcal{X}_2$ such that:

$$\big(v(t),\Lambda v(t)+\dot v(t)\big)\in \mathrm{Dom}(L) \text{ for almost all } t\in[0,T] \tag{59}$$

$$I_{\ell,L,\Lambda}(v) = \inf_{u\in\mathcal{X}_2} I_{\ell,L,\Lambda}(u) = 0, \tag{60}$$

$$L(t,v(t),\Lambda v(t)+\dot v(t)) + \langle v(t),\dot v(t)\rangle = 0 \text{ for almost all } t\in[0,T], \tag{61}$$

$$\ell(v(0),v(T)) = \frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2), \tag{62}$$

$$(-\dot v(t)-\Lambda v(t), -v(t)) \in \partial L(t,v(t),\dot v(t)+\Lambda v(t)). \tag{63}$$

(2) In particular, for every $v_0\in H$ the following functional

$$I_{v_0,L,\Lambda}(u) = \int_0^T L(t,u(t),\Lambda u(t)+\dot u(t))dt + \frac{1}{2}\|u(0)\|^2 - 2\langle v_0,u(0)\rangle + \|v_0\|^2 + \frac{1}{2}\|u(T)\|^2$$

has minimum equal to zero on L^2_X. It is attained at a unique path v such that $v(0)=v_0$, verifying (59- 63) and in particular

$$\|v(t)\|_H^2 = \|v_0\|^2 - 2\int_0^t L(s,v(s),\Lambda v(s)+\dot v(s))ds \text{ for every } t\in[0,T]. \tag{64}$$

Proof: First apply Proposition 5.2 to get that the Lagrangian

$$\mathcal{M}_L(u,p) = \begin{cases} \int_0^T L(t,u(t),p(t)+\dot{u}(t))dt + \ell(u(0),u(T)) & \text{if} \quad u \in \mathcal{X}_2 \\ +\infty & \quad\text{otherwise} \end{cases}$$

is anti-self dual on L^2_X. It is now sufficient to apply Corollary 3.5 to conclude that the infimum of $\mathcal{M}_L(u,\Lambda u)$ is equal 0 and is achieved. This yields claim (59) and (60).
Since $L(t,v(t),\dot{v}(t)) \geq -\langle v(t),\dot{v}(t)\rangle$ for all $t\in[0,T]$, and since $\ell(v(0),v(T)) \geq \frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2)$, claims (61) and (62) follow from the following identity

$$0 = I_{\ell,L,\Lambda}(v) = \int_0^T L(t,v(t),\Lambda v(t)+\dot{v}(t)+\langle v(t),\dot{v}(t)\rangle)dt - \frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2) + \ell(v(0),v(T)).$$

To prove (63), use (61), the fact that L is anti-selfdual and that Λ is conservative to write:

$$L(s,v(s),\Lambda v(s)+\dot{v}(s)) + L^*(s,-\Lambda v(s)-\dot{v}(s),-v(s)) + \langle (v(s),\Lambda v(s)+\dot{v}(s)),(\Lambda v(s)+\dot{v}(s),v(s))\rangle = 0$$

and conclude by the limiting case of the Legendre-Fenchel duality in the space $X\times X^*$.
For (2) it suffices to apply the first part with the boundary Lagrangian

$$\ell(r,s) = \frac{1}{2}\|r\|^2 - 2\langle v_0,r\rangle + \|v_0\|^2 + \frac{1}{2}\|s\|^2.$$

which is clearly self-dual. We then get

$$I_{\ell,L,\Lambda}(u) = \int_0^T \left[L(t,u(t),\Lambda u(t)+\dot{u}(t)) + \langle u(t),\dot{u}(t)\rangle\right]dt + \|u(0)-v_0\|^2.$$

Note also that (61) yields

$$\frac{d(|v(s)|^2)}{ds} = -2L(s,v(s),\Lambda v(s)+\dot{v}(s)),$$

which readily implies (64).

We now apply the results of the last section to the particular class of ASD Lagrangian of the form $L(x,p) = \varphi(x) + \varphi^*(Ax-p)$ to obtain variational formulations and proofs of existence for various nonlinear parabolic equations.

Proposition 5.3 *Let $X\subset H\subset X^*$ be an evolution triple and consider for each $t\in[0,T]$ a bounded linear operator $A_t : X\to X^*$ and $\varphi:[0,T]\times X\to\bar{\mathbb{R}}$ such that for each t the functional $\psi(t,x) := \varphi(t,x)+\frac{1}{2}\langle A_t x,x\rangle$ is convex, lower semi-continuous and satisfies for some $C>0$, $m,n>1$ the following growth condition:*

$$\frac{1}{C}\left(\|x\|^m_{L^2_X}-1\right) \leq \int_0^T\{\varphi(t,x(t))+\tfrac{1}{2}\langle A_t x(t),x(t)\rangle\}\,dt \leq C\left(\|x\|^n_{L^2_X}+1\right) \text{ for every } x\in L^2_X. \tag{65}$$

If $\Lambda : D(\Lambda)\subset X\to X^*$ is a regular conservative operator and $v_0\in X$, we consider on $\mathcal{X}_2$ the functional

$$I(x) = \int_0^T\{\psi(t,x(t))+\psi^*(t,-\Lambda x(t)-A^a_t x(t)-\dot{x}(t))\}\,dt + \frac{1}{2}(|x(0)|^2+|x(T)|^2) - 2\langle x(0),v_0\rangle + |v_0|^2,$$

where for each $t\in[0,T]$, A^a_t is the anti-symmetric part of the operator A_t. Then there exists a path $v\in\mathcal{X}_2$ such that

$$I(v) = \inf_{x\in\mathcal{X}_2} I(x) = 0. \tag{66}$$

$$\begin{array}{rcl} -\dot{v}(t)-A_t v(t)-\Lambda v(t) & \in & \partial\varphi(t,v(t)) \quad \text{for a.e. } t\in[0,T] \\ v(0) & = & v_0. \end{array} \tag{67}$$

Proof: The Lagrangian $L(t,x,p) := \psi(t,x)+\psi^*(t,-A^a x-p)$ is an ASD Lagrangian on $X\times X^*$ by Proposition 2.5. Consider ℓ on $H\times H$ to be $\ell(r,s)=\frac{1}{2}(|r|^2+|s|^2)-2\langle r,v_0\rangle+|v_0|^2$, and lift Λ to a regular conservative operator $\tilde{\Lambda}$ from its domain in $L^2_X([0,T])$ into $L^2_{X^*}([0,T])$ by setting $(\tilde{\Lambda}x)(t)=\Lambda(x(t))$. It is easy to check that all the conditions of Theorem 5.2 are satisfied by L, ℓ, B and $\tilde{\Lambda}$, hence there exists $v\in\mathcal{X}_2$ such that $I(v)=0$. We obtain

$$0=\int_0^T \big(\psi(t,v(t))+\psi^*(t,-\Lambda v(t)-A^a_t v(t)-\dot{v}(t))+\langle v(t),\Lambda v(t)+A_t v(t)+\dot{v}(t)\rangle\big)\,dt+\frac{1}{2}\|v(0)-v_0\|^2_H$$

which yields since the integrand is non-negative for each t and since we are now in the limiting case of Legendre-Fenchel duality that

$$\begin{aligned} -\dot{v}(t)-A^a_t v(t)-\Lambda v(t) &\in \partial\varphi(t,v(t))+A^s_t v(t) \quad \text{for a.e. } t\in[0,T] \\ v(0) &= v_0. \end{aligned} \tag{68}$$

Example 5: Navier-Stokes evolutions

We now consider the evolution equation associated to a fluid driven by its boundary.

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u+f &= \nu\Delta u-\nabla p \quad \text{on } [0,T]\times\Omega \\ \operatorname{div} u &= 0 \quad \text{on } [0,T]\times\Omega \\ u(t,x) &= u^0(x) \quad \text{on } [0,T]\times\partial\Omega \\ u(0,x) &= u_0(x) \quad \text{on } \Omega \end{cases} \tag{69}$$

where $\int_{\partial\Omega} u^0\cdot\mathbf{n}\,d\sigma=0$, $\nu>0$ and $f\in L^p(\Omega;\mathbb{R}^3)$. Assuming that $u^0\in H^{3/2}(\partial\Omega)$ and that $\partial\Omega$ is connected, Hopf's extension theorem again yields the existence of $v^0\in H^2(\Omega)$ such that

$$v^0=u^0 \text{ on } \partial\Omega,\quad \operatorname{div} v^0=0 \quad\text{and}\quad \int_\Omega \Sigma^3_{j,k=1} u_k \frac{\partial v^0_j}{\partial x_k} u_j\,dx \le \epsilon\|u\|^2_V \text{ for all } u\in V \tag{70}$$

where $V=\{u\in H^1(\Omega;\mathbf{R}^3);\operatorname{div} v=0\}$. Setting $v=u+v^0$, then solving (69) reduces to finding a solution in the Banach space $V_0=\{u\in H^1_0(\Omega;\mathbf{R}^3);\operatorname{div} v=0\}$ for

$$\begin{aligned} \frac{\partial u}{\partial t}+(u\cdot\nabla)u+(v^0\cdot\nabla)u+(u\cdot\nabla)v^0+g &\in -\partial\Phi(u) \\ u(0) &= u_0-v^0. \end{aligned} \tag{71}$$

where Φ is again the convex Dirichlet energy functional $\Phi(u)=\frac{\nu}{2}\int_\Omega \Sigma^3_{j,k=1}(\frac{\partial u_j}{\partial x_k})^2\,dx$ and where

$$g := f-\nu\Delta v^0+(v^0\cdot\nabla)v^0\in V^*.$$

In other words, this is an equation of the form

$$\frac{\partial u}{\partial t}+\Lambda u+Bu+g\in -\partial\Phi(u) \tag{72}$$

where $\Lambda u=(u\cdot\nabla)u$ is a regular conservative operator, and $Bu=(v^0\cdot\nabla)u+(u\cdot\nabla)v^0$ is a bounded linear operator on V. The component $B^1 u:=(v^0\cdot\nabla)u$ of B is skew-symmetric which means that Hopf's estimate implies

$$C\|u\|^2_V\ge\Psi(u):=\Phi(u)+\frac{1}{2}\langle Bu,u\rangle\ge\frac{1}{2}(\nu-\epsilon)\|u\|^2 \quad \text{for all } u\in V.$$

Letting A^a be the antisymmetric part of the operator $Au=(u\cdot\nabla)v^0$, we can now apply Proposition 5.3 to obtain

Theorem 5.3 *Under the above hypothesis on u^0, and for $f\in L^p(\Omega,\mathbb{R}^3)$ with $p>\frac{6}{5}$ and $u_0\in V$, the minimum of the functional*

$$
\begin{aligned}
I(u) &= \int_0^T \left\{ \Psi(u(t)) + \Psi^*(-(u\cdot\nabla)u - B^a u + f - \dot u) - \int_\Omega \langle f, u\rangle dx \right\} dt \\
&\quad + \int_\Omega \left\{ \frac{1}{2}(|u(0,x)|^2 + |u(x,T)|^2) - 2\langle u(0,x), u_0(x) - v^0(x)\rangle + |u_0(x) - v^0(x)|^2 \right\} dx
\end{aligned}
$$

on A_V^2 is zero and is attained at a solution of the equation (71).

References

[1] G. Auchmuty. *Saddle points and existence-uniqueness for evolution equations*, Differential Integral Equations, **6** (1993), 1161–1171.

[2] G. Auchmuty. *Variational principles for operator equations and initial value problems*, Nonlinear Analysis, Theory, Methods and Applications Vol. **12**, No.5, pp. 531-564 (1988).

[3] C. Bardos. *Problèmes aux limites pour les equations aux dérivées partielles du premier ordre a coefficients réels; Théorèmes d'approximation; Application à l'équation de transport*, Ann. scient. Ec. Norm. Sup., 4^e série, t. 3, (1970), p. 185-233.

[4] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 971–974.

[5] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 1197–1198.

[6] H. Brezis, L. Nirenberg, G. Stampachia, *A remark on Ky Fan's Minimax Principle*, Bollettino U. M. I (1972), 293-300

[7] N. Ghoussoub, *A theory of anti-selfdual Lagrangians: Dynamical case*, C.R. Acad. Sci., Paris, Ser. I 340 (2005) p. 325-330

[8] N. Ghoussoub, *Anti-selfdual Lagrangians: Variational resolutions of non self-adjoint equations and dissipative evolutions*, (2005). Submitted, 50 pages

[9] N. Ghoussoub, A. Moameni, *A general nonlinear variational principle and its applications to partial differential equations and systems*, In preparation (2005)

[10] N. Ghoussoub, L. Tzou. *Anti-self dual Lagrangians II: Unbounded non self-adjoint operators and evolution equations*, (2005). Submitted, 30 pages

[11] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Applied mathematical sciences, 68, Springer-Verlag (1997).