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Abstract

We develop a concept of *anti-self dual Lagrangians* that seems inherent to many problems in mathematical physics, Riemannian geometry, and differential equations. On one hand, they represent gradients of convex functions which usually drive dissipative systems, and on the other, their structure is rich enough to also cover – certain representations of– skew-symmetric operators which normally generate unitary flows. These Lagrangians provide variational formulations and resolutions for several non-potential boundary value problems many of which do not fit in the Euler-Lagrange theory. Solutions are minima of functionals of the form $L(u, \Lambda u)$ where L is an anti-self dual Lagrangian and where Λ is a skew-adjoint operator. However, and just like the self (antiself) dual equations of quantum field theory (e.g. Yang-Mills, Seiberg-Witten and Ginzburg-Landau) the equations associated to minimal solutions of our variational problems are not derived from the fact they are critical points of the associated functionals, but because they are also zeroes of the corresponding Lagrangians.

Résumé

Une théorie des Lagrangiens anti-autoduaux: Cas stationnaire: On introduit et développe la notion de *Lagrangien anti-autodual* qui apparait dans plusieurs problèmes de géométrie et de physique théorique. Cette classe inclut les champs de gradient de fonctions convexes qui sont à la base de systèmes dissipatifs, mais aussi contient les opérateurs anti-symétriques qui, par contre, engendrent des flots conservatifs. Comme pour les équations autoduales de Yang-Mills, Seiberg-Witten et Ginzburg-Landau, ces Lagrangiens permettent la résolution variationnelle de plusieurs équations différentielles du premier ordre qui ne rentrent pas donc dans le cadre de la théorie de Euler-Lagrange.

Version francaise abrégée: On montre que plusieurs équations de la forme $Au + \partial\varphi(u) = f$ avec φ convexe s.c.i. sur un reflexif X, $f \in X^*$ et $A : X \to X^*$ étant un opérateur linéaire borné positif, peuvent être résolues en minimisant des fonctionelles de la forme $I(u) = L(u, \Lambda u)$ où L est un *Lagrangien anti-autodual* (i.e $L^*(p, x) = L(-x, -p)$) sur $X \times X^*$ et où Λ est un opérateur antisymétrique de X dans X^*. Ces Lagrangiens permettent des formulations et des résolutions variationnelles de plusieurs équations différentielles qui ne rentrent pas normalement dans le cadre de la théorie classique de Euler-Lagrange, puisque

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l'opérateur A n'est pas supposé être auto-adjoint. Comme application, on considère des équations de transport non-linéaires et des inégalités variationnelles. L'approche s'étend au cas où il y a des conditions au bord via un opérateur frontière (b_1, b_2) de X dans un produit d'espace de Hilbert $H_1 \times H_2$. Dans ce cas, la fonctionnelle à minimiser est de la forme $I(u) = L(u, \Lambda u) + \ell(b_1(u), b_2(u))$, où ℓ est un Lagrangien *autodual* sur cette frontière, au sens que $\ell^*(h_1, h_2) = \ell^*(-h_1, h_2)$ sur $H_1 \times H_2$.

1 Basic properties of anti-selfdual Lagrangians

Let X be a reflexive Banach space and consider $L : X \times X^* \to \boldsymbol{R} \cup \{+\infty\}$ to be a convex lower semi-continuous function, that is valued in $\boldsymbol{R} \cup \{+\infty\}$ but not being identically $+\infty$. Its Legendre-Fenchel dual (in both variables) is defined at each $(q, y) \in X^* \times X$ as:
$L^*(q, y) = \sup\{\langle q, x\rangle + \langle p, y\rangle - L(x, p); x \in X, p \in X^*\}$.

Definition 1.1 *Say that L is an anti-self dual Lagrangian on X, if*

$$L^*(p, x) = L(-x, -p) \quad \text{for all } (p, x) \in X^* \times X. \tag{1}$$

Our basic premise is that many boundary value problems can be solved by minimizing functionals of the form $I(x) = L(x, 0)$ where L is an anti-selfdual Lagrangian. They satisfy

$$L(x, p) \geq -\langle x, p\rangle \quad \text{for every } (x, p) \in X \times X^*, \tag{2}$$

which means that $L(x, 0) \geq 0$ for all $x \in X$. However, their main relevance to our study is because generically, the infimum is actually equal to 0. This latter property allows for novel variational formulations and resolutions of several basic PDEs and evolution equations, which –often because of lack of self-adjointness– do not normally fit the Euler-Lagrange framework. What is remarkable is that it is the very presence of skew-adjoint operators in certain equations that make the anti-self dual Lagrangian framework suitable for a variational approach. Details, proofs and more applications will be given in [7] in the setting of bounded linear operators. The case of linear unbounded operators is dealt with in [10]. Nonlinear but appropriately defined "skew-adjoint" operators such as those appearing in the Navier-Stokes and other equations of hydrodynamics will be considered in [8].

Theorem 1.2 *Let L be an anti-self dual Lagrangian on a reflexive Banach space X and let $\Lambda : X \to X^*$ be a skew-adjoint operator (i.e., $\Lambda^* = -\Lambda$).*
(1) If $\lim\limits_{\|x\| \to \infty} \frac{L(x, \Lambda x)}{\|x\|} = +\infty$ (coercivity), then there exists $\bar{x} \in X$, such that:

$$\begin{cases} L(\bar{x}, \Lambda\bar{x}) & = & \inf\limits_{x \in X} L(x, \Lambda x) = 0. \\ (-\Lambda\bar{x}, -\bar{x}) & \in & \partial L(\bar{x}, \Lambda\bar{x}). \end{cases} \tag{3}$$

(2) The same conclusion holds if the map $x \to L(x, 0)$ is bounded above on a neighborhood of the origin of X and if Λ is an invertible operator.

The class $\mathcal{L}_{\rm AD}(X)$ of anti-selfdual Lagrangians on a given Banach space X satisfies several permanence properties: For $\lambda > 0$ and $L \in \mathcal{L}_{\rm AD}(X)$, we have
(1) Scalar multiplication: $\lambda \cdot L(x, p) := \lambda^2 L(\frac{x}{\lambda}, \frac{p}{\lambda})$ is also in $\mathcal{L}_{\rm AD}(X)$.
(2) Convolution: If $M_\lambda(x, p) = \frac{\|x\|^2}{2\lambda^2} + \frac{\lambda^2\|p\|^2}{2}$, then $L_\lambda = L \star M_\lambda$ is also in $\mathcal{L}_{\rm AD}(X)$, where

$$(L \star M_\lambda)(x, p) := \inf\{L(z, p) + \frac{\|x - z\|^2}{2\lambda^2} + \frac{\lambda^2\|p\|^2}{2}; z \in X\}.$$

Note that L_λ is a λ-regularization of the Lagrangian L, which is reminescent of the Yosida theory for operators and for convex functions.
(3) Iteration with Skew-adjoint operators: If $L \in \mathcal{L}_{\rm AD}(X)$ and $\Lambda : X \to X^*$ is a skew-adjoint operator, then the Lagrangian $M(x, p) = L(x, \Lambda x + p)$ is also in $\mathcal{L}_{\rm AD}(X)$.
If in addition $\Lambda : X \to X^*$ is invertible, then $N(x, p) = L(x + \Lambda^{-1}p, \Lambda x)$ is in $\mathcal{L}_{\rm AD}(X)$.

2 ASD Lagrangians as extensions of certain maximal monotone operators

ASD Lagrangians are natural extensions of operators of the form $A+\partial\varphi$, where A is positive and φ is convex. This is an important subclass of monotone operators which can now be resolved variationally. Indeed, first consider the cone $\mathcal{C}(X)$ of all bounded below, proper convex l.s.c functions on X, and let $\mathcal{A}(X)$ be the cone of all positive bounded linear operators from X into X^* (i.e., $\langle Ax, x\rangle \geq 0$ for all $x \in X$). Consider the subclasses

$$\mathcal{C}_0(X) = \{\varphi \in \mathcal{C}(X); \inf_{x\in X} \varphi(x) = 0\} \quad \text{and} \quad \mathcal{A}_0(X) = \{A \in \mathcal{A}(X); A^* = -A\}.$$

Lemma 2.1 *There is a projection* $\Pi : (\mathcal{C}(X), \mathcal{A}(X)) \to (\mathcal{C}_0(X), \mathcal{A}_0(X))$ *such that if* (φ_0, A_0) *is the image of* (φ, A) *by* Π*, then a pair* $(x, f) \in X \times X^*$ *satisfy* $(A+\partial\varphi)(x) = f$ *if and only if* $(A_0 + \partial\varphi_0)(x) = f$*.*

Proof: For $(\varphi, A) \in (\mathcal{C}(X), \mathcal{A}(X))$, decompose A into a symmetric A^s and an anti-symmetric part A^a, by simply writing $A^s = \frac{1}{2}(A + A^*)$ and $A^a = \frac{1}{2}(A - A^*)$. Let φ_0 be the convex functional $\psi + \psi^*(0)$, where $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x)$. Define now the projection as $\Pi(\varphi, A) = (\varphi_0, A^a)$.

Theorem 2.2 *(Variational formulation and proof of a nonlinear Lax-Milgram theorem) For any pair* $(\varphi, A) \in \mathcal{C}(X) \times \mathcal{A}(X)$ *and any* $f \in X^*$*, there exists a Lagrangian* $L \in \mathcal{L}_{\rm AD}(X)$ *such that:*

1. *For any* $f \in X^*$*, the equation* $(A + \partial\varphi)(x) = f$ *has a solution* $\bar{x} \in X$ *if and only if the functional* $I(x) = L(x, 0)$ *attains its minimum at* $\bar{x}$*.*
2. *If* $\lim\limits_{\|x\|\to\infty} \frac{\varphi(x)+\frac{1}{2}\langle Ax,x\rangle}{\|x\|} = +\infty$*, then for any* $f \in X^*$*, the equation* $-Ax + f \in \partial\varphi(x)$ *has a solution* $\bar{x} \in X$ *that is obtained as a minimizer of the problem:*

$$\inf_{x\in X} \{\psi(x) + \psi^*(-A^a x)\} = 0 \tag{4}$$

where ψ *is the convex functional* $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x) - \langle f, x\rangle$*, and* A^a *is the anti-symmetric part of* A*.*

Proof: Associate to each $(\varphi, A) \in \mathcal{C}(X) \times \mathcal{A}(X)$, the anti-selfdual Lagrangian

$$L_{(\varphi,A)}(x, p) = L_{(\varphi_0,A_a)}(x, p) = \varphi_0(x) + \varphi_0^*(-A^a x - p), \quad \text{for } (x, p) \in X \times X^*,$$

where (φ_0, A^a) is the projection of (φ, A). The fact that the minimum in (4) is attained at $\bar{x} \in X$, means that $\psi(\bar{x}) + \psi^*(-A^a\bar{x}) = 0 = -\langle A^a\bar{x}, \bar{x}\rangle$ which yields, in view of Legendre-Fenchel duality that $-A^a\bar{x} \in \partial\psi(\bar{x}) = A^s\bar{x} + \partial\varphi(\bar{x}) - f$, hence $\bar{x}$ satisfies $-Ax + f \in \partial\varphi(x)$.

Example 1 - A variational principle for a non-symmetric Dirichlet problem: Let $\mathbf{a} : \Omega \to \mathbf{R^n}$ be a smooth function on a bounded domain Ω of $\mathbf{R^n}$, and consider the first order linear operator $Av = \mathbf{a} \cdot \nabla v = \Sigma_{i=1}^n a_i \frac{\partial v}{\partial x_i}$ assumed to be the restriction of a smooth vector field $\Sigma_{i=1}^n \bar{a}_i \frac{\partial v}{\partial x_i}$ defined on an open neighborhood of $\bar{\Omega}$. Consider the Dirichlet problem:

$$\begin{cases} \Delta u + \Sigma_{i=1}^n a_i \frac{\partial u}{\partial x_i} = u^3 + f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

If $a_i = 0$, then to find a solution, it is sufficient to minimize the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega |u|^4 dx + \int_\Omega f u dx$$

and to write that the minimizer $\bar{u}$ satisfies $\partial\Phi(\bar{u}) = 0$. However, if the non self-adjoint term a is not zero, we can use the above to get:

Theorem 2.3 *Assume* $\mathrm{div}(\mathbf{a}) \geq 0$ *on* Ω *and consider on* $H_0^1(\Omega)$*, the functional*

$$I(u) = \Psi(u) + \Psi^*(\mathbf{a}.\nabla u + \frac{1}{2}\mathrm{div}(\mathbf{a})\, u)$$

where $\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega |u|^4 dx + \int_\Omega fudx + \frac{1}{4}\int_\Omega \mathrm{div}(\mathbf{a})\, |u|^2 dx$*, and* Ψ^* *is its Legendre transform. Then, there exists* $\bar{u} \in H_0^1(\Omega)$ *such that* $I(\bar{u}) = \inf\{I(u); u \in H_0^1(\Omega)\} = 0$*, and* $\bar{u}$ *is a solution of* (5)*.*

A variational resolution for variational inequalities: Given a bilinear continuous functional a on $X \times X$ so that $a(v,v) \geq \lambda\|v\|^2$ and letting $\varphi : X \to \mathbf{R}$ be a convex l.s.c, then solving the corresponding variational inequality amounts to constructing for any $f \in X^*$, a point $\bar{x} \in X$ such that for all $z \in X$,

$$a(\bar{x}, \bar{x} - z) + \varphi(\bar{x}) - \varphi(z) \leq \langle \bar{x} - z, f\rangle. \tag{6}$$

It is easy to see that this problem can be rewritten as: $f \in Ay + \partial\varphi(y)$, where A is the bounded linear operator from X into X^* defined by $a(u,v) = \langle Au, v\rangle$. This means that the variational inequality (6) can be rewritten and solved using the variational principle (4). For example, one can solve the following typical "obstacle " problem in the following way.

Corollary 2.4 *Let a be bilinear continuous functional on a reflexive Banach space* $X \times X$ *so that* $a(v,v) \geq \lambda\|v\|^2$*, and let* K *be a convex closed subset of* X*. Then, for any* $f \in X^*$*, there is* $\bar{x} \in K$ *where the following minimum is attained:*

$$\varphi(\bar{x}) + (\varphi + \psi_K)^*(-\Lambda\bar{x}) = \inf_{x \in X}\{\varphi(x) + (\varphi + \psi_K)^*(-\Lambda x)\} = 0.$$

Here $\varphi(x) = \frac{1}{2}a(x,x) - \langle f, x\rangle$*,* $\Lambda : X \to X^*$ *is the skew-adjoint operator defined by* $\langle \Lambda u, v\rangle = \frac{1}{2}(a(u,v) - a(v,u))$ *and* $\psi_K(x) = 0$ *on* K *and* $+\infty$ *elsewhere. Furthermore,* $\bar{x}$ *is a solution of the variational inequality:* $a(\bar{x}, \bar{x} - z) \leq \langle \bar{x} - z, f\rangle$ for all $z \in K$*.*

3 ASD Lagrangians in boundary value problems

Definition 3.1 *A boundary operator will be any pair* $(b_1, b_2) : X \to H_1 \times H_2$ *of continuous linear maps from* X *into Hilbert spaces* H_1 *and* H_2*. We shall say that* (b_1, b_2) *is a regular boundary operator if there is a projection* $\Pi : X \to X_0 := Ker(b_1, b_2)$ *so that the bounded linear map* $(\Pi, b_1, b_2) : X \to Ker(b_1, b_2) \oplus H_1 \oplus H_2$ *is an isomorphism.*
We shall then identify X^* *with the space* $X_0^* \oplus H_1 \oplus H_2$ *in such a way that the duality between* X *and* X^* *is given by:* $\langle x, p\rangle = \langle x, (p_0, p_1, p_2)\rangle = \langle x, p_0\rangle + \langle b_1(x), p_1\rangle + \langle b_2(x), p_2\rangle$*.*

Definition 3.2 *An operator* $\Lambda : X \to X^*$ *is said to be skew-symmetric modulo the boundary operator* (b_1, b_2)*, if for every* $x, y \in X$*,*

$$\langle \Lambda x, y\rangle_{(X,X^*)} = -\langle \Lambda y, x\rangle_{(X,X^*)} + \langle b_2(x), b_2(y)\rangle_{H_2} - \langle b_1(x), b_1(y)\rangle_{H_1} \tag{7}$$

That is if the operator $\Lambda - \frac{1}{2}(b_2^* b_2 - b_1^* b_1)$ *is skew-symmetric. We then say that we have a skew symmetric triplet* (Λ, b_1, b_2)*.*

Definition 3.3 *Say that* $\ell : H_1 \times H_2 \to \boldsymbol{R} \cup \{+\infty\}$ *is a self-dual boundary Lagrangian if*

$$\ell^*(-h_1, h_2) = \ell(h_1, h_2) \quad \text{for all } (h_1, h_2) \in H_1 \times H_2. \tag{8}$$

The basic example of a self dual boundary Lagrangian is given by a function ℓ_a on $H_1 \times H_2$ of the form $\ell_a(x,p) = \frac{1}{2}\|x\|^2 - 2\langle a, x\rangle + \|a\|^2 + \frac{1}{2}\|p\|^2$, where a is given in H_1. Boundary conditions require new ASD Lagrangians.

Theorem 3.4 *Let L be an anti-self dual Lagrangian on a reflexive Banach space X. Let $(\Lambda, b_1, b_2) : X \to X^* \times H_1 \times H_2$ be a regular skew symmetric triplet, and let ℓ be a self dual boundary Lagrangian on $H_1 \times H_2$. Then,*

1. *The Lagrangian $M(x,p) = L(x, \Lambda x) + \ell(b_1(x), b_2(x))$ is anti-self dual on X.*
2. *If $x \to L(x, \Lambda x) + \ell(b_1(x), b_2(x))$ is coercive on X, then there exists $\bar{x} \in X$ such that:*
$$L(\bar{x}, \Lambda\bar{x}) + \ell(b_1\bar{x}, b_2\bar{x}) = \inf_{x\in X} \{L(x, \Lambda x) + \ell(b_1 x, b_2 x)\} = 0. \tag{9}$$
3. *For any $a \in H_1$, there exists $\bar{x} \in X$ such that:*
$$\begin{cases} L(\bar{x}, \Lambda\bar{x}) + \langle \bar{x}, \Lambda\bar{x}\rangle &= 0. \\ (-\Lambda\bar{x}, -\bar{x}) &\in \partial L(\bar{x}, \Lambda\bar{x}) \\ b_1(\bar{x}) &= a \end{cases} \tag{10}$$
It is obtained as a minimizer for $I(x) = L(x, \Lambda x) + \langle x, \Lambda x\rangle + \|b_1(x) - a\|^2$ over X.

Variational principle for a Lax-Milgram type theorem with prescribed boundary

Definition 3.5 *Say that $A : X \to X^*$ is positive modulo the boundary operator (b_1, b_2) if the operator $A - \frac{1}{2}(b_2^* b_2 - b_1^* b_1)$ is positive.*

Corollary 3.6 *Assume $A : X \to X^*$ is positive modulo the boundary operator (b_1, b_2), and that $\varphi \in \mathcal{C}(X)$. If $\lim_{\|x\|\to\infty} \|x\|^{-1}\big\{\varphi(x) + \frac{1}{2}\langle Ax, x\rangle - \frac{1}{4}(\|b_2 x\|^2 - \|b_1 x\|^2)\big\} = +\infty$, then for any $a \in H_1$ and any $f \in X^*$, there is $\bar{x} \in X$ where the following minimum is attained:*
$$\inf_{x\in X} \left\{\psi(x) + \psi^*(-\Lambda x) + \frac{1}{2}\|b_1(x)\|^2 - 2\langle a, b_1(x)\rangle + \|a\|^2 + \frac{1}{2}\|b_2(x)\|^2\right\} = 0.$$
Here $\psi(x) = \varphi(x) + \frac{1}{2}\langle Ax, x\rangle - \frac{1}{4}(\|b_2 x\|^2 - \|b_1 x\|^2) + \langle f, x\rangle$ and $\Lambda = A^a + \frac{1}{2}(b_2^* b_2 - b_1^* b_1)$. Furthermore, $\bar{x}$ is a solution to the equation
$$\begin{cases} -Ax &\in \partial\varphi(x) + f \\ b_1(x) &= a. \end{cases} \tag{11}$$

Example 2 - A variational principle for first order non-linear transport equations: As in example 1, let $\mathbf{a} : \Omega \to \mathbf{R^n}$ and $a_0 : \Omega \to \mathbf{R}$ be two smooth functions on a bounded domain Ω of $\mathbf{R^n}$, and consider the first order linear operator $Av = \mathbf{a} \cdot \nabla v = \Sigma_{i=1}^n a_i \frac{\partial v}{\partial x_i}$ and $\Lambda v = \mathbf{a} \cdot \nabla v + a_0 v$. Assume that the boundary of Ω is piecewise C^1, in such a way that the outer normal $\mathbf{n}$ is defined almost everywhere on $\partial\Omega$. Denoting
$$\Sigma_- = \{x \in \partial\Omega;\, \mathbf{n}(x)\cdot\mathbf{a}(x) < 0\} \quad \text{and } \Sigma_+ = \partial\Omega \setminus \Sigma_- = \{x \in \partial\Omega;\, \mathbf{n}(x)\cdot\mathbf{a}(x) \geq 0\},$$
then a trace $u_{|\Sigma_-}$ makes sense in $L^2_{\mathrm{loc}}(\Sigma_-)$ as soon as $u \in L^2(\Omega)$ and $\Lambda u \in L^2(\Omega)$.
Let now $\beta : \mathbf{R} \to \mathbf{R}$ be a continuous nondecreasing function so that its antiderivative j is convex, and let $f \in L^2(\Omega)$. We are interested in finding variationally solutions for the nonlinear transport equation:
$$\begin{cases} -\mathbf{a}\cdot\nabla u + a_0 u &= \beta(u) + f \ \text{ on } \ \Omega \\ u(x) &= 0 \qquad\quad \text{on } \ \Sigma_-. \end{cases} \tag{12}$$
The appropriate space in our setting is
$$H^1_A(\Omega) = \{u \in L^2(\Omega);\, Au \in L^2(\Omega), \text{ and } \int_{\Sigma_-} |u(x)|^2 |\mathbf{n}(x)\cdot\mathbf{a}(x)| d\sigma < +\infty\}.$$

equipped with the norm $\|u\|_{H^1_A} = \|u\|_2 + \|Au\|_2 + \|u_{|\Sigma_-}\|_{L^2_A(\Sigma_-)}$.
If now $a_0(x) - \frac{1}{2}\text{div}a(x) \geq 0$ on Ω, then Λ is positive modulo the boundary operators $u \to (u_{|\Sigma_-}, u|_{\Sigma_+}) \in L^2(\Sigma_-) \times L^2(\Sigma_+)$, and the operator

$$\Lambda_1(u) := \mathbf{a} \cdot \nabla u + \frac{1}{2}\text{div}(\mathbf{a})u = \Lambda(u) - (a_0 - \frac{1}{2}\text{div}\,\mathbf{a})u$$

is therefore skew-adjoint modulo that boundary. We can now state:

Theorem 3.7 *Assume the coercivity condition $a_0(x) - \frac{1}{2}\text{div}a(x) \geq \alpha > 0$ on Ω and consider the following functional on the space $H^1_A(\Omega)$*

$$I(u) = \psi(u) + \psi^*(-\Lambda_1 u) + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma \tag{13}$$

where ψ is the convex functional on $L^2(\Omega)$ defined by:

$$\psi(u) = \int_\Omega \left\{ j(u(x)) + f(x)u(x) + \frac{1}{2}(a_0 - \frac{1}{2}\text{div a})u^2) \right\} dx$$

where ψ^* is its Legendre conjugate. Then there exists a solution $\bar{u}$ for (12) that is obtained as a minimizer $I(\bar{u}) = \inf\{I_G(u); u \in H^1_A(\Omega)\} = 0$.

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