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A Theory of Anti-Selfdual Lagrangians: Dynamical case

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Abstract

We consider the class of *time-dependent anti-selfdual Lagrangians*, which –just like the stationary case announced in [5]– enjoys remarkable permamence properties and provides variational formulations and resolutions for several initial-value parabolic equations including gradient flows and other dissipative systems. Even though these evolutions do not fit in the standard Euler-Lagrange theory, we show that their solutions –as well as those of related parabolic variational inequalities– can be obtained as minima –but also more importantly as zeroes– of action functionals of the form $\int_0^T L(t, u(t), \dot{u}(t) + \Lambda_t u(t))dt)$ where L is a time-dependent anti-selfdual Lagrangian and where Λ_t is a flow of skew-adjoint operators. Details, proofs and more applications will be given in [7] in the setting of bounded linear operators. The case of linear unbounded operators is dealt with in [11]. Nonlinear but appropriately defined "skew-adjoint" operators will be considered in [8].

Résumé

Lagrangiens anti-autoduaux: Le cas dynamique. On considère le cas des Lagrangiens anti-autoduaux qui dépendent du paramètre temps. Comme dans le cas stationnaire annoncé dans [5], cette classe possède des propriétés de permanence remarquables qui permettent une formulation et une résolution variationnelle de plusieurs équations paraboliques dissipatives qui ne sont pas normalement de type Euler-Lagrange.

Version francaise abrégée: À tout Lagrangien anti-autodual autonome L sur $X \times X^*$ (où X est reflexif), on associe un semi-group de contractions $(T_t)_{t\in\mathbf{R}^+}$ tel que $x(t) = T_t x$ est la solution de $(-\dot{x}(t), -x(t)) \in \partial L(x(t), \dot{x}(t))$ avec $x(0) = x$. On associe un nouveau principe variationnel à une classe importante d'équations –ainsi que des inéquations– paraboliques dissipatives. Les solutions sont obtenues comme minima –mais aussi surtout comme des racines– de fonctionnelles d'action de la forme $\int_0^T L(t, u(t), \dot{u}(t) + \Lambda_t u(t))dt$, où L est un Lagrangien anti-autodual et où Λ_t est un flow d'opérateurs antisymmétriques. Ces équations peuvent être des flots de gradients à potentiel convexe, comme dans l'équation de la chaleur et celle des médias poreux, mais aussi des évolutions nonlinéaires associées à des opérateurs du premier ordre, et donc non-autoadjoints.

1 Time-dependent anti-selfdual Lagrangians

Let H be a Hilbert space with $\langle\, ,\, \rangle$ as scalar product and let $[0, T]$ be a fixed real interval $(0 < T < +\infty)$. Consider the classical space L^2_H of Bochner integrable functions from $[0, T]$ into H with norm denoted by

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$\|\cdot\|_2$, as well as the Hilbert space $A^2_H = \{u : [0,T] \to H;\ \dot u \in L^2_H\}$ consisting of all absolutely continuous arcs $u : [0,T] \to H$, equipped with the norm $\|u\|_{A^2_H} = (\|u(0)\|^2_H + \int_0^T \|\dot u\|^2 dt)^{\frac{1}{2}}$.

Definition 1.1 *Let $L : [0,T]\times H\times H \to \boldsymbol{R}\cup\{+\infty\}$ be measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in $H\times H$. Say that L is an anti-self dual Lagrangian (ASD) on $[0,T]\times H\times H$ if $L_t : (x,p) \to L(t,x,p)$ is in $\mathcal{L}_{\rm AD}(H)$ for any $t\in[0,T]$: that is if $L^*(t,p,x) = L(t,-x,-p)$ for all $(x,p)\in H\times H$, where L^* is the Legendre transform in the last two variables.*

The most basic time-dependent ASD-Lagrangians are of the form $L(t,x,p) = \varphi(t,x) + \varphi^*(t,-p)$ where for each t, the function $x \to \varphi(t,x)$ is convex and lower semi-continuous.

Definition 1.2 *Say that a convex lower semi-continuous function $\ell : H\times H \to \boldsymbol{R}\cup\{+\infty\}$ is a self-dual time-boundary Lagrangian if $\ell^*(-h_1,h_2) = \ell(h_1,h_2)$ for all $(h_1,h_2)\in H\times H$.*

The basic example of a self dual boundary Lagrangian is given by a function ℓ on $H\times H$, of the form $\ell(x,p) = \frac{1}{2}\|x\|^2 - 2\langle a,x\rangle + \|a\|^2 + \frac{1}{2}\|p\|^2$, where a is given in H. A remarkable permanence property of ASD Lagrangians is that it "lifts" to path spaces.

Proposition 1.1 *Suppose that L is an anti-self dual Lagrangian on $[0,T]\times H\times H$, then*

1. *For each $\omega\in\mathbf{R}$, the Lagrangian $M(u,p) := \int_0^T e^{2wt}L(t,e^{-wt}u(t),e^{-wt}p(t))dt$ is anti-self dual on L^2_H.*
2. *If ℓ is a self-dual boundary Lagrangian on $H\times H$, then the Lagrangian*
$$M(u,p) = \begin{cases} \int_0^T L(t,u(t),p(t)+\dot u(t))dt + \ell(u(0),u(T)) & \text{if} \quad u\in A^2_H \\ +\infty & \text{otherwise}\end{cases}$$
is anti-self dual on L^2_H, provided $u \to \int_0^T L(t,u(t),p(t))dt$ is continuous on L^2_H.
3. *The Lagrangian defined on $A^2_H\times(A^2_H)^*$ by*
$$N(u,p) = \int_0^T L(t,u(t) - \textstyle\int_t^T p(s)ds,\ \dot u(t))dt + \ell(u(0) + \int_0^T p(s)ds,\ u(T))$$
is anti-selfdual on $A^2_H\times X_0^*$ where $X_0^* = \{q\in(A^2_H)^*;\ \int_0^T q(s)ds = 0\}$.

Theorem 1.3 *Suppose L is an anti-self dual Lagrangian on $[0,T]\times H\times H$ and ℓ is a self-dual boundary Lagrangian on $H\times H$, and suppose there exists $C>0$ such that for all $x\in L^2_H$,*
$$\int_0^T L(t,x(t),0)dt \le C(1+\|x\|^2_{L^2_H}).$$
Then, there exists $v\in A^2_H$ such that $\big(v(t),\dot v(t)\big)\in \mathrm{Dom}(L)$ for almost all $t\in[0,T]$ and
$$\int_0^T L(t,v(t),\dot v(t))dt + \ell(v(0),v(T)) = \inf_{u\in A^2_H}\int_0^T L(t,u(t),\dot u(t))dt + \ell(u(0),u(T)) = 0.$$
In particular, for every $v_0\in H$ the following functional on A^2_H
$$I(u) = \int_0^T L(t,u(t),\dot u(t))dt + \frac{1}{2}\|u(0)\|^2 - 2\langle v_0,u(0)\rangle + \|v_0\|^2 + \frac{1}{2}\|u(T)\|^2$$
has a minimum equal zero. It is attained at a unique path v which then satisfies the following:
$$v(0) = v_0 \text{ and } \big(v(t),\dot v(t)\big)\in\mathrm{Dom}(L) \quad \text{for almost all } t\in[0,T], \tag{1}$$
$$\frac{d}{dt}\partial_p L(t,v(t),\dot v(t)) = \partial_x L(t,v(t),\dot v(t)) \tag{2}$$
$$(-\dot v(t),-v(t))\in\partial L(t,v(t),\dot v(t)), \tag{3}$$
$$\|v(t)\|^2_H = \|v_0\|^2 - 2\int_0^t L(s,v(s),\dot v(s))ds \quad \text{for every } t\in[0,T]. \tag{4}$$

2 Semigroups associated to autonomous anti-selfdual Lagrangians

When the Lagrangian $L(x,p)$ is autonomous, the situation is much nicer since we can associate a flow without stringent boundedness or coercivity conditions. Indeed, we can then use a Yosida-type λ-regularization of ASD-Lagrangians $L_\lambda = L \star T_\lambda$ where $T_\lambda(x,p) = \frac{\|x\|^2}{2\lambda^2} + \frac{\lambda^2\|p\|^2}{2}$. Then L_λ satisfies the conditions of Theorem 1.3, and we can then find for each initial point $v \in H$, a path $v_\lambda \in A_H^2$, with $v_\lambda(0) = v$, which verify properties (1)-(4). Letting $\lambda \to 0$, we can recover a semi-group of 1-Lipschitz operators T_t defined on the *Partial Domain* of ∂L defined as follows

$$\mathrm{Dom}_1(\partial L) = \{x \in X; \text{there exists } p, q \in X^* \text{ such that } (p, 0) \in \partial L(x, q)\}.$$

Note that if $L(x,p) = \varphi(x) + \varphi^*(-p)$ with 0 assumed to be in the domain of $\partial\varphi$, then x_0 belongs to $\mathrm{Dom}_1(\partial L)$ if and only if it belongs to the usual domain of $\partial\varphi$. We then obtain the following result.

Theorem 2.1 *Let L be an anti-selfdual Lagrangian on a Hilbert space H that is uniformly convex in the first variable. Assuming $\mathrm{Dom}_1(\partial L)$ is non-empty, then there exists a semi-group of 1-Lipschitz operators $(T_t)_{t\in\mathbf{R}^+}$ on $\mathrm{Dom}_1(\partial L)$ denoted by e^{tL} such that $T_0 = Id$ and for any $x \in \mathrm{Dom}_1(\partial L)$, the path $x(t) = e^{tL}x$ satisfies the following:*

$$\frac{d}{dt}\partial_p L(x(t), \dot{x}(t)) = \partial_x L(x(t), \dot{x}(t)) \tag{5}$$

$$(-\dot{x}(t), -x(t)) \in \partial L(t, x(t), \dot{x}(t)) \tag{6}$$

$$\|x(t)\|_H^2 = \|x\|^2 - 2\int_0^t L(x(s), \dot{x}(s))ds \quad \text{for every } t \in [0,T]. \tag{7}$$

2.1 Variational resolution of initial value problems

The following was established in [10] in the case of gradient flows of convex potentials (i.e., when $A = 0$ and $\omega = 0$), and in [9] in the case of gradient flows of semi-convex functions (i.e., when $A = 0$ and $\omega > 0$).

Theorem 2.2 *Let φ be a proper, bounded below, convex lower semi-continuous functional on H such that $0 \in \mathrm{Dom}\partial\varphi$ and let A be a positive bounded linear operator on H. For any $\omega \in \mathbf{R}$ and $v_0 \in \mathrm{Dom}\partial\varphi$, consider the following functional on A_H^2:*

$$I(u) = \int_0^T e^{-2\omega t}\left\{\psi(e^{\omega t}u(t)) + \psi^*(e^{\omega t}(-A^a u(t) - \dot{u}(t))\right\}dt + \frac{1}{2}\|u(0)\|^2 - 2\langle v_0, u(0)\rangle + \|v_0\|^2 + \frac{1}{2}\|u(T)\|^2$$

where A^a is the anti-symmetric part of A, and $\psi(u) = \varphi(u) + \frac{1}{2}\langle Au, u\rangle$. The minimum of I is then zero and is attained at a path $x(t)$, in such a way that $v(t) = e^{\omega t}x(t)$ is a solution of

$$\begin{cases} -Av(t) + \omega v(t) - \dot{v}(t) & \in & \partial\varphi(v(t)) \quad \text{a.e. } t \in [0,T] \\ v(0) & = & v_0 \end{cases} \tag{8}$$

2.2 Variational resolution for coupled flows and wave-type equations

ASD Lagrangians are suited to treat variationally coupled evolution equations.

Proposition 2.1 *Let φ be a proper convex lower semi-continuous function on $X \times Y$ and let $A : X \to Y^*$ be any bounded linear operator. Assume $B_1 : X \to X$ (resp., $B_2 : Y \to Y$) are positive operators, then for any $(x_0, y_0) \in \mathrm{dom}(\partial\varphi)$ and any $(f, g) \in X \times Y$, there exists a path $(x(t), y(t)) \in A_X^2 \times A_Y^2$ such that*

$$\begin{aligned} -\dot{x}(t) - A^*y(t) - B_1x(t) + f &\in \partial_1\varphi(x(t), y(t)) \\ -\dot{y}(t) + Ay(t) - B_2y(t) + g &\in \partial_2\varphi(x(t), y(t)) \\ x(0) &= x_0 \\ y(0) &= y_0. \end{aligned}$$

The solution is obtained as a minimizer on $A^2_X \times A^2_Y$ of the following functional

$$\begin{aligned} I(x,y) &= \int_0^T \{\psi(x(t),y(t)) + \psi^*(-A^*y(t) - B_1^a x(t) - \dot{x}(t), Ax(t) - B_2^a y(t) - \dot{y}(t))\}\, dt \\ &\quad + \frac{1}{2}\|x(0)\|^2 - 2\langle x_0, x(0)\rangle + \|x_0\|^2 + \frac{1}{2}\|x(T)\|^2 \\ &\quad + \frac{1}{2}\|y(0)\|^2 - 2\langle y_0, y(0)\rangle + \|y_0\|^2 + \frac{1}{2}\|y(T)\|^2. \end{aligned}$$

whose infimum is zero. Here B_1^a (resp., B_2^a) are the skew-symmetric parts of B_1 and B_2 and

$$\psi(x,y) = \varphi(x,y) + \frac{1}{2}\langle B_1 x, x\rangle - \langle f, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle - \langle g, x\rangle$$

Proof: It is enough to apply Theorem 2.1 to the ASD Lagrangian

$$L((x,y),(p,q)) = \psi(x,y) + \psi^*(-A^*y - B_1^a x - p, Ax - B_2^a y - q).$$

3 Variational resolution for general parabolic equations

For $t \in [0,T]$, consider $(b_1^t, b_2^t) : X_t \to H_1^t \times H_2^t$ to be *regular boundary operators* from a reflexive Banach space X_t into Hilbert spaces H_1^t, H_2^t and we let $\Lambda_t : X_t \to X_t^*$ be *skew-adjoint operators modulo the boundary* (b_1^t, b_2^t): that is for every $x, y \in X_t$, we have $\langle \Lambda_t x, y\rangle_H = -\langle \Lambda_t y, x\rangle_H + \langle b_2^t(x), b_2^t(y)\rangle_{H_2^t} - \langle b_1^t(x), b_1^t(y)\rangle_{H_1^t}$. We refer to ([5], [7]) for the details. Suppose now H is a Hilbert space such that:

$$X_t \subset H \subset X_t^* \text{ is an evolution triple with } \mathrm{Ker}(b_1^t, b_2^t) \text{ being dense in } H. \tag{9}$$

$$\Lambda_t(X_t) \subset H \quad \text{and } X_t = \{x \in H; \sup\{\langle x, \Lambda_t y\rangle_H; y \in X_t, \|y\|_H \le 1\} < +\infty\} \tag{10}$$

We then call (X_t, H, Λ_t) a *maximal evolution triple.* Starting now with a time-dependent ASD Lagrangian L on H, and selfdual *state-boundary Lagrangians* $m_t : H_1^t \times H_2^t \to \boldsymbol{R} \cup \{+\infty\}$ one can prove (see [7]) that

$$M(t,x,p) = \begin{cases} L(t,x,\Lambda_t x + p) + m_t(b_1^t(x), b_2^t(x)) & \text{if} \quad x \in X_t \\ +\infty & \qquad\quad \text{otherwise} \end{cases}$$

is also anti-self dual on $H \times H$ for each $t \in [0,T]$.
If now ℓ is a self-dual time-boundary Lagrangian on H, then the following Lagrangian

$$\tilde{M}(u,p) = \int_0^T \{M(t,u(t),p(t) + \dot{u}(t))\}\, dt + \ell(u(0), u(T))$$

is anti-self dual Lagrangian on the elements of $A^2_H \times \{0\}$ which is sufficient to get that

$$I(u) = \tilde{M}(u,0) = \int_0^T \left\{L(t,u(t),\Lambda_t u(t) + \dot{u}(t)) + m(t, b_1^t u(t), b_2^t u(t))\right\} dt + \ell(u(0), u(T))$$

has a minimum at $\bar{v}(t)$, and that the minimal value is zero. Applying the theorem with the time boundary Lagrangian on H, $\ell(x,p) = \frac{1}{2}\|x\|^2 - 2\langle a, x\rangle + \|a\|^2 + \frac{1}{2}\|p\|^2$, where a is a given initial value in H, and with a state boundary Lagrangian

$$m(t,x,p) = \frac{1}{2}\|x\|^2 - 2\langle b(t), x\rangle + \|b(t)\|^2 + \frac{1}{2}\|p\|^2,$$

where $b(t)$ is prescribed in H_1^t for each t, we get that $\bar{v}(t)$ satisfies:

$$\left\{ \begin{array}{rcll} L(t,v(t),\Lambda_t v(t) + \dot{v}(t)) + \langle v(t), \Lambda_t v(t) + \dot{v}(t)\rangle &=& 0 & \text{a.e. } t \in [0,T] \\ (-\Lambda_t v(t) - \dot{v}(t), -v(t)) &\in& \partial L(t, v(t), \dot{v}(t)) & \\ b_1^t(v(t)) &=& b(t) & \text{a.e } t \in [0,T] \\ v(0) &=& a & \end{array} \right. \tag{11}$$

Theorem 3.1 *Under the above conditions on $(X_t, H, H_1^t, H_2^t, b_1^t, b_2^t)$, consider bounded linear operators $A_t : X_t \to X_t^*$ such that $A_t - \frac{1}{2}((b_2^t)^* b_2^t - (b_1^t)^* b_1^t)$ is positive and denote by Λ_t the operator $\Lambda_t = \frac{1}{2}(A_t - A_t^*) + \frac{1}{2}((b_2^t)^* b_2^t - (b_1^t)^* b_1^t)$ which is skew-adjoint modulo the boundary. For each $t \in [0,T]$, suppose (X_t, H, Λ_t) is a maximal evolution triple and that $\varphi(t,\cdot)$ is a convex continuous function on H. For $f \in L^2([0,T]; H)$, $a \in H$ and $b(t) \in H_t^1$ consider the following functional on A_H^2,*

$$\begin{aligned} I(u) \quad &= \quad \int_0^T \left\{ \psi(t,u(t)) + \psi^*(t, -\Lambda_t u(t) - \dot{u}(t)) + \frac{1}{2}(\|b_1^t u(t)\|^2 + \|b_2^t u(t)\|^2) - 2\langle b(t), u(t)\rangle + \|b(t)\|^2) \right\} dt \\ &\quad + \frac{1}{2}(\|u(0)\|^2 + \|u(T)\|^2) - 2\langle u(0), a\rangle + \|a\|^2, \end{aligned}$$

where $\psi(t,x) = \varphi(t,x) + \frac{1}{2}\langle A_t x, x\rangle - \frac{1}{4}(\|b_2^t x\|^2 - \|b_1^t x\|^2) + \langle f(t), x\rangle$. Suppose there is $C > 0$ so that for every $x \in L_H^2$,

$$\int_0^T \psi(t,x(t)) + \psi^*(t, -\Lambda_t x(t) dt \leq C(1 + \|x\|_{L_H^2}^2).$$

Then there exists $v \in A_H^2$ such that $I(v) = \inf\limits_{u \in A_H^2} I(u) = 0$. Moreover, v solves

$$\left\{ \begin{array}{rcll} -A_t v(t) - \dot{v}(t) & \in & \partial\varphi(t, v(t)) + f(t) & \text{a.e. } t \in [0,T] \\ b_1^t(v(t)) & = & b(t) & \text{a.e } t \in [0,T] \\ v(0) & = & a & \end{array} \right. \tag{12}$$

Non linear Transport evolutions: With the notation of example 1 of [5], we consider the equation

$$\left\{ \begin{array}{rcll} \frac{\partial u}{\partial t} - \Sigma_{i=1}^n a_i \frac{\partial u}{\partial x_i} - a_0 u & = & \beta(u) + f & \text{on} \quad [0,T] \times \Omega \\ u(t,x) & = & b(t,x) & \text{on} \quad [0,T] \times \Sigma_-. \\ u(0,x) & = & u_0(x) & \text{on } \Omega \end{array} \right. \tag{13}$$

where $u_0 \in H_A^1(\Omega)$, $f \in H_A^1(\Omega)^*$ and where $b(t) \in L^2(\Sigma, \mathbf{n} \cdot \mathbf{a} dx)$ for each $t \in [0,T]$. Let

$$\psi(u) = \int_\Omega \left\{ j(u(x)) + f(x)u(x) + \frac{1}{2}(a_0 - \frac{1}{2}\text{div a})u^2) \right\} dx$$

Theorem 3.2 *Assume $a_0(x) - \frac{1}{2}\text{div}a(x) \geq \alpha > 0$ on Ω, and consider the following functional on the space $X := A^2([0,T]; H_A^1(\Omega))$.*

$$\begin{aligned} I(u) \quad &= \quad \int_0^T \left\{ \psi(u(t)) + \psi^*(-\mathbf{a} \cdot \nabla_x u(t) - \frac{1}{2}\text{div a}\, u(t) - \dot{u}(t)) \right\} dt \\ &\quad + \int_0^T \left\{ \frac{1}{2}\int_{\Sigma_+} |u(t,x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(t,x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma + \int_{\Sigma_-} (|b(t,x)|^2 - 2b(t,x)u(t,x))\mathbf{n} \cdot \mathbf{a}\, d\sigma \right\} dt \\ &\quad + \int_\Omega \left\{ \frac{1}{2}(|u(0,x)|^2 + |u(x,T)|^2) - 2\langle u(0,x), u_0(x)\rangle + |u_0(x)|^2 \right\} dx. \end{aligned}$$

There exists then $\bar{u} \in X$ such that $I(\bar{u}) = \inf_{u \in X} I(u) = 0$ and which solves equation (13).

4 Variational resolution for parabolic variational inequalities

Consider for each time t, a bilinear continuous functional a_t on a Hilbert space $H \times H$ and a convex l.s.c function $\varphi(t,\cdot) : H \to \mathbf{R} \cup \{+\infty\}$. Solving the corresponding parabolic variational inequality amounts to constructing for a given $f \in L^2([0,T]; H)$ and $x_0 \in H$, a path $x(t) \in A_H^2([0,T])$ such that for all $z \in H$,

$$\langle \dot{x}(t), x(t) - z) + a_t(x(t), x(t) - z) + \varphi(t, x(t)) - \varphi(t,z) \leq \langle x(t) - z, f(t)\rangle. \tag{14}$$

for almost all $t \in [0,T]$. This problem can be rewritten as: $f(t) \in \dot{y}(t) + A_t y(t) + \partial\varphi(t,y)$, where A_t is the bounded linear operator on H defined by $a_t(u,v) = \langle A_t u, v\rangle$. This means that the variational inequality (14) can be rewritten and solved using the variational principle in Theorem 1.3. For example, one can then solve variationally the following "obstacle " problem.

Corollary 4.1 *Let $(a_t)_t$ be bilinear continuous functionals on $H \times H$ satisfying:*

- *For some $\lambda > 0$, we have $a_t(v,v) \geq \lambda \|v\|^2$ on H for every $t \in [0,T]$.*
- *The map $u \to \int_0^T a_t(u(t),u(t))dt$ is continuous on L^2_H.*

If K is a convex closed subset of H, then for any $f \in L^2([0,T];H)$ and any $x_0 \in K$, there exists a path $x \in A^2_H([0,T])$ such that $x(0) = x_0$, $x(t) \in K$ for almost all $t \in [0,T]$ and

$$\langle \dot{x}(t), x(t) - z\rangle + a_t(x(t), x(t) - z) \leq \langle x(t) - z, f\rangle \quad \text{for all } z \in K.$$

The path $x(t)$ is obtained as a minimizer of the following functional on $A^2_H([0,T])$:

$$I(y) = \int_0^T \{\varphi(t,y(t)) + (\varphi(t,\cdot) + \psi_K)^*(-\dot{y}(t) - \Lambda_t y(t))\}\, dt + \frac{1}{2}(|y(0)|^2 + |y(T)|^2) - 2\langle y(0), x_0\rangle + |x_0|^2.$$

Here $\varphi(t,y) = \frac{1}{2}a_t(y,y) - \langle f(t), y\rangle$, $\Lambda_t : H \to H$ is the skew-adjoint operator defined by $\langle \Lambda_t u, v\rangle = \frac{1}{2}(a_t(u,v) - a_t(v,u))$, and $\psi_K(y) = 0$ on K and $+\infty$ elsewhere.

References

[1] G. Auchmuty. *Saddle points and existence-uniqueness for evolution equations*, Differential Integral Equations, **6** (1993), 1161–1171.

[2] C. Bardos. *Problèmes aux limites pour les equations aux dérivées partielles du premier ordre a coefficients reéls; Théorèmes d'approximation; Application à l'équation de transport*, Ann. scient. Ec. Norm. Sup., 4^e série, t. 3, (1970), p. 185-233.

[3] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 971–974.

[4] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 1197–1198.

[5] N. Ghoussoub, *A theory of anti-selfdual Lagrangians: Stationary case*, C.R. Acad. Sci. Paris (2004)

[6] N. Ghoussoub, *A variational principle for non-linear transport equations*, Communications on Pure and Applied Analysis, To appear (2004) 10 p.

[7] N. Ghoussoub, *Anti-self dual Lagrangians: Variational resolutions for non self-adjoint equations and dissipative evolutions*, (2004). Submitted.

[8] N. Ghoussoub, *Anti-self dual Lagrangians II: Variational resolution for Navier-Stokes equations and other nonlinear evolutions*, In preparation (2004)

[9] N. Ghoussoub, R. McCann *A least action principle for steepest descent in a non-convex landscape*, Contemporary Math. Vol 362 (2004) p. 177-187.

[10] N. Ghoussoub, L. Tzou. *A variational principle for gradient flows*, Math. Annalen, Vol 30, 3 (2004) p. 519-549

[11] N. Ghoussoub, L. Tzou. *Anti-self dual Lagrangians III: Unbounded non self-adjoint operators and evolution equations*, (2004). In preparation.