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A variational principle for gradient flows

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Abstract We verify - after appropriate modifications - an old conjecture of Brezis-Ekeland ([3], [4]) concerning the feasibility of a global and variational approach to the problems of existence and uniqueness of gradient flows for convex energy functionals. Our approach is based on a concept of "self-duality" inherent in many parabolic evolution equations, and motivated by Bolza-type problems in the classical calculus of variations. The modified principle allows to identify the extremal value –which was the missing ingredient in [3]– and so it can now be used to give variational proofs for the existence and uniqueness of solutions for the heat equation (of course) but also for quasi-linear parabolic equations, porous media, fast diffusion and more general dissipative evolution equations.

1 Introduction

Second order boundary value problems have often been connected to variational principles since many of the basic ones arise as Euler-Lagrange equations associated to certain energy or action functionals. In 1976, Brezis and Ekeland formulated in [3] an intriguing minimization principle associated to certain first order initial value problems including gradient flows of convex energy functionals on infinite dimensional spaces (as in the case of the heat equation), which are not equations of Euler-Lagrange type. This is because the equations are derived in this case from the fact that they correspond to "zeroes" of the functionals and

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not because they are critical points (actually minima). This meant that the applicability of this principle for establishing existence and uniqueness of solutions for associated equations, depends crucially on the verification that the value of the infimum is actually zero: a fact they could not establish unless the existence of solutions was a priori known.

In this paper, we offer a variant of the Brezis-Ekeland principle in which many of the shortcomings are removed. With it we could prove global existence and uniqueness of solutions for several basic first order linear and nonlinear evolution equations. We only deal here with questions of existence and uniqueness of gradient flows, but we believe that –like with many new variational principles– it will prove useful. Here is the framework:

Consider the following evolution equation

$$\begin{cases} \dot{u}(t) + \partial\varphi(u(t)) = f(t) \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ u(0) = u_0 \end{cases} \tag{1}$$

where $\varphi : H \to I\!R \cup \{+\infty\}$ is a proper convex and lower semi-continuous functional on a Hilbert space H and where $\partial\varphi$ denotes its subdifferential map. It is well known [2] that for any $f \in L^2([0,T];H)$ and any u_0 in the proper domain $\mathrm{Dom}(\varphi)$ of φ, there exists a unique solution $u \in C([0,T];H)$ for (1) such that $\dot{u}(t) \in L^2([0,T];H)$ and $u(t) \in \mathrm{Dom}(\partial\varphi)$ a.e.

In the mid-seventies, Brezis and Ekeland [3] formulated the following variational approach to obtain existence and uniqueness for equation (1). Let φ^* be the Legendre conjugate of φ on H defined as:

$$\varphi^*(y) = \sup\{\langle y, z\rangle - \varphi(z);\, z \in H\}, \tag{2}$$

and –assuming for simplicity that $f = 0$– we consider the set

$$K = \{v \in C([0,T];H);\, \varphi^*(-\frac{dv}{dt}) \in L^1(0,T), v(0) = u_0\}, \tag{3}$$

then the solution of (1) is the unique minimizer of the variational problem

$$\text{Minimize} \quad I_{BE}(v) := \int_0^T [\varphi(v(t)) + \varphi^*(-\dot{v}(t))]\, dt + \frac{1}{2}\|v(T)\|_H^2 \text{ over } v \in K. \tag{4}$$

The proof is based on the Fenchel-Young inequality:

$$\varphi(u(t)) + \varphi^*(-\dot{u}(t)) \geq \langle u(t), -\dot{u}(t)\rangle = -\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 \quad \text{a.e. on } [0,T] \tag{5}$$

with equality holding if and only if u satisfies

$$-\dot{u}(t) \in \partial\varphi(u(t)) \quad \text{a.e. on } [0,T] \tag{6}$$

hence equation (1). But equality in (5) is assured only if one can show that

$$\mathrm{Min}\{I_{BE}(v); v \in K\} = \frac{\|u_0\|_H^2}{2}, \tag{7}$$

which is however not so obvious to prove, unless we already know by different methods, that (1) has a solution.
For example, in the case of the homogeneous heat equation in a smooth bounded domain Ω of $\mathbb{R}^n$, the approach of Brezis-Ekeland amounts to minimizing the functional

$$I_{BE}(u) = \frac{1}{2}\int_0^T \left(\int_\Omega (|\nabla u(t,x)|^2 + |\nabla \Delta^{-1}\frac{\partial u}{\partial t}(t,x)|^2)dx\right) dt + \frac{1}{2}\int_\Omega |u(T,x)|^2 dx \tag{8}$$

on the set

$$K = \{u \in C([0,T]; L^2(\Omega));\ \int_\Omega |\nabla \Delta^{-1}\frac{\partial u}{\partial t}(\cdot,x)|^2 dx \in L^1(0,T), u(0) = u_0\}. \tag{9}$$

This corresponds to the case where $\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx$ on $H_0^1(\Omega)$ and $+\infty$ elsewhere on $L^2(\Omega)$. Here $w = \Delta^{-1} g$ is defined as the solution of the Dirichlet problem $\Delta w = g$ on Ω with $w = 0$ on the boundary $\partial\Omega$.
Unless one shows that the infimum is actually equal to $\frac{1}{2}\int_\Omega |u_0(x)|^2 dx$, then we can only use the Euler-Lagrange equation associated to that minimization problem, in which case one only obtains a solution to the following equation:

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)(\frac{\partial}{\partial t} + \Delta)u = 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ \qquad u(0) \qquad = u_0. \end{cases} \tag{10}$$

To remedy the situation, we change the Brezis-Ekeland principle in two fundamental ways:

- First, we isolate and exploit a concept of self-dual variational problems that seems to be inherent to this type of evolution equations. For that we consider a new convex energy $\psi(u) = \varphi(u + u_0) - \langle u, f\rangle$ associated to (1), then we define the functional

$$I(u) = \int_0^T [\psi(u(t)) + \psi^*(-\dot{u}(t))]\, dt + \frac{1}{2}(\|u(0)\|_H^2 + \|u(T)\|_H^2) \tag{11}$$

  which corresponds to the readily "self-dual" Lagrangian pair (L, ℓ) defined by:

$$\ell(c_0, c_T) = \frac{1}{2}\|c_0\|_H^2 + \frac{1}{2}\|c_T\|_H^2 \quad \text{and} \quad L(u,v) = \psi(u) + \psi^*(-v). \tag{12}$$

- This has the added advantage of changing the variational formulation to a boundary-free one, allowing us to change the constraint set to a Banach space –typically $A_H^\alpha = \{u : [0,T] \to H;\ u \,\&\, \dot{u} \in L_H^\alpha\}$ for some $1 < \alpha < \infty$– so that standard methods from the calculus of variations –properly extended to an infinite dimensional framework– can be applied to establish the existence of a unique minimizer.

This self-dual setting will then always lead to zero as minimal value, so that under the right conditions, there is a unique $\hat{u}$ such that:

$$I(\hat{u}) = \inf_{A_H^\alpha} I(u) = 0. \tag{13}$$

On the other hand, the Fenchel-Young inequality gives that:

$$I(u) \geq \|u(0)\|_H^2 \quad \text{for any } u \in A_H^\alpha. \tag{14}$$

It follows that $\hat{u}(0) = 0$, while the limiting case of Young's inequality applied to ψ, implies –as above– that the path $u(t) = \hat{u}(t) + u_0$ is a weak solution for the evolution equation (1).
In summary, we are proposing the following principle established in Theorem 3 below: Assume φ is proper convex and lower semi-continuous on a Hilbert space H, with a non-empty subdifferential at 0. For any $u_0 \in \mathrm{Dom}(\varphi)$ and any $f \in L^2([0,T]; H)$, the following functional:

$$\begin{aligned} I(u) := &\int_0^T \left[\varphi(u(t)+u_0) + \varphi^*(f(t) - \dot{u}(t)) - \langle u(t)+u_0, f(t)\rangle + \langle \dot{u}(t), u_0\rangle\right] dt \\ &+\frac{1}{2}(\|u(0)\|_H^2 + \|u(T)\|_H^2) \end{aligned} \tag{15}$$

on A_H^2, has a unique minimum $v \in C([0,T]; H)$ such that:

$$\dot{v} \in L_H^2, \quad v(t) \in \mathrm{Dom}(\partial\varphi) - u_0 \quad \text{for almost all } t \in [0,T], \tag{16}$$

$$I(v) = \inf_{A_H^2} I(u) = 0, \tag{17}$$

and the path $u(t) = v(t) + u_0$ is a weak solution for the evolution equation (1).
In the case of the heat equation, this translates to the following:

Corollary 1 *For any $u_0 \in H_0^1(\Omega)$ and any $f \in L^2([0,T] \times \Omega)$, the infimum of the functional*

$$\begin{aligned} I(u) = &\frac{1}{2}\int_0^T \int_\Omega \left(|\nabla(u(t,x)+u_0(x))|^2 + |\nabla\Delta^{-1}(f(t,x) - \frac{\partial u}{\partial t}(t,x))|^2\right) dxdt \\ &+ \int_0^T \int_\Omega \left[u_0(x)\frac{\partial u}{\partial t}(t,x) - u_0(x)f(t,x) - u(t,x)f(t,x)\right] dxdt \\ &+\frac{1}{2}\int_\Omega (|u(0,x)|^2 + |u(T,x)|^2) dx \end{aligned} \tag{18}$$

on the space $A_{L^2(\Omega)}^2$ is equal to zero and is attained uniquely at a path $v \in A_{L^2(\Omega)}^2$ with $v(t) \in H_0^1 \cap H^2$ for all $t \in [0,T]$, and in such a way that $u(t) = v(t) + u_0$ is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f & \text{on} \quad \Omega \times [0,T] \\ u(0,x) = u_0 & \text{on} \quad \Omega \\ u(t,x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{19}$$

As mentioned above, behind this principle lies a far-reaching concept of self-duality inherent to gradient flows, but also applicable in other situations. In section 2, we formulate and analyze general self-dual variational problems in a Hilbertian setting that will be applied in section 3, to establish existence of gradient flows for non-time dependent potentials. To get the most general result, one still needs to go through a regularization procedure reminescent of the Hille-Yosida theory. However, the variational context makes the approximation much simpler since only weak –as opposed to uniform– convergence arguments are needed. In section 4 and 5, we develop another approach to cover time-dependent convex energies. Here, certain intermediate Banach spaces (the so-called "evolution triples" which appear naturally in the applications) play a key role. An extension of our approach is given in [7] to cover the case of gradient flows of semi-convex potentials. We also mention that, several months after the completion of the first version of this paper, Auchmuty informed us about his paper [1], where he also considers the Brezis-Ekeland variational problem. There he uses min-max methods to identify the value of the infimum and to establish existence and uniqueness under suitable growth conditions on the convex potential.

2 Self-dual Lagrangian on Hilbert spaces

Let H be a Hilbert space with $\langle\, ,\, \rangle$ as scalar product and let $[0,T]$ be a fixed real interval ($0 < T < +\infty$). Consider the classical space L^2_H of Bochner integrable functions from $[0,T]$ into H with norm denoted by $\|\cdot\|_2$, as well as the Hilbert space

$$A^2_H = \{u : [0,T] \to H;\ \dot{u} \in L^2_H\}$$

consisting of all absolutely continuous arcs $u : [0,T] \to H$, equipped with the norm

$$\|u\|_{A^2_H} = (\|u(0)\|^2_H + \int_0^T \|\dot{u}\|^2 dt)^{\frac{1}{2}}.$$

It is clear that A^2_H can be identified with the product space $H \times L^2_H$, and that its dual $(A^2_H)^*$ can also be identified with $H \times L^2_H$ via the formula:

$$\langle u, (a,p)\rangle_{A^2_H, H\times L^2_H} = (u(0), a)_H + \int_0^T \langle \dot{u}(t), p(t)\rangle dt.$$

We consider the following action functional on A^2_H:

$$I_{\ell,L}(u) = \int_0^T L(t, u(t), \dot{u}(t))dt + \ell(u(0), u(T))$$

where

$$\ell : H \times H \to I\!R \cup \{+\infty\} \quad \text{and} \quad L : [0,T] \times H \times H \to I\!R \cup \{+\infty\}$$

are two appropriate Lagrangians. We shall always assume that L is measurable with respect to the σ-field in $[0,T]\times H\times H$ generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in $H\times H$.

We assume throughout that ℓ and $L(t,\cdot,\cdot)$ are convex, lower semi-continuous valued in $I\!R\cup\{+\infty\}$ but not identically $+\infty$. In this case, we can associate to the pair (ℓ, L), the following "Bolza-dual" functionals:

$$M(t,p,s)=L_t^*(s,p) \text{ and } \quad m(r,s)=\ell^*(r,-s)$$

where L_t^* and ℓ^* denote the Legendre duals of $L_t=L(t,\cdot,\cdot)$ and ℓ respectively. In other words, M and l are the convex and lower semi-continuous functions on $H\times H$ defined by:

$$M(t,p,s)=\sup\{\langle u,s\rangle+\langle v,p\rangle-L(t,u,v);\, u,v\in H\}$$

and

$$m(r,s)=\sup\{\langle u,r\rangle+\langle v,-s\rangle-\ell(u,v);\, u,v\in H\}.$$

Writing

$$I_{m,M}(u)=\int_0^T M(t,u,\dot u)dt+m(u(0),u(T))$$

for $u\in A_H^2$, the relevance of the "Bolza-dual" functionals starts becoming apparent from the following –easy to establish– "weak duality" formula:

$$\inf_{u\in A_H^2} I_{\ell,L}(u)\geq-\inf_{u\in A_H^2} I_{m,M}(u). \tag{20}$$

A key aspect of the finite dimensional theory is that equality holds provided $I_{m,M}$ and $I_{\ell,L}$ "behave lower semicontinuously with respect to certain perturbations. To analyse that in our context, we associate the following "variation function" $J_{\ell,L}$ defined on $(A_H^2)^*=H\times L_H^2$ as:

$$J_{\ell,L}(a,y)=\inf\{\int_0^T L(t,u(t)+y(t),\dot u(t))dt+\ell(u(0)+a,u(T))\ ;\ u\in A_H^2\}.$$

Proposition 1 *Under the above conditions, the functional $J_{\ell,L}$ is convex and satisfies*

$$J_{\ell,L}^*(p)=I_{m,M}(p)\quad \text{for all } p\in A_H^2.$$

where $J_{\ell,L}^*$ is the Legendre transform of $J_{\ell,L}$ in the duality $(H\times L_H^2, A_H^2)$.

The convexity of $J_{\ell,L}$ is easy to establish. For the rest, we need the following lemma.

Lemma 1 *Let $E_M: L_H^2\times L_H^2\to I\!R\cup\{+\infty\}$ be defined as $E_M(p,s)=\int_0^T M(t,p(t),s(t))dt$, then*

$$E_M(p,s)=\sup\left\{\int_0^T(\langle s(t),u(t)\rangle+\langle p(t),v(t)\rangle-L(t,u(t),v(t)))dt\ ;\ (u,v)\in L_H^2\times L_H^2\right\}.$$

Proof: For all $u, v \in L^2_H$ and $p, s \in L^2_H$, we have:

$$\int_0^T L(t,u(t),v(t))dt + \int_0^T M(t,p(t),s(t))dt \geq \int_0^T (\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle dt,$$

which implies

$$\int_0^T M(t,p(t),s(t))dt \geq \sup\left\{\int_0^T \left[\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t)\right] dt; (u,v) \in L^2_H \times L^2_H\right\}.$$

For the reverse inequality, let (p,s) be in $L^2_H \times L^2_H$ in such a way that $\beta < E_M(p,s)$ and let $\mu(t)$ be such that $\mu(t) < M(t,p(t),s(t))$ for all t while $\int_0^T \mu(t)dt > \beta$. We then have for all t,

$$-\mu(t) > -M(t,p(t),s(t)) = \inf\{L(t,u,v) - \langle u,s(t)\rangle - \langle v,p(t)\rangle; (u,v) \in H \times H\}.$$

By a standard measurable selection theorem (see [5]), there exists a measurable pair $(u_1,u_2) \in L^2_H \times L^2_H$ such that

$$-\mu(t) \geq L(t,u_1(t),u_2(t)) - \langle u_1(t),s(t)\rangle - \langle u_2(t),p(t)\rangle.$$

Therefore

$$\begin{aligned}\beta &< \int_0^T \mu(t)dt \leq \int_0^T \left[-L(t,u_1(t),u_2(t)) + \langle u_1(t),s(t)\rangle + \langle u_2(t),p(t)\rangle\right] dt \\ &\leq \sup\left\{\int_0^T \left[\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))\right] dt \;;\; (u,v) \in L^2_H \times L^2_H\right\}\end{aligned}$$

which implies that

$$E_M(p,s) \leq \sup\left\{\int_0^T \left[\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))\right] dt \;;\; (u,v) \in L^2_H \times L^2_H\right\}.$$

Proof of Proposition 1: For $p \in A^2_H$, write:

$$J^*_{\ell,L}(p) = \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\left\{\langle a,p(0)\rangle + \int_0^T \left[\langle y(t),\dot p(t)\rangle - L(t,u(t)+y(t),\dot u)\right] dt - \ell(u(0)+a,u(T))\right\}.$$

Make a substitution

$$u(0) + a = a' \in H \quad \text{and } u + y = y' \in L^2_H,$$

we obtain

$$J^*_{\ell,L}(p) = \sup_{a'\in H}\sup_{y'\in L^2_H}\sup_{u\in A^2_H}\left\{\langle a'-u(0),p(0)\rangle - \ell(a',u(T)) + \int_0^T \left[\langle y'(t)-u(t),\dot p(t)\rangle - L(t,y'(t),\dot u(t)\right] dt\right\}.$$

Since $\dot{u} \in L^2_H$ and $u \in L^2_H$, we have:

$$\int_0^T \langle u, \dot{p} \rangle = -\int_0^T \langle \dot{u}, p \rangle + \langle p(T), u(T) \rangle - \langle p(0), u(0) \rangle,$$

which implies

$$J^*_{\ell,L}(p) = \sup_{a' \in H} \sup_{y' \in L^2_H} \sup_{u \in A^2_H} \{\langle a', p(0) \rangle + \int_0^T \{\langle y', \dot{p} \rangle + \langle \dot{u}, p \rangle - L(t, y'(t), \dot{u}(t))\} dt$$
$$- \langle u(T), p(T) \rangle - \ell(a', u(T))\}.$$

It is now convenient to identify A^2_H with $H \times L^2_H$ via the correspondence:

$$(c, v) \in H \times L^2_H \mapsto c + \int_t^T v(s)\, ds \in A^2_H$$
$$u \in A^2_H \mapsto (u(T), -\dot{u}(t)) \in H \times L^2_H.$$

We finally obtain

$$\begin{aligned} J^*_{\ell,L}(p) &= \sup_{a' \in H} \sup_{c \in H} \{\langle a', p(0) \rangle + \langle -c, p(T) \rangle - \ell(a', c)\} \\ &\quad + \sup_{y' \in L^2_H} \sup_{v \in L^2_H} \left\{ \int_0^T \left[\langle y', \dot{p} \rangle + \langle v, p \rangle - L(t, y'(t), v(t)) \right] dt \right\} \\ &= E_M(p, \dot{p}) + m(p(0), p(T)) \\ &= I_{M,m}(p). \end{aligned}$$

Proposition 2 *An arc $p \in A^2_H$ belongs to $\partial J_{\ell,L}(0,0)$ if and only if it satisfies:*

$$I_{m,M}(p) = \inf_{A^2_H} I_{m,M} = -\inf_{A^2_H} I_{\ell,L}$$

Dually, an arc $x \in A^2_H$ belongs to $\partial J_{m,M}(0,0)$ if and only if it satisfies:

$$I_{\ell,L}(x) = \inf_{A^2_H} I_{\ell,L} = -\inf_{A^2_H} I_{m,M}$$

Proof: As noted above, the definition of m, M and weak duality, yield:

$$\inf_{u \in A^2_H} I_{\ell,L}(u) \geq -\inf_{u \in A^2_H} I_{m,M}(u).$$

In view of Proposition 1, if $p \in \partial J_{\ell,L}(0,0) \in A^2_H$, then

$$\begin{aligned} \inf_{A^2_H} I_{\ell,L}(u) \geq -\inf_{A^2_H} I_{m,M}(u) &= \sup_{A^2_H} -I_{m,M}(u) \\ &= \sup_{A^2_H} -J^*_{\ell,L}(u) \geq -J^*_{\ell,L}(p) \\ &= J_{\ell,L}(0,0) = \inf_{u \in A^2_H} I_{\ell,L}(u). \end{aligned}$$

The following concept is at the heart of our approach.

Definition 1 *Say that the pair (L,ℓ) is self-dual if for all $(p,s)\in H\times H$, we have*

$$m(r,s)=\ell(-r,-s) \quad \text{and } M(s,p)=L(-s,-p),$$

or equivalently

$$\ell^*(r,s)=\ell(-r,s) \quad \text{and } L^*(p,s)=L(-s,-p)$$

Theorem 1 *Suppose that L and l are two proper convex and lower semi-continuous functions from $H\times H$ to $\mathbb{R}\cup\{+\infty\}$ such that the pair (L,ℓ) is self-dual, then*

$$I_{m,M}(u)=I_{\ell,L}(-u) \quad \text{for any } u \text{ in } A_H^2 \tag{21}$$

and

$$\inf_{A_H^2} I_{\ell,L}\geq 0\geq \sup_{A_H^2} -I_{m,M}=-\inf_{A_H^2} I_{\ell,L}. \tag{22}$$

Suppose in addition that for some p_0 and $q_0\in H$, $C>0$ and $\alpha\in L^\infty[0,T]$, we have for all $(t,x)\in[0,T]\times H$,

$$L(t,x,p_0)\leq \alpha(t)(1+\|x\|_H^2) \quad \text{and } \ell(x,q_0)\leq C(1+\|x\|_H^2). \tag{23}$$

Then, there exists $v\in A_H^2$ such that $(v(t),\dot v(t))\in \mathrm{Dom}(L)$ for almost all $t\in[0,T]$ and

$$I_{\ell,L}(v)=\inf_{u\in A_H^2} I_{\ell,L}(u)=0. \tag{24}$$

Proof of Theorem 1: Proposition 1 and the fact that (L,ℓ) is self-dual implies immediately that $I_{m,M}(u)=I_{\ell,L}(-u)$ for any u. This combined with the weak duality inequality and the fact that the constraint set is a vector space, gives that

$$\inf_{u\in A_H^2} I_{\ell,L}(u)\geq -\inf_{u\in A_H^2} I_{m,M}(u)=-\inf_{A_H^2} I_{\ell,L}(u)$$

which means that $\inf_{A_H^2} I_{\ell,L}$ is necessarily non-negative.

To prove (24) we need to show that the convex functional $J_{\ell,L}$ is sub-differentiable at $(0,0)$, so as to conclude using Proposition 2. For that, we show that J is bounded in a neighborhood of $(0,0)$ in the space $H\times L_H^2$. Indeed, by considering the path $\gamma(t)=q_0t+\frac{t}{T}(p_0-q_0)$ joining p_0 to q_0, we get:

$$\begin{aligned} J_{\ell,L}(a,y) &\leq \int_0^T L(t,y(t)+\gamma(t),p_0)dt+\ell(p_0+a,q_0)\\ &\leq \int_0^T \alpha(t)(C_1t+\|y(t)\|_H^2)dt+C_2(1+\|a\|_H^2). \end{aligned}$$

This means that $J_{\ell,L}$ is convex and bounded in a neighborhood of $(0,0)$ in the space $H\times L_H^2$, hence it is subdifferentiable at $(0,0)$, and Proposition 2 therefore applies.

Remark 1 Note that all what we needed for the proof is the existence of a path γ joining p_0 and $q_0 \in H$, and $C > 0$ such that for any $(y, a) \in L^2_H \times H$ with $\|a\|_H \leq 1$ and $\|y\|_{L^2_H} \leq 1$, we have

$$\int_0^T L(t, y(t) + \gamma(t), p_0)dt + \ell(p_0 + a, q_0) \leq C. \tag{25}$$

This is a much weaker assumption on the Lagrangian.

Corollary 2 *Suppose that L is a proper convex and lower semi-continuous Lagrangian from $[0,T] \times H \times H$ to $\mathbb{R} \cup \{+\infty\}$ such that for all $(t, x, p) \in [0,T] \times H \times H$:*

$$L^*(t, x, p) = L(t, -p, -x) \tag{26}$$

and for some $p_0 \in H$ and $\alpha \in L^\infty[0,T]$, we have

$$L(t, x, p_0) \leq \alpha(t)(1 + \|x\|^2_H) \quad \text{for all } (t, x) \in [0,T] \times H. \tag{27}$$

Then there exists $v \in A^2_H$ such that:

$$(v(t), \dot{v}(t)) \in \mathrm{Dom}(L) \quad \text{for almost all } t \in [0,T], \tag{28}$$

$$\frac{d}{dt}\partial_p L(t, v(t), \dot{v}(t)) = \partial_x L(t, v(t), \dot{v}(t)) \tag{29}$$

and

$$\|v(t)\|^2_H = -2\int_0^t L(s, v(s), \dot{v}(s))ds \quad \text{for every } t \in [0,T]. \tag{30}$$

Proof: Consider on $H \times H$ the convex function $\ell(x, y) = \frac{1}{2}(\|x\|^2 + \|y\|^2)$ and notice that the pair $(L.\ell)$ is then self-dual. By Theorem 1, the functional

$$I_{\ell,L}(u) = \int_0^T L(t, u(t), \dot{u}(t)dt + \ell(u(0), u(T))$$

has zero as infimum and it is attained at some $v \in A^2_H$. Writing the corresponding Euler-Lagrange equation gives (29).
Now note that

$$I_{\ell,L}(u) = \int_0^T \left[L(t, u(t), \dot{u}(t) + \langle u(t), \dot{u}(t)\rangle\right] dt + \|u(0)\|^2.$$

By (27) also implies that

$$L(t, x, p) \geq -\langle x, p\rangle \quad \text{for all } (t, x, p) \in [0,T] \times H \times H, \tag{31}$$

and so from the fact that $I_{\ell,L}(v) = \inf_{u \in A^2_H} I_{\ell,L}(u) = 0$, follows that $v(0) = 0$ and

$$L(s, v(s), \dot{v}(s) + \langle v(s), \dot{v}(s)\rangle = 0 \quad \text{for almost all } s \in [0,T].$$

This clearly yields (30).

3 Self-dual Lagrangians associated to gradient flows

We now give the arch-typical example of a self-dual Lagrangian, from which we deduce a variational formulation of gradient flows.

Proposition 3 *Let $\varphi : [0,T] \times H \to I\!R \cup \{+\infty\}$ be a measurable function with respect to the σ-field in $[0,T] \times H$ generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in H. Assume $\varphi(t,\cdot)$ is convex and lower semicontinuous on H for every $t \in [0,T]$, such that for some increasing function $\tau : [0,+\infty) \to [0,+\infty)$ we have*

$$-\infty < \int_0^T \varphi(t,y(t))dt \le \tau(\|y\|_{L^2}) \quad \text{for every } y \in L^2_H. \tag{32}$$

Then, for any $u_0 \in H$ and any $f \in L^2([0,T];H)$ such that $\int_0^T \varphi^*(t,-f(t))dt < \infty$, consider the convex potential

$$\psi(t,x) = \varphi(t,x+u_0) - \langle f(t),x\rangle$$

and the functional

$$I(u) = \int_0^T [\psi(t,u(t)) + \psi^*(t,-\dot u(t))]\,dt + \frac{1}{2}(\|u(0)\|^2 + \|u(T)\|^2). \tag{33}$$

on A^2_H. Then, there exists a unique minimizer v in A^2_H such that

$$I(v) = \inf_{u \in A^2_H} I(u) = 0. \tag{34}$$

Moreover, $u(t) := v(t) + u_0$ is the unique solution in A^2_H to the equation

$$\begin{cases} -\dot u(t) \in \partial\varphi(t,u(t)) + f(t) \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ u(0) = u_0. \end{cases} \tag{35}$$

Proof: The above variational problem corresponds to the readily self-dual Lagrangian pair (L,ℓ) defined by:

$$\ell(c_0,c_T) = \frac{1}{2}\|c_0\|^2_H + \frac{1}{2}\|c_T\|^2_H \quad \text{and} \quad L(t,x,p) = \psi(t,x) + \psi^*(t,-p). \tag{36}$$

Note again that

$$I(u) = \int_0^T [\psi(t,u(t)) + \psi^*(t,-\dot u(t)) + \langle u(t),\dot u(t)\rangle]\,dt + \|u(0)\|^2. \tag{37}$$

The Fenchel-Young inequality yields:

$$\psi(t,u(t)) + \psi^*(t,-\dot u(t)) \ge \langle u(t),-\dot u(t)\rangle = -\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_H \quad \text{a.e. on } [0,T] \tag{38}$$

with equality holding if and only if u satisfies

$$-\dot{u}(t) \in \partial\psi(t, u(t)) \quad \text{a.e. on } [0, T]. \tag{39}$$

The hypothesis insure that Theorem 1 applies. Indeed, in view of Remark 1, by taking the arc which is identically zero, we have for $(a, y) \in H \times L^2_H$,

$$\begin{aligned}\int_0^T L(t, y(t), 0)dt + \ell(a, 0) &= \int_0^T \psi(t, y(t))dt + (\psi|_X)^*(t, 0) + \frac{\|a\|^2_H}{2} \\ &= \int_0^T [\varphi(t, u_0 + y(t)) + \varphi^*(t, -f(t))]\, dt \\ &\quad + \int_0^T [|\langle y(t), f(t)\rangle| + |\langle u_0, f(t)\rangle|]\, dt + \frac{\|a\|^2_H}{2} \\ &\leq \tau(\|y + u_0\|_{L^2_H}) + C_1\|y\|_{L^2_H}) + C_2 + \frac{\|a\|^2_H}{2}\end{aligned}$$

which means that its is bounded in a neighborhood of $(0, 0)$ in the space $H \times L^2_H$. By Theorem 1, there is a unique $v \in A^2_H$ such that:

$$I(v) = \inf_{A^2_H} I = 0. \tag{40}$$

This will then insure equality in (38) and that $v(0) = 0$. It follows that the path $u(t) = v(t) + u_0$ is a weak solution for (35).

Yosida's regularization: The boundedness condition (32) on $\varphi(t, \cdot)$ in Proposition 3, is quite restrictive and actually not satisfied by most potentials of interest. We offer two ways to deal with such a difficulty. The first is to assume similar bounds on ψ but in Banach norms that are stronger –hence easier to satisfy– than the Hilbert norm of the ambiant space. This will be the subject of sections 4 and 5.

Another way to remedy this is to regularize ψ by using inf-convolution. That is, for each $\lambda > 0$ we define

$$\psi_\lambda(t, x) = \inf\{\psi(t, y) + \frac{1}{2\lambda}\|x - y\|^2_H;\ y \in H\},$$

so that

$$\psi_\lambda(t, x) \leq \psi(t, 0) + \frac{1}{2\lambda}\|x\|^2$$

while its conjugate is given by

$$\psi^*_\lambda(t, y) = \psi^*(t, y) + \frac{\lambda}{2}\|y\|^2. \tag{41}$$

The ψ_λ now satisfy the hypothesis of Proposition 3, as long as

$$-\infty < \int_0^T \varphi(t, u_0) + \varphi^*(t, -f(t))dt < \infty,$$

and therefore the corresponding evolution equations

$$\begin{cases} \dot{v}(t) + \partial\varphi_\lambda(t, v(t)) = f(t) \quad \text{a.e. on } [0, T] \\ \qquad\qquad v(0) \quad = u_0 \end{cases} \tag{42}$$

have unique solutions $v_\lambda(t)$ in A_H^2 that minimize

$$I_\lambda(u) := \int_0^T [\psi_\lambda(t, u(t)) + \psi_\lambda^*(t, -\dot{u}(t)) + \langle u(t), \dot{u}(t)\rangle]\, dt + \|u(0)\|^2. \tag{43}$$

Now we need to argue that $(v_\lambda)_\lambda$ converges as $\lambda \to 0$ to a solution of the original problem. For that, an upper bound on the L^2-norm of $(\dot{v}_\lambda(t))_\lambda$ is needed, but this is not always possible for general time-dependent potentials. However we shall show next that this is indeed possible at least for when φ does not evolve in time. In a forthcoming paper of Ghoussoub-McCann ([7]), it is shown that such estimates hold for certain interesting cases of time-dependent potentials.

This analysis is reminiscent of the approach via the resolvent theory of Hille-Yosida, but is much easier here since the variational argument does not require uniform convergence of $(v_\lambda)_\lambda$ or their time-derivatives.

Here is the main application of our method.

Theorem 2 *Let $\varphi : H \to I\!R \cup \{+\infty\}$ be a bounded below convex and lower semi-continuous function on H and let $\varphi^* : H \to I\!R \cup \{+\infty\}$ be its Fenchel conjugate. For any $u_0 \in \mathrm{Dom}(\partial\varphi)$ and $f \in \mathrm{Dom}(\varphi^*)$, consider on A_H^2 the functional:*

$$\begin{aligned} I(u) := \int_0^T &[\varphi(u(t) + u_0) + \varphi^*(f - \dot{u}(t)) - \langle u(t), f\rangle + \langle \dot{u}(t), u_0\rangle]dt \\ &+ \frac{1}{2}(\|u(0)\|_H^2 + \|u(T)\|_H^2) - T\langle f, u_0\rangle. \end{aligned} \tag{44}$$

Then, there exists a unique v in A_H^2 such that

$$I(v) = \inf_{A_H^2} I(u) = 0. \tag{45}$$

Moreover, the path $u(t) = v(t) + u_0$ is valued in $\mathrm{Dom}(\partial\varphi)$ for almost all $t \in [0, T]$ and is a solution for

$$\begin{cases} \dot{u}(t) + \partial\varphi(u(t)) \ = f \quad \text{a.e. on } [0, T] \\ \qquad\qquad u(0) = u_0. \end{cases} \tag{46}$$

We first establish the existence and semi-group property of the solutions, under a stronger bound on the potential φ.

Proposition 4 *let $\varphi : H \to I\!R$ be a bounded below convex and lower semi-continuous function on H. Assume that φ satisfies for some $C > 0$,*

$$\varphi(x) \le C(1 + \|x\|_H^2) \quad \textit{for } u \in H. \tag{47}$$

Then, for any $u_0, f \in H$ such that $\varphi^*(-f) < \infty$, the functional I defined in (44) achieves its minimum on A^2_H at a path v, and $u(t) = v(t) + u_0$ is a solution for (46).

Moreover, the formula $P_T(u_0) = u(T)$ defines unambiguously a 1-Lipschitz semi-group of operators $\{P_t\}_{t\in\mathbf{R}^+}$ on H.

Proof: Again, the functional I can be written as

$$I(u) = \int_0^T [\psi(u(t)) + \psi^*(-\dot{u}(t))]\,dt + \frac{1}{2}(\|u(0)\|^2_{H} + \|u(T)\|^2_{H}) \tag{48}$$

where $\psi(u) = \varphi(u+u_0) - \langle u, f\rangle$. Proposition 3 then applies to yield the existence of a unique solution of (46).

To establish the semi-group and other properties of the solutions, note that the solution $\{u(t);\, t \in [0,T]\}$ can be also characterized as the unique path in A^2_H such that $u(0) = u_0$ while for any $t \le T$,

$$\int_0^t \varphi(u(s)) + \varphi^*(f - \dot{u}(s))\,ds + \frac{\|u(t)\|^2_H}{2} = \frac{\|u_0\|^2_H}{2}.$$

This means that one can define unambiguously a one-parameter family of operators $\{P_t\}_{t\in\mathbf{R}^+}$ on H by $P_t(u_0) = u(t)$.
Showing that it is a 1-Lipschitz semi-group of operators is standard: take any two initial conditions u_0 and v_0 in H, and write:

$$\begin{aligned}
0 &\le \langle P_t(u_0) - P_t(v_0), \partial\varphi(P_t(u_0)) - \partial\phi(P_t(v_0))\rangle \\
&= -\langle P_t(u_0) - P_t(v_0), \frac{d}{dt}(P_t(u_0) - P_t(v_0))\rangle \\
&= -\frac{1}{2}\frac{d}{dt}\|P_t(u_0) - P_t(v_0)\|^2_H
\end{aligned}$$

which means that $\frac{d}{dt}\|P_t(u_0) - P_t(v_0)\|^2_H \le 0$, and consequently

$$\|P_t(u_0) - P_t(v_0)\|^2_H \le \|u_0 - v_0\|^2_H.$$

For the semi-group property, first take $u_0 \in H$ and let $v_0 = P_t(u_0)$ Then

$$\int_0^s \varphi(P_\tau(v_0)) + \varphi^*\left(f - \frac{d}{d\tau}P_\tau(v_0)\right)d\tau + \frac{\|P_s(v_0)\|^2_H}{2} = \frac{\|v_0\|^2_H}{2}$$

and

$$\int_0^t \varphi(P_\tau(u_0)) + \varphi^*\left(f - \frac{d}{d\tau}P_\tau(u_0)\right)d\tau + \frac{\|P_t(u_0)\|^2_H}{2} = \frac{\|u_0\|^2_H}{2}.$$

Adding the two, we get:

$$\int_0^s \varphi(P_\tau(v_0)) + \varphi^*\left(f - \frac{d}{d\tau}P_\tau(v_0)\right)d\tau + \frac{\|P_s(P_t(u_0))\|^2}{2} + \int_0^t \varphi(P_\tau(u_0)) + \varphi^*\left(f - \frac{d}{d\tau}P_\tau(u_0)\right) = \frac{\|u_0\|^2}{2}.$$

Let now

$$W(\tau) = \begin{cases} P_\tau(u_0) & \text{if} \quad \tau \in [0,t] \\ P_{\tau-t}(P_t(u_0)) & \text{if} \quad \tau \in [t,t+s] \end{cases}$$

then

$$\int_0^{s+t} \varphi(W(\tau)) + \varphi^*(f - \dot{W}(\tau))d\tau + \frac{\|W(t+s)\|_H^2}{2} = \frac{\|u_0\|_H^2}{2}.$$

But we know that:

$$\int_0^{s+t} \varphi(P_\tau(u_0)) + \varphi^* \left(f - \frac{d}{d\tau} P_\tau(u_0) \right) d\tau + \frac{\|P_{s+t}(u_0)\|_H^2}{2} = \frac{\|u_0\|_H^2}{2}$$

which means that $P_\tau(u_0) = W_\tau$ for all $\tau \in [0, s+t]$ and $P_{s+t}(u_0) = P_s(P_t(u_0))$.

End of proof of Theorem 2: Consider as before $\psi(u) = \varphi(u+u_0) - \langle u, f\rangle$ and for each $\lambda > 0$, let

$$\psi_\lambda(x) = \inf\{\psi(y) + \frac{1}{2\lambda}\|x-y\|_H^2;\ y \in H\}.$$

The functional ψ_λ now satisfy the hypothesis of Proposition 4 and therefore the corresponding evolution equation

$$\begin{cases} \dot{u}_\lambda(t) + \partial\psi_\lambda(u_\lambda(t)) = 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ \qquad\qquad u_\lambda(0) \quad = 0 \end{cases} \tag{49}$$

have a solution $u_\lambda(t)$ in A_H^2 that minimizes

$$I_\lambda(u) = \int_0^T [\psi_\lambda(u(t)) + \psi_\lambda^*(-\dot{u}(t))]\, dt + \frac{1}{2}(\|u(0)\|_H^2 + \|u(T)\|_H^2). \tag{50}$$

Now we need to argue that $(u_\lambda)_\lambda$ converges as $\lambda \to 0$ to a solution of the original problem. Define $J_\lambda(x)$ to be the unique point in H such that

$$\psi_\lambda(x) = \psi(J_\lambda(x)) + \frac{1}{2\lambda}\|x - J_\lambda(x)\|^2.$$

It is easy to check that for every $\lambda > 0$, the map $x \to J_\lambda(x)$ is 1-Lipschitz on H. We now establish the following estimates:

Lemma 2 *For any $\lambda > 0$, we have:*

$$\dot{u}_\lambda(t) + \partial\psi(J_\lambda(u_\lambda(t)) = 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \tag{51}$$

$$-\dot{u}_\lambda(t) = \frac{u_\lambda(t) - J_\lambda(u_\lambda(t))}{\lambda} \quad \text{for all } t \in [0,T], \tag{52}$$

and

$$\|\dot{u}_\lambda(t)\| \le \|\dot{u}_\lambda(0)\| = \frac{\|J_\lambda(u_\lambda(0))\|}{\lambda} \le \inf\{\|z\|; z \in \partial\psi(0)\}. \tag{53}$$

Proof: Denote $v_\lambda(t) = J_\lambda(u_\lambda(t))$ and note that by two applications of Young-Fenchel duality, we have

$$\begin{aligned}0 &= \int_0^T \psi(v_\lambda(t)) + \psi^*(-\dot u_\lambda(t)) + \tfrac{1}{2\lambda}\|u_\lambda(t) - v_\lambda(t)\|^2 + \tfrac{\lambda}{2}\|\dot u_\lambda(t)\|^2 dt + \tfrac12\|u_\lambda(T)\|^2 \\ &\geq \int_0^T \langle v_\lambda(t), -\dot u_\lambda(t)\rangle + \tfrac{1}{2\lambda}\|u_\lambda(t) - v_\lambda(t)\|^2 + \tfrac{\lambda}{2}\|\dot u_\lambda(t)\|^2 dt + \tfrac12\|u_\lambda(T)\|^2 \\ &\geq \int_0^T \langle v_\lambda(t), -\dot u_\lambda(t)\rangle + \langle v_\lambda(t) - u_\lambda(t), \dot u_\lambda(t)\rangle + \tfrac12\|u_\lambda(T)\|^2 \\ &= \tfrac12\|u_\lambda(0)\|^2 = 0.\end{aligned}$$

This implies that

$$\psi(v_\lambda(t)) + \psi^*(-\dot u_\lambda(t)) = \langle v_\lambda(t), -\dot u_\lambda(t)\rangle \quad \text{a.e.}$$

and

$$\frac{1}{2\lambda}\|u_\lambda(t) - v_\lambda(t)\|^2 + \frac{\lambda}{2}\|\dot u_\lambda(t)\|^2 = \langle v_\lambda(t) - u_\lambda(t), \dot u_\lambda(t)\rangle.$$

It follows that

$$-\dot u_\lambda(t) \in \partial\psi(v_\lambda(t)) \quad \text{and} \quad -\dot u_\lambda(t) = \frac{u_\lambda(t) - v_\lambda(t)}{\lambda} \quad \text{a.e.,}$$

and since $x \to J_\lambda(x)$ is continuous for each $\lambda > 0$, the latter is true for every $t \in [0,T]$.

Pick now $z \in \partial\psi(0)$ and note that

$$\begin{aligned}0 &\leq \langle 0 - v_\lambda(0), z - (-\dot u_\lambda(0))\rangle \\ &= \langle 0 - v_\lambda(0), z - (\tfrac{u_\lambda(0) - v_\lambda(0)}{\lambda})\rangle \\ &= \langle -v_\lambda(0), z + \tfrac{v_\lambda(0)}{\lambda}\rangle\end{aligned}$$

which implies that $\frac{\|v_\lambda(0)\|}{\lambda} \leq \|z\|$. Use now the 1-Lipschitz semi-group property to get for $\lambda > 0$ and each $t \in [0,T]$,

$$\|u_\lambda(t+h) - u_\lambda(t)\| = \|P_t(u_\lambda(h)) - P_t(0)\| \leq \|u_\lambda(h)\|.$$

This implies

$$\|\dot u_\lambda(t)\| \leq \|\dot u_\lambda(0)\| = \frac{\|v_\lambda(0)\|}{\lambda} \leq \|z\|,$$

and the lemma is established.

The above estimate now yields that a subsequence $(u_{\lambda_j})_j$ is converging weakly in A^2_H to a path u. This implies that for each $t \in [0,T]$, $u_{\lambda_j}(t) \to u(t)$ weakly in H, and since $\|u_\lambda(t) - v_\lambda(t)\| = |\lambda|\|\dot u(t)\| \leq |\lambda|\|z\|$, we get that $v_{\lambda_j}(t) \to u(t)$ weakly in H for every $t \in [0,T]$.

Since ψ and ψ^* are weakly lower semi-continuous on H, one can easily deduce that:

$$\int_0^T \psi(u(t))dt \leq \underline{\lim}_j \int_0^T \psi(v_{\lambda_j}(t))dt,$$

$$\int_0^T \psi^*(-\dot{u}(t))dt \le \underline{\lim}_j \int_0^T \psi^*(-\dot{u}_{\lambda_j}(t))dt,$$

as well as

$$\|u(0)\|^2 \le \underline{\lim}_j \|u_{\lambda_j}(0)\|^2 = 0 \quad \text{and} \quad \|u(T)\|^2 \le \underline{\lim}_j \|u_{\lambda_j}(T)\|^2 = 0.$$

Moreover,

$$\int_0^T \frac{\|u_{\lambda_j}(t) - v_{\lambda_j}(t)\|^2}{\lambda_j} \le \frac{\lambda_j^2 \|z\|^2 T}{\lambda_j} \to 0,$$

and

$$\int_0^T \lambda_j \|\dot{u}_{\lambda_j}(t)\|^2 dt \le \|z\|^2 T \lambda_j \to 0.$$

It follows that

$$\begin{aligned}
\textstyle\int_0^T \psi(u(t) + \psi^*(-\dot{u}(t))\, dt + \frac{\|u(T)\|^2}{2} &\le \underline{\lim}_j \textstyle\int_0^T \psi(v_{\lambda_j}(t))dt + \underline{\lim}_j \int_0^T \psi^*(-\dot{u}_{\lambda_j}(t))dt \\
&\quad + \underline{\lim}_j \textstyle\int_0^T \big(\frac{\|u_{\lambda_j}(t) - v_{\lambda_j}(t)\|^2}{2\lambda_j} + \frac{\lambda_j}{2}\|\dot{u}_{\lambda_j}(t)\|^2\big)dt + \underline{\lim}_j \frac{\|u_{\lambda_j}(T)\|^2}{2} \\
&\le \underline{\lim}_j \textstyle\int_0^T \psi_{\lambda_j}(u_{\lambda_j}(t)) + \psi^*_{\lambda_j}(-\dot{u}_\lambda(t))dt + \underline{\lim}_j \frac{\|u_{\lambda_j}(T)\|^2}{2} \\
&\le \underline{\lim}_j \big(\textstyle\int_0^T \psi_{\lambda_j}(u_{\lambda_j}(t)) + \psi^*_{\lambda_j}(-\dot{u}_\lambda(t))dt + \frac{\|u_{\lambda_j}(T)\|^2}{2}\big) \\
&= 0.
\end{aligned}$$

This means that u solves the minimization problem and that $-\dot{u}(t) \in \partial\psi(u(t))$ a.e. while $u(0) = 0$. The path $u_0 + u(t)$ then solves the original problem (36).

Quasi-linear parabolic equations

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $p \ge \frac{n-2}{n+2}$, we have that the Sobolev space $W_0^{1,p+1}(\Omega) \subset H := L^2(\Omega)$, and so we define on $L^2(\Omega)$ the functional $\varphi(u) = \frac{1}{p+1}\int_\Omega |\nabla u|^{p+1}$ on $W_0^{1,p+1}(\Omega)$ and $+\infty$ elsewhere. Its conjugate is then $\varphi^*(v) = \frac{p}{p+1}\int_\Omega |\nabla \Delta^{-1} v|^{\frac{p+1}{p}} dx$. We then obtain for any $u_0 \in W_0^{1,p+1}(\Omega)$ and any $f \in W^{-1,\frac{p+1}{p}}(\Omega)$, that the infimum of the functional

$$\begin{aligned}
I(u) = \frac{1}{p+1}\int_0^T \int_\Omega \Big(|\nabla(u(t,x) + u_0(x))|^{p+1} + p|\nabla \Delta^{-1}(f(x) - \frac{\partial u}{\partial t}(t,x))|^{\frac{p+1}{p}} \Big)\, dxdt \\
+ \int_0^T \int_\Omega \Big[u_0(x)\frac{\partial u}{\partial t}(t,x) - f(x)u(x,t) \Big]\, dxdt \\
-T\int_\Omega f(x)u_0(x)\, dx + \frac{1}{2}\int_\Omega (|u(0,x)|^2 + |u(T,x)|^2) dx
\end{aligned}$$

on the space A^2_H is equal to zero and is attained uniquely at an $W_0^{1,p+1}(\Omega)$-valued path $\tilde{u}$ such that $\int_0^T \|\dot{u}(t)\|_2^2 dt < +\infty$. Moreover, the path $u(t) = \tilde{u}(t) + u_0$ is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta_p u + f \text{ on } \quad \Omega \times [0,T] \\ u(0,x) \quad = u_0 \quad \text{on} \quad \Omega \\ u(t,0) \quad = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{54}$$

Porous media equations

Let $H = H^{-1}(\Omega)$ equipped with the norm induced by the scalar product

$$\langle u, v\rangle = \int_\Omega u(-\Delta)^{-1} v dx = \langle u, v\rangle_{H^{-1}(\Omega)}.$$

For $m \geq \frac{n-2}{n+2}$, we have $L^{m+1}(\Omega) \subset H^{-1}$, and so we can consider the functional

$$\varphi(u) = \begin{cases} \frac{1}{m+1}\int_\Omega |u|^{m+1} \text{ on } \quad L^{m+1}(\Omega) \\ +\infty \quad\quad\quad\quad\quad \text{elsewhere} \end{cases} \tag{55}$$

and its conjugate

$$\varphi^*(v) = \frac{m}{m+1}\int_\Omega |\Delta^{-1} v|^{\frac{m+1}{m}} dx. \tag{56}$$

Then, for any $u_0 \in L^{m+1}(\Omega)$ and any $f \in L^2(\Omega)$, the infimum of the functional

$$\begin{aligned} I(u) = {} & \frac{1}{m+1}\int_0^T\int_\Omega \left(|(u(t,x)+u_0(x))|^{m+1} dx + m\Delta^{-1}(f(x) - \frac{\partial u}{\partial t}(t,x))|^{\frac{m+1}{m}} \right) dxdt \\ & + \int_0^T\int_\Omega \left[u_0(x)(\Delta^{-1}\frac{\partial u}{\partial t})(t,x) - u(x,t)(\Delta^{-1} f)(x) \right] dxdt \\ & - T\int_\Omega u_0(x)(-\Delta)^{-1} f(x)\, dx + \frac{1}{2}\left(\|u(0)\|^2_{H^{-1}} + \|u(T)\|^2_{H^{-1}} \right) \end{aligned}$$

on the space A^2_H is equal to zero and is attained uniquely at an $L^{m+1}(\Omega)$-valued path $\tilde{u}$ such that $\int_0^T \|\dot{u}(t)\|_H^2 dt < +\infty$. Moreover, the path $u(t) = \tilde{u}(t) + u_0$ is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u^m + f \text{ on } \quad \Omega \times [0,T] \\ u(0,x) \quad = u_0 \quad \text{on} \quad \Omega. \end{cases} \tag{57}$$

4 Intermediate spaces and self-dual variational problems

The approach we use in the rest of the paper, consists of introducing natural Banach spaces whose norm is stronger than the Hilbertian norm and on which the energy functional has a better chance to be bounded. The framework –known as an evolution triple setting– is well known, and the intermediate Banach spaces appear naturally in the applications. Here is a brief description of this framework.

Let H be a Hilbert space with $\langle , \rangle$ as scalar product. Let X be a dense vector subspace of H and assume that X is equipped with a norm $\|\cdot\|$ that makes it a reflexive Banach space. Also assume that the canonical injection $X \to H$ is continuous. We identify the Hilbert space H with its dual H^* and we "inject" H in X^* via the following procedure. For each $h \in H$, the map $Sh : u \in X \to \langle h, u\rangle_H$ is a continuous linear functional on X in such a way that

$$\langle Sh, u\rangle_{X^*,X} = \langle h, u\rangle_H \quad \text{for all } h \in H \text{ and all } u \in X$$

One can easily see that $S : H \to X^*$ is continuous, one-to-one, and that $S(H)$ is dense in X^*. In other words, one can then place H in X^* in such a way that $X \subset H = H^* \subset X^*$ where the injections are continuous and with dense range. We note that with such an identification the duality $\langle f, u\rangle_{X^*,X}$ coincides with the scalar product $\langle f, u\rangle_H$ as soon as $f \in H$ and $u \in X$. In other words, we are representing the dual X^* of X as the completion of H for the dual norm $\|h\| = \sup\{\langle h, u\rangle_H; \|u\|_X \leq 1\}$. We shall sometimes say that the space X is anchored on the Hilbert space H.

For each α $(1 < \alpha < \infty)$, we consider the Banach space

$$A^\alpha_{H,X^*} = \{u : [0, T] \to X^*; u(0) \in H, \quad u \,\&\, \dot u \in L^\alpha_{X^*}\}$$

equipped with the norm

$$\|u\|_{A^\alpha_{H,X^*}} = \|u(0)\|_H + (\int_0^T \|\dot u\|^\alpha_{X^*} dt)^{\frac{1}{\alpha}}.$$

It is clear that A^α_{H,X^*} is a reflexive Banach space that can be identified with the product space $H \times L^\alpha_{X^*}$, while its dual $(A^\alpha_{H,X^*})^* \simeq H \times L^\beta_X$ where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The duality is then given by the formula:

$$\langle u, (a, p)\rangle_{A^\alpha_{H,X^*}, H\times L^\beta_X} = (u(0), a)_H + \int_0^T \langle \dot u(t), p(t)\rangle dt$$

where $\langle\cdot,\cdot\rangle$ is the duality on X, X^* and $(\cdot,\cdot)$ is the inner product on H.

Let $\ell : X^* \times X^* \to \mathbb{R} \cup \{+\infty\}$ be convex and weak*-lower semi-continuous on $X^* \times X^*$, and let $L : [0, T] \times X^* \times X^* \to \mathbb{R} \cup \{+\infty\}$ be measurable with respect to the σ-field in $[0, T] \times X^* \times X^*$ generated by the products of Lebesgue sets in $[0, T]$ and Borel sets in $X^* \times X^*$, in such a way that for each $t \in [0, T]$, $L(t, \cdot, \cdot)$

is convex and weak*-lower semi-continuous on $X^* \times X^*$. To any such a pair, we associate the action functional on A^α_{H,X^*} by:

$$I_{\ell,L}(u) = \int_0^T L(t,u(t),\dot u(t))dt + \ell(u(0),u(T))$$

as well as the corresponding "variation function" $J_{\ell,L}$ defined on $(A^\alpha_{H,X^*})^* = H \times L^\beta_X$ by

$$J^\alpha_{\ell,L}(a,y) = \inf\{\int_0^T L(t,u+y,\dot u)dt + \ell(u(0)+a,u(T))\ ;\ u \in A^\alpha_{H,X^*}\}$$

Now associate to the pair (ℓ, L), the following "Bolza-dual" functionals:

$$M(t,p,s) = (L_t|_{X\times X})^*(s,p) \text{ and } \ m(p,s) = (\ell|_{X\times X})^*(p,-s)$$

where $(L_t|_{X\times X})^*$ and $(\ell|_{X\times X})^*$ denote the Legendre duals of the restrictions of $L_t = L(t,\cdot,\cdot)$ and ℓ to $X \times X$. In other words, M and l are the convex and lower semi-continuous functions on $X^* \times X^*$ defined by:

$$M(t,p,s) = \sup\{\langle u,s\rangle_{X,X^*} + \langle v,p\rangle_{X,X^*} - L(t,u,v);\ u,v \in X\}$$

and

$$m(p,s) = \sup\{\langle u,p\rangle_{X,X^*} + \langle v,-s\rangle_{X,X^*} - \ell(u,v);\ u,v \in X\}$$

Definition 2 *We again say that the pair (L,ℓ) is self-dual if for all $(p,s) \in X^* \times X^*$, we have*

$$m(p,s) = \ell(-p,-s) \quad \text{and } M(t,s,p) = L(t,-s,-p),$$

or equivalently

$$(\ell|_{X\times X})^*(p,s) = \ell(-p,s) \quad \text{and } (L_t|_{X\times X})^*(t,p,s) = L(t,-s,-p).$$

Theorem 3 *Suppose that for each $t \in [0,T]$, the Lagrangians $L(t,\cdot)$ and l are two proper convex and weak*-lower semi-continuous functions on $X^* \times X^*$ such that the pair (L,ℓ) is self-dual. Suppose that for some $\alpha \in (1,2]$, $J^\alpha_{\ell,L} : H \times L^{\alpha^*}_X \to \mathbb{R} \cup \{+\infty\}$ is sub-differentiable at $(0,0)$, then there exists $v \in A^\alpha_{H,X^*}$ such that:*

$$(v(t),\dot v(t)) \in \mathrm{Dom}(L) \quad \text{for almost all } t \in [0,T],$$

and

$$I_{\ell,L}(v) = \inf_{A^\alpha_{H,X^*}} I_{\ell,L}(u) = 0.$$

Again, we need the following lemmas.

Lemma 3 *Define* $E_M(\cdot,\cdot) : L^\alpha_{X^*}\times L^\alpha_{X^*} \to \mathbf{R}\cup\{+\infty\}$ *by* $E_M(p,s) = \int_0^T M(t,p(t),s(t))dt$, *then*

$$E_M(p,s) = \sup\left\{\int_0^T [\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))]dt\ ;\ (u,v)\in L^\beta_X\times L^\beta_X\right\}.$$

Proof: For $u,v\in L^\beta_X$ and $p,s\in L^\alpha_{X^*}$, we have:

$$\int_0^T L(t,u(t),v(t))dt + \int_0^T M(t,p(t),s(t))dt \geq \int_0^T (\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle)dt,$$

which implies

$$\int_0^T M(t,p(t),s(t))dt \geq \sup\left\{\int_0^T [\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))]dt; (u,v)\in L^\beta_X\times L^\beta_X\right\}.$$

For the reverse inequality, Let (p,s) be in $L^\alpha_{X^*}\times L^\alpha_{X^*}$ in such a way that $\beta < E_M(p,s)$ and let $\mu(t)$ be such that $\mu(t) < M(t,p(t),s(t))$ for all t while $\int_0^T\mu(t)dt > \beta$. We then have for all t,

$$-\mu(t) > -M(t,p(t),s(t)) = \inf\{L(t,u(t),v(t)) - \langle u(t),s(t)\rangle - \langle v(t),p(t)\rangle; (u,v)\in X\times X\}$$

Again, by ([5]), there exists a measurable pair $(u_1,u_2)\in L^\beta_X\times L^\beta_X$ such that

$$-\mu(t) \geq L(t,u_1(t),u_2(t)) - \langle u_1(t),s(t)\rangle_{X,X^*} - \langle u_2(t),p(t)\rangle_{X,X^*}.$$

Therefore

$$\begin{aligned}\beta &< \int_0^T\mu(t)dt \leq \int_0^T[-L(t,u_1(t),u_2(t)) + \langle u_1(t),s(t)\rangle_{X,X^*} + \langle u_2(t),p(t)\rangle_{X,X^*}]dt\\ &\leq \sup\left\{\int_0^T [\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))]dt\ ;\ (u,v)\in L^\beta_X\times L^\beta_X\right\}\end{aligned}$$

which implies that

$$E_M(p,s) \leq \sup\left\{\int_0^T [\langle s(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),v(t))]dt\ ;\ (u,v)\in L^\beta_X\times L^\beta_X\right\}.$$

For the next lemma, recall that $I_{m,M}(u) = \int_0^T M(t,u(t),\dot u(t))dt + m(u(0),u(T))$ for $u\in A^\alpha_{H,X^*}$.

Lemma 4 *Under the above conditions, we have* $J^*_{\ell,L}(p) \geq I_{m,M}(p)$ *for all* $p\in A^\alpha_{H,X^*}$.

Proof: For $p \in A^{\alpha}_{H,X^*}$, write:

$$J^*_{\ell,L}(p) = \sup_{a\in H}\sup_{y\in L^\beta_X}\sup_{u\in A^\alpha_{H,X^*}}\left\{(a,p(0)) + \int_0^T [\langle y,\dot p\rangle - L(t,u+y,\dot u)]dt - \ell(u(0)+a,u(T))\right\}.$$

Set

$$F \overset{\text{def}}{=} \left\{u \in A^\alpha_{H,X^*}\ ;\ u\in L^\beta_X\right\} \subseteq A^\alpha_{H,X^*}.$$

Then

$$J^*_{\ell,L}(p) \geq \sup_{a\in H}\sup_{y\in L^\beta_X}\sup_{u\in F}\left\{(a,p(0)) + \int_0^T [-L(t,u+y,\dot u) + \langle y,\dot p\rangle]dt - \ell(u(0)+a,u(T))\right\}$$

Make a substitution

$$u(0)+a = a' \in H \quad \text{and } u+y = y' \in L^\beta_X,$$

we obtain

$$J^*_{\ell,L}(p) \geq \sup_{a'\in H}\sup_{y'\in L^\beta_X}\sup_{u\in F}\left\{(a'-u(0),p(0)) - \ell(a',u(T)) + \int_0^T [\langle y'-u,\dot p\rangle - L(t,y',\dot u)]dt\right\}.$$

Set now

$$S = \{u:[0,T]\to X; u\in L^\beta_X,\ \dot u\in L^\beta_X,\ u(0)\in X\}.$$

Since $\beta \geq 2 \geq \alpha$ and $\|\cdot\|_{X^*} \leq C\|\cdot\|_X$, we have $S \subseteq A^{\alpha}_{H,X^*}\cap L^\beta_X = F$ and

$$J^*_{\ell,L}(p) \geq \sup_{a'\in H}\sup_{y'\in L^\beta_X}\sup_{u\in S}\left\{(a'-u(0),p(0)) + \int_0^T [\langle y',\dot p\rangle - \langle u,\dot p\rangle - L(t,y',\dot u)]dt - \ell(a',u(T))\right\}$$

Since $\dot u \in L^\beta_X$ and $u\in L^\beta_X$, we have:

$$\int_0^T \langle u,\dot p\rangle dt = -\int_0^T \langle \dot u,p\rangle dt + \langle p(T),u(T)\rangle - \langle p(0),u(0)\rangle.$$

But $p(0)\in H$, so $\langle p(0),u(0)\rangle = (p(0),u(0))$ which implies

$$J^*_{\ell,L}(p) \geq \sup_{a'\in H}\sup_{y'\in L^\beta_X}\sup_{u\in S}\left\{(a',p(0)) + \int_0^T \left\{\langle y',\dot p\rangle + \langle \dot u,p\rangle - L(t,y',\dot u)\right\}dt - \langle u(T),p(T)\rangle - \ell(a',u(T))\right\}.$$

It is now convenient to identify $S = \{u:[0,T]\to X;\ u\in L^\beta_X,\ \dot u\in L^\beta_X,\ u(0)\in X\}$ with $X\times L^\beta_X$ via the correspondence:

$$(c,v)\in X\times L^\beta_X \mapsto c + \int_t^T v(s)\,ds \in S$$
$$u\in S\mapsto (u(T),-\dot u(t))\in X\times L^\beta_X.$$

We finally obtain

$$\begin{aligned}
J^*_{\ell,L}(p) &\geq \sup_{a'\in H}\sup_{c\in X}\{(a',p(0))+\langle -c,p(T)\rangle-\ell(a',c)\}\\
&\quad+\sup_{y'\in L^\beta_X}\sup_{v\in L^\beta_X}\left\{\int_0^T[\langle y',\dot p\rangle+\langle v,p\rangle-L(t,y',v)]dt\right\}\\
&=E_M(p,\dot p)+m(p(0),p(T))\\
&=I_{M,m}(p).
\end{aligned}$$

Proof of Theorem 3: By the definition of m, M and by the self-duality hypothesis, we have

$$\inf_{u\in A^\alpha_{H,X^*}} I_{\ell,L}(u)\geq -\inf_{u\in A^\alpha_{H,X^*}} I_{m,M}(u)=-\inf_{u\in A^\alpha_{H,X^*}} I_{\ell,L}(u).$$

If v is an element in $-\partial J_{\ell,L}(0,0)$, it then follows from Lemma 4 that

$$\begin{aligned}
0&\geq -\inf_{A^\alpha_{H,X^*}} I_{\ell,L}=-\inf_{A^\alpha_{H,X^*}} I_{m,M}\\
&\geq \sup_{A^\alpha_{H,X^*}} -J^*_{\ell,L}\\
&=\sup_{u\in A^\alpha_{H,X^*}}\inf_{(a,y)\in H\times L^\beta_X}\left\{J_{\ell,L}(a,y)-\langle (a,y),u\rangle_{H\times L^\beta_X,A^\alpha_{H,X^*}}\right\}\\
&\geq \inf_{H\times L^\beta_X}\left\{J_{\ell,L}(a,y)-\langle (a,y),v\rangle_{H\times L^\beta_X,A^\alpha_{H,X^*}}\right\}\\
&\geq J_{\ell,L}(0,0)=\inf_{u\in A^\alpha_{H,X^*}} I_{\ell,L}(u)\geq 0.
\end{aligned}$$

Note that any v in $-\partial J_{\ell,L}(0,0)\subset A^\alpha_{H,X^*}$ is a solution since

$$\begin{aligned}
0&\leq I_{\ell,L}(-v)=I_{m,M}(v)\\
&\leq \sup_{(a,y)\in H\times L^\beta_X}\left\{\langle v,(a,y)\rangle_{A^\alpha_{H,X^*},H\times L^\beta_X}-J_{\ell,L}(a,y)\right\}\\
&\leq -J_{\ell,L}(0,0)=-\inf_{u\in A^\alpha_{H,X^*}} I_{\ell,L}(u)=0.
\end{aligned}$$

5 Variational formulation of the gradient flow of an evolving convex landscape

Here is the main result of this section.

Theorem 4 *Let X be a reflexive Banach space anchored on a Hilbert space H, and suppose $\varphi : [0,T] \times H \to I\!R \cup \{+\infty\}$ is a jointly measurable function such that for every $t \in [0,T]$, the function $\varphi(t,\cdot)$ is convex and lower semi-continuous functional on H. Assume*

$$\varphi(t,\cdot) \text{ is coercive on } H \text{ i.e., } \lim_{\|v\|_H \to \infty} \varphi(t,v) = +\infty, \tag{58}$$

and that for some $\gamma > 1$, there is an increasing function $\tau : [0,+\infty) \to [0,+\infty)$ so that

$$-\infty < \int_0^T \varphi(t,y(t))dt \leq \tau(\|y\|_{L^\gamma_X}) \quad \text{for every } y \in L^\gamma_X. \tag{59}$$

Setting $\alpha = \min\{2,\gamma^*\}$ where $\frac{1}{\gamma} + \frac{1}{\gamma^*} = 1$, then for any $u_0 \in \mathrm{Dom}(\varphi)$ and any $f \in L^{\alpha^*}_{X^*}$ such that $\int_0^T \varphi^*(t,-f(t))dt < \infty$, the functional

$$\begin{aligned} I(u) : = \int_0^T &\left[\varphi(t,u(t)+u_0) + (\varphi|_X)^*(t,f(t)-\dot u(t)) - \langle u(t)+u_0, f(t)\rangle + \langle \dot u(t), u_0\rangle\right] dt \\ &+\frac{1}{2}(\|u(0)\|^2_H + \|u(T)\|^2_H) \end{aligned} \tag{60}$$

attains its infimum on the set $K = \{u \in A^\alpha_{H,X^*}; u(t) \in H \text{ a.e}\}$ uniquely at a point v such that $v(t) + u_0 \in \mathrm{Dom}(\varphi)$ for all $t \in [0,T]$. Moreover,

$$I(v) = \inf_K I(u) = 0, \tag{61}$$

and the path $u(t) = v(t) + u_0$ is a solution for the equation

$$\begin{cases} \dot u(t) + \partial\varphi(u(t)) = f(t) \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ u(0) = u_0. \end{cases} \tag{62}$$

Proof: Note that I can be written as

$$I(u) = \int_0^T \left[\psi(t,u(t)) + (\psi|_X)^*(t,-\dot u(t))\right] dt + \frac{1}{2}(\|u(0)\|^2_H + \|u(T)\|^2_H) \tag{63}$$

where $\psi(t,u) = \varphi(t,u+u_0) - \langle u, f(t)\rangle$ on $[0,T] \times X$, and $+\infty$ elsewhere on $[0,T] \times X^*$.

Define $\ell : X^* \times X^* \to I\!R \cup \{+\infty\}$ by

$$\ell(c_0,c_T) = \begin{cases} \frac{1}{2}\|c_0\|^2_H + \frac{1}{2}\|c_T\|^2_H \text{ if } c_0 \text{ and } c_T \in H \\ +\infty \quad \text{otherwise,} \end{cases} \tag{64}$$

and L on $[0,T] \times X^* \times X^*$ as $L(t,u,v) = \psi(t,u) + (\psi|_X)^*(t,-v)$. We need to show that ℓ and L satisfy the hypothesis of Theorem 3.

Since the functions $u \to \psi(t,u)$ and $u \to \|u\|^2_H$ are convex, lower semi-continuous and coercive on H, it follows that ℓ and $L(t,\cdot,\cdot)$ are convex and weak*-lower semi-continuous on $X^* \times X^*$. Indeed, to show that L is weak*-lower semi-continuous, let $u_n \in X^*$ go to u in the weak*-topology. We may as well assume that $\liminf_n \tilde{\psi}(t, u_n) < \infty$ which means that $(u_n)_n$ is eventually living in $X \subset H$. Since $\psi(t, \cdot)$ is coercive on H, the sequence $(u_n)_n$ is eventually bounded there and therefore converging weakly in H -up to a subsequence- to an element $\tilde{u} \in H$. Since X is anchored on a Hilbert space H, the convergence of $(u_n)_n$ to $\tilde{u}$ is also in the weak-star topology of X^* and therefore $\tilde{u} = u \in X$. The rest follows from the lower semi-continuity of $\varphi(t, \cdot)$ in the weak topology of H.

To establish self-duality, let m and M be their associated Bolza-dual functionals. We need to show that

$$m(p,s) = \ell(p,s) \quad \text{and} \quad M(t,s,p) = L(t,-s,-p) \text{ for all } (p,s) \in X^* \times X^*.$$

Recall that $m(\cdot,\cdot) : X^* \times X^* \to I\!R \cup \{+\infty\}$ is defined as:

$$\begin{aligned} m(d_0, d_T) &= \sup\{\langle c_0, d_0\rangle + \langle c_T, d_T\rangle - \ell(c_0, c_T);\ c_0, c_T \in X\} \\ &= \sup\{\langle c_0, d_0\rangle - \frac{\|c_0\|_H^2}{2} | c_0 \in X\} + \sup\{\langle c_T, d_T\rangle - \frac{\|c_T\|_H^2}{2};\ c_T \in X\} \end{aligned}$$

To prove that $m(d_0, d_T) = \ell(d_0, d_T)$, we distinguish two cases:

If $d_0 \in X^*$ but $d_0 \notin H$, then there exists $\{c_j\} \subseteq X \subseteq H$ that is bounded in H, yet $\langle c_j, d_0\rangle \to \infty$ as $j \to \infty$. It follows that $\sup\{\langle c_0, d_0\rangle - \frac{\|c_0\|_H^2}{2}\ ;\ c \in X\} \geq \infty$. The same obviously holds for the case where $d_T \in X^*$, but $d_T \notin H$.

On the other hand, $d_0 \in H$ yields

$$\begin{aligned} \sup\{\langle d_0, c_0\rangle - \frac{\|c_0\|_H^2}{2}\ ;\ c_0 \in X\} &= \sup\{(d_0, c_0) - \frac{\|c_0\|_H^2}{2}\ ;\ c_0 \in X\} \\ &= \sup\{(d_0, c_0) - \frac{\|c_0\|_H^2}{2}\ ;\ c_0 \in \bar{X} = H\} \\ &= \|d_0\|_H^2/2 \end{aligned}$$

and therefore $m(d_0, d_T) = \ell(d_0, d_T)$ in all cases.

To establish the self-duality of L, write

$$\begin{aligned} M(t,s,p) &= \sup\{\langle s, v\rangle_{X^*,X} + \langle p, u\rangle_{X^*,X} - L(t,u,v); (v,u) \in X \times X\} \\ &= \sup\{\langle v, -s\rangle_{X,X^*} - (\psi|_X)^*(t,v); v \in X\} + \sup\{\langle u, p\rangle_{X,X^*} - \psi(t,u); u \in X\} \\ &= \psi^{**}(t,-s) + (\psi|_X)^*(t,p) \\ &= \psi(t,-s) + (\psi|_X)^*(t,p) \\ &= L(t,-s,-p) \end{aligned}$$

Here we have used that $(\psi|_X)^* = (\psi)^*$ on $[0,T] \times X$, and that $\psi(t, \cdot)$ is weak*-lower semi-continuous on X^*.

It remains to show that the functional $J_{\ell,L}$ is subdifferentiable at $(0,0)$ in the $H\times L^\beta_X$-topology where $\beta=\alpha^*$. But note that for $(a,y)\in H\times L^\beta_X$

$$\begin{aligned}
J_{\ell,L}(a,y) &= \inf\{\int_0^T L(t,u(t)+y(t),\dot u(t))dt+\ell(u(0)+a,u(T));\ u\in A^\alpha_{H,X^*}\}\\
&\le \int_0^T L(t,y(t),0)dt+\ell(a,0)\\
&= \int_0^T \psi(t,y(t))dt+(\psi|_X)^*(t,0)+\frac{\|a\|_H^2}{2}\\
&= \int_0^T [\varphi(t,u_0+y(t))+\varphi^*(t,-f(t))]\,dt\\
&\quad +\int_0^T [|\langle y(t),f(t)\rangle|+|\langle u_0,f(t)\rangle|]\,dt+\frac{\|a\|_H^2}{2}\\
&\le \tau(\|y+u_0\|_{L^\beta_X})+C_1\|y\|_{L^\beta_X})+C_2+\frac{\|a\|_H^2}{2}
\end{aligned}$$

which means that $J_{\ell,L}$ is bounded in a neighborhood of $(0,0)$ in the space $H\times L^{\alpha^*}_X$, hence it is subdifferentiable at $(0,0)$.

Apply now Theorem 3 to find $v\in A^\alpha_{H,X^*}$ such that: $(v(t),\dot v(t))\in \mathrm{Dom}(L)$ for almost all $t\in[0,T]$, and

$$I_{\ell,L}(v)=\inf_{A^\alpha_{H,X^*}} I_{\ell,L}(u)=0.$$

Note that $\mathrm{Dom}(L)\subset \mathrm{Dom}(\psi)\times \mathrm{Dom}(\psi|_X)^*$. Write now

$$0=I_{\ell,L}(v)=\int_0^T[\psi(t,v(t))+\psi^*(t,-\dot v(t))]dt+\frac12\|v(0)\|_H^2+\frac12\|v(T)\|_H^2$$

which means that both $t\to\psi(t,v(t))$ and $t\to(\psi|_X)^*(t,-\dot v(t))$ are in $L^1[0,T]$ and therefore $v(t)\in X$ a.e. Moreover, the path $v\in C([0,T];X)$, and

$$\psi(t,v(t))+(\psi|_X)^*(t,-\dot v(t))\ge\langle v(t),-\dot v(t)\rangle=-\frac12\frac{d}{dt}\|v(t)\|_H^2$$

with equality if and only if $-\dot v(t)\in\partial\psi(t,v(t))$. Write now again

$$\begin{aligned}
0=I_{\ell,L}(v) &= \int_0^T[\psi(t,v(t))+\psi^*(t,-\dot v(t))]dt+\frac12\|v(0)\|_H^2+\frac12\|v(T)\|_H^2\\
&\ge \int_0^T[\psi(t,v(t))+\psi^*(t,-\dot v(t))+\langle v(t),\dot v(t)\rangle dt\\
&\quad -\int_0^T\frac12\frac{d}{dt}\|v(t)\|_H^2dt+\frac12\|v(0)\|_H^2+\frac12\|v(T)\|_H^2\\
&= \int_0^T[\psi(t,v(t))+\psi^*(t,-\dot v(t))+\langle v(t),\dot v(t)\rangle dt+\|v(0)\|_H^2\\
&= 0.
\end{aligned}$$

It follows that $v(0) = 0$ and that $\psi(t, v(t)) + \psi^*(t, -\dot{v}(t)) = \langle v(t), -\dot{v}(t)\rangle$ for almost all $t \in [0, T]$. This means that $v(t)$ satisfies

$$\begin{cases} \dot{v}(t) + \partial\psi(t, v(t)) = 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ \qquad\qquad v(0) \ = 0 \end{cases} \tag{65}$$

and that $u(t) = v(t) + u_0$ is a weak solution for equation (62).

6 Applications to nonlinear evolution equations

Fast diffusion equations

This is the case when we have $0 < m < 1$ in equation (68). But now $(-\Delta)^{-1}u$ is not necessarily in $L^{\frac{m+1}{m}}$ when $u \in L^{m+1}(\Omega)$, hence we need to change the setting and consider the space X defined as

$$X = \{u \in L^{m+1}(\Omega);\ (-\Delta)^{-1}u \in L^{\frac{m+1}{m}}(\Omega)\}$$

equipped with the norm

$$\|u\|_X = \|u\|_{m+1} + \|(-\Delta)^{-1}u\|_{\frac{m+1}{m}}.$$

X is anchored on the Hilbert space H^{-1} obtained by completing X for the norm induced by the scalar product

$$\langle u, v\rangle = \int_\Omega u(-\Delta)^{-1}v dx = \langle u, v\rangle_{H^{-1}(\Omega)}$$

That is $X \subset H^{-1}(\Omega) \subset X^*$ is an evolution triple. Consider the functional

$$\varphi(u) = \begin{cases} \frac{1}{m+1}\int_\Omega |u|^{m+1} & \text{on} \quad L^{m+1}(\Omega) \\ +\infty & \text{on} \quad H^{-1} \setminus L^{m+1}(\Omega). \end{cases} \tag{66}$$

Clearly $X \subset \mathrm{Dom}(\varphi)$ and the conjugate of its restriction to X is:

$$(\varphi|_X)^*(v) = \begin{cases} \frac{m}{m+1}\int_\Omega |\Delta^{-1}v|^{\frac{m+1}{m}} dx. \text{ if} & \Delta^{-1}v \in L^{\frac{m+1}{m}}(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{67}$$

Theorem 4 therefore applies with $\gamma = m+1$ and $\gamma^* = \frac{m+1}{m} \geq 2$, which means that $\alpha = 2$ and we get by considering the space

$$A^2_{H,X^*} = \left\{u : [0,T] \to X^*; u(0) \in H^{-1};\ \int_0^T \|\dot{u}(t)\|^2_{X^*} dt < +\infty\right\}.$$

Corollary 3 *Assume $0 < m < 1$, then for any $u_0 \in L^{m+1}(\Omega)$ and any $f \in L^{\frac{m+1}{m}}$ such that $\Delta^{-1} f \in L^{\frac{m+1}{m}}$ the infimum of the functional*

$$\begin{aligned} I(u) = &\frac{1}{m+1}\int_0^T \int_\Omega \left(|(u(t,x)+u_0(x))|^{m+1} dx + m\Delta^{-1}(f(x) - \frac{\partial u}{\partial t}(t,x))|^{\frac{m+1}{m}} \right) dxdt \\ &+ \int_0^T \int_\Omega \left[u_0(x)(\Delta^{-1}\frac{\partial u}{\partial t})(t,x) - u(x,t)(\Delta^{-1} f)(x) \right] dxdt \\ &- T\int_\Omega u_0(x)(-\Delta)^{-1} f(x)\, dx + \frac{1}{2}\left(\|u(0)\|^2_{H^{-1}} + \|u(T)\|^2_{H^{-1}} \right) \end{aligned}$$

on the space A^2_{H,X^*} is equal to zero and is attained uniquely at an X-valued path $\tilde{u}$. Moreover, $u(t) = \tilde{u}(t) + u_0$ is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u^m + f & \text{on} \quad \Omega \times [0,T] \\ u(0,x) \ = u_0 \quad \text{on} & \Omega. \end{cases} \tag{68}$$

Note that in order to conclude that $u(t) \in X$ and not just in $\mathrm{Dom}(\varphi)$ as implied by Theorem 4, one needs to use the easy that:

$$\|\Delta^{-1} u(t)\|_{\frac{m+1}{m}} \leq \|\Delta^{-1} u(0)\|_{\frac{m+1}{m}} + \int_0^t \|\Delta^{-1}\dot{u}(s)\|_{\frac{m+1}{m}} ds.$$

More general parabolic equations

Consider an equation of the form

$$\frac{\partial u}{\partial t} - \Sigma_{j=1}^n \frac{\partial}{\partial x_j}\left(a_j(x,t)|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j} \right) + a_0(x,t)|u|^{p-2}u = f(x,t) \tag{69}$$

on $(0,T] \times I\!\!R^n$ and subject to the inititial condition

$$u(x,0) = u_0(x). \tag{70}$$

Corollary 4 *Assume $p \geq 2$, $f \in L^p([0,T]; W^{-1,p^*}(I\!\!R^n))$, $u_0 \in W^{1,p}(I\!\!R^n)$, and that each a_j $(0 \leq j \leq n)$ is a non-negative measurable function such that*

$$0 < c_0 \leq a_j(t.x) \leq c_1 < \infty \quad \text{a.e on } (0,T] \times I\!\!R^n. \tag{71}$$

Let $\varphi : [0,T] \times W^{1,p}(I\!\!R^n)$ be defined as

$$\varphi(t,u) = \frac{1}{p}\int_{R^n}\left(\Sigma_{j=1}^n a_j(x,t)|\frac{\partial u}{\partial x_j}|^p + a_0(t,x)|u|^p \right) dx$$

and let $\varphi^*(t,u)$ *be its conjugate for each* $t \in [0,T]$*. Then, the functional*

$$\begin{aligned} I(u) := & \int_0^T \left[\varphi(t, u(t)+u_0) + \varphi^*(t, f(t) - \frac{\partial u}{\partial t})\right] dt \\ & + \int_0^T \int_{R^n} \left[u_0(x)\frac{\partial u}{\partial t}(t,x) - f(t,x)(u(x,t)+u_0(x))\right] dx dt \\ & + \frac{1}{2}(\int_{R^n} |u(0,x)|^2 + |u(T,x)|^2) dx \end{aligned}$$

has infimum zero on the space

$$A = \left\{ u : [0,T] \to W^{-1,p^*}(\mathbb{R}^n); u(0) \in L^2(\mathbb{R}^n);\ \int_0^T \|\dot{u}(t)\|_{W^{-1,p^*}}^{\frac{p}{p-1}} dt < +\infty \right\}.$$

This infimum is attained uniquely at a path v *such that* $v(t) \in W^{1,p}(\mathbb{R}^n)$ *for all* $t \in [0,T]$*, and* $u(t) = v(t) + u_0$ *is a weak solution for the equation (69) and (70).*

Indeed, here the evolution triple is obvioulsy

$$X = W^{1,p} \subset L^2 \subset W^{-1,\frac{p}{p-1}} = X^*.$$

In this case, $\alpha = \frac{p}{p-1}$ and $\alpha^* = p$, in such a way that the hypothesis of Theorem 4 are readily verified as soon as $p \geq 2$. Note that condition (71) can be weakened considerably.

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