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Abstract

Anti-selfdual Lagrangians on a state space lift to path space provided one adds a suitable selfdual boundary Lagrangian. This process can be iterated by considering the path space as a new state space for the newly obtained anti-selfdual Lagrangian. We give here two applications for these remarkable permanence properties. In the first, we establish for certain convex-concave Hamiltonians \mathcal{H} on a –possibly infinite dimensional– symplectic space H^2 , the existence of a solution for the Hamiltonian system $-J\dot{u}(t) = \partial\mathcal{H}(u(t))$ that connects in a given time $T > 0$, two Lagrangian submanifolds. Another application deals with the construction of a multiparameter gradient flow for a convex potential. Our methods are based on the new variational calculus for anti-selfdual Lagrangians developed in [4], [5] and [7].

1 Introduction

Given two convex and lower semi-continuous functions (φ_1, φ_2) on \mathbb{R}^n , we consider the Hamiltonian \mathcal{H} on \mathbb{R}^{2n} defined by $\mathcal{H}(x, y) = \varphi_1(x) - \varphi_2(y)$ and we look for solutions for the Hamiltonian system $-J\dot{u}(t) = \partial\mathcal{H}(u(t))$ that connects in time $T > 0$, the Lagrangian submanifolds

$$L_1 = \{(x, y) \in \mathbb{R}^{2n}; -y \in A_1x + \partial\psi_1(x)\} \quad \text{to} \quad L_2 = \{(x, y) \in \mathbb{R}^{2n}; y \in A_2x + \partial\psi_2(x)\}.$$

where ψ_1, ψ_2 are convex lower semi-continuous functions on \mathbb{R}^n and A_1, A_2 are positive (but not necessarily self-adjoint) matrices. In other words, we are looking for a solution on $[0, T]$ for the Hamiltonian system:

$$\begin{aligned} \dot{x}(t) &\in \partial_2\mathcal{H}(x(t), y(t)) \\ -\dot{y}(t) &\in \partial_1\mathcal{H}(x(t), y(t)) \end{aligned} \tag{1}$$

with the following boundary conditions

$$-y(0) - A_1x(0) \in \partial\psi_1(x(0)) \quad \text{and} \quad y(T) - A_2x(T) \in \partial\psi_2(x(T)). \tag{2}$$

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We shall show that a solution can indeed be obtained by minimizing the following functional

$$I(x, y) = \int_0^T \Phi((x(t), y(t)) + \Phi^*(-\dot{y}(t), -\dot{x}(t))) dt + \psi_1(x(0)) + \psi_1^*(-y(0) - A_1 x(0)) + \psi_2(x(T)) + \psi_2^*(y(T) - A_2 x(T)).$$

on the space $A^2([0, T]; \mathbb{R}^{2n}) = \{u = (x, y) : [0, T] \rightarrow \mathbb{R}^{2n}; \dot{u} \in L^2_{\mathbb{R}^{2n}}\}$, where here Φ is the convex function $\Phi(x, y) = \varphi_1(x) + \varphi_2(y)$ on \mathbb{R}^{2n} and Φ^* is its Legendre transform. The equation is obtained from the fact that the infimum is actually 0, which is the main point of the exercise.

Actually, this is a particular case of a much more general result. For one, the method is infinite dimensional and \mathbb{R}^n can be replaced by any Hilbert space H and for PDE puposes, the domain can be an evolution pair $X \subset H \subset X^*$ where X is a Banach space dense in H . More importantly, the theorem is really about the existence of a path connecting in prescribed time T , two given “anti-selfdual” Lagrangian submanifolds in H^2 through an ”anti-selfdual” Lagrangian submanifold in phase space H^4 . Let us first recall the following notions from [4].

Definition 1.1 (1) A convex lower semi-continuous functional $L : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ (resp., $\ell : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$) is said to be *R-antiselfdual* (resp., *R-selfdual*) for some automorphism $R : H \rightarrow H$ if

$$L^*(p, x) = L(-Rx, -Rp) \quad (\text{resp., } \ell^*(x, p) = \ell(-Rx, Rp)) \text{ for any } (x, p) \in H \times H.$$

(2) An *R-antiselfdual manifold* M in $H \times H$ is a set of the form

$$M = \{(x, p) \in H \times H; L(x, p) + \langle Rx, p \rangle = 0\}$$

where L is an *R-antiselfdual Lagrangian* on H .

Typical examples are

$$M_{+, \psi} = \{(x, p) \in H \times H; \psi(x) + \psi^*(-p) + \langle x, p \rangle = 0\} = \{(x, p) \in H \times H; p \in -\partial\psi(x)\}.$$

and

$$M_{-, \psi} = \{(x, p) \in H \times H; \psi(x) + \psi^*(p) - \langle x, p \rangle = 0\} = \{(x, p) \in H \times H; p \in \partial\psi(x)\}.$$

where ψ is a convex lower semi-continuous function on H and where $R(x) = x$ for $M_{+, \psi}$ and $R(x) = -x$ for $M_{-, \psi}$.

Moreover, if $A : H \rightarrow H$ is a bounded skew-adjoint operator on H , then the following manifolds are also $(+I) - ASD$ (resp., $(-I) - ASD$) (See [4]).

$$M_{+, \psi, A} = \{(x, p) \in H \times H; \psi(x) + \psi^*(-Ax - p) + \langle x, p \rangle = 0\} = \{(x, p) \in H \times H; -p \in (A + \partial\psi)(x)\}.$$

and

$$M_{-, \psi, A} = \{(x, p) \in H \times H; \psi(x) + \psi^*(-Ax + p) - \langle x, p \rangle = 0\} = \{(x, p) \in H \times H; p \in (A + \partial\psi)(x)\}.$$

The condition that A is a skew-adjoint operator can be replaced by the hypothesis that it is merely positive, i.e., that $\langle Ax, x \rangle \geq 0$ for every $x \in H$. Indeed, one can decompose A into its symmetric part $A^s = \frac{1}{2}(Ax + A^*x)$ and its skew-symmetric part $A^a = \frac{1}{2}(Ax - A^*x)$. Then, the manifold

$$M_{+, \psi, A} = \{(x, p) \in H \times H; -p - Ax \in \partial\psi(x)\}$$

is equal to the $(+I)$ -ASD manifold

$$\mathcal{M}_{+,\tilde{\psi},A^a} = \{(x,p) \in H \times H; -p - A^a x \in \partial\tilde{\psi}(x)\}$$

where $\tilde{\psi}(x) = \psi(x) + \frac{1}{2}\langle Ax, x \rangle$, while the manifold

$$\mathcal{M}_{-,\psi,A} = \{(x,p) \in H \times H; p - Ax \in \partial\psi(x)\}$$

is equal to the $(-I)$ -ASD manifold

$$\mathcal{M}_{-,\tilde{\psi},A^a} = \{(x,p) \in H \times H; p - A^a x \in \partial\tilde{\psi}(x)\}$$

This will allow us –in the sequel– to reduce many of the proofs for statements concerning bounded positive operators to the case where they are skew adjoint.

Consider now a convex lower semi-continuous function Φ on $H \times H$ and let $S : H \times H \rightarrow H \times H$ be the automorphism $S(p, q) = (q, p)$, then one can easily check that the following manifold

$$\mathcal{M}_{S,\Phi} := \{((x_1, x_2), (p_1, p_2)) \in H^2 \times H^2; (-p_2, -p_1) \in (\partial_1\Phi(x_1, x_2), \partial_2\Phi(x_1, x_2))\}$$

is S -antiselfdual, and can be written as

$$\mathcal{M}_{S,\Phi} := \{((x_1, x_2), (p_1, p_2)) \in H^2 \times H^2; \Phi(x_1, x_2) + \Phi^*(-S(p_1, p_2)) + \langle (x_1, x_2), S(p_1, p_2) \rangle = 0\}$$

Our main theorem in section 2 below asserts that under very general conditions, one should be able for any time $T > 0$, to connect any given $(+I)$ -ASD submanifold in H^2 to a given $(-I)$ -ASD submanifold in H^2 through a path in phase space $(x(t), \dot{x}(t))$ that lies on a given S -ASD submanifold in H^4 .

The proof relies on the extremely useful fact that if L is an R -antiselfdual Lagrangian on state space and if ℓ is an R -selfdual boundary Lagrangian then the following Lagrangian defined by

$$\mathcal{L}(x, p) := \begin{cases} \int_0^T L(t, x(t), \dot{x}(t) + p(t)) dt + \ell(x(0), x(T)) & \text{if } \dot{x} \in L_H^2 \\ +\infty & \text{elsewhere} \end{cases} \quad (3)$$

is also an R -antiselfdual Lagrangian on path space $L_H^2[0, T]$.

In section 3, we exploit the antiselfduality of this new Lagrangian to lift it to another ASD Lagrangian on a new path space $L^2([0, S]; L_H^2([0, T])$. Applied to the basic ASD Lagrangian $L(x, p) = \varphi(x) + \varphi^*(-p)$ associated to a given convex lower semi-continuous function φ , this leads to the construction for any $x_0 \in H$, $T > 0$ and $S > 0$, of surfaces $\hat{x}(t, s)$ verifying for almost all $(s, t) \in [0, S] \times [0, T]$

$$\frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t) \in -\partial\varphi(\hat{x}(s, t))$$

$$\hat{x}(0, t) = x_0 \text{ a.e. } t \in [0, T]$$

$$\hat{x}(s, 0) = x_0 \text{ a.e. } s \in [0, S].$$

It is clear that this process can be iterated to obtain some kind of a multiparameter gradient flow for any convex potential.

2 Connecting Lagrangian submanifolds

As mentioned above, the key ingredient in what follows is the fact that if L is an R -ASD Lagrangian on a space H , then –under suitable boundedness conditions– the Lagrangian \mathcal{L} defined in (3) is then R -ASD on the path space L_H^2 . The proof of the main result in this section requires however that \mathcal{L} be only *partially R -antiselfdual* on path space (See [4]) which holds –as proved below– without additional boundedness conditions. The infinite dimensional framework required by the applications to PDE can be formulated in many settings. We describe some of them in varying levels of detail.

2.1 The Hilbertian framework

Let H be a Hilbert space with $\langle \cdot, \cdot \rangle$ as scalar product and let $[0, T]$ be a fixed real interval. For $\alpha \in (1, +\infty)$, we consider the classical space L_H^α of Bochner integrable functions from $[0, T]$ into H with norm denoted by $\|\cdot\|_\alpha$, as well as the reflexive Banach space $A_H^\alpha = \{u : [0, T] \rightarrow H; \dot{u} \in L_H^\alpha\}$ consisting of all absolutely continuous arcs $u : [0, T] \rightarrow H$, equipped with the norm

$$\|u\|_{A_H^\alpha} = \|u(0)\|_H + \left(\int_0^T \|\dot{u}\|^\alpha dt \right)^{\frac{1}{\alpha}}.$$

It is clear that A_H^α can be identified with the product space $H \times L_H^\alpha$, and that its dual $(A_H^\alpha)^*$ can also be identified with $H \times L_H^\beta$ (where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) via the formula:

$$\langle u, (a, p) \rangle_{A_H^\alpha, H \times L_H^\beta} = \langle u(0), a \rangle_H + \int_0^T \langle \dot{u}(t), p(t) \rangle dt.$$

We consider the following action functional on A_H^α :

$$I_{\ell, L}(u) = \int_0^T L(t, u(t), \dot{u}(t)) dt + \ell(u(0), u(T))$$

where

$$\ell : H \times H \rightarrow \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad L : [0, T] \times H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$$

are two appropriate Lagrangians. We shall always assume that L is measurable with respect to the σ -field in $[0, T] \times H \times H$ generated by the products of Lebesgue sets in $[0, T]$ and Borel sets in $H \times H$, and that ℓ and $L(t, \cdot, \cdot)$ are convex, lower semi-continuous valued in $\mathbb{R} \cup \{+\infty\}$ but not identically $+\infty$.

Theorem 2.1 *Assume that R is an automorphism of H , that $L(t, \cdot, \cdot) : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ is R -antiselfdual for each $t \in [0, T]$ and that ℓ is R -selfdual. Assume*

$$L(t, y, 0) \leq C(1 + \|y\|_H^\beta) \text{ for } y \in H \text{ and } a \mapsto \ell(a, 0) \text{ is bounded on the bounded sets of } H. \quad (4)$$

Then there exists $\hat{x} \in A_H^\alpha$ such that

$$I_{L, \ell}(\hat{x}) = \inf_{x \in A_H^\alpha} I_{L, \ell}(x) = 0. \quad (5)$$

For the proof, we consider the functional $J_{L,\ell}^\alpha : (A_H^\alpha)^* \cong H \times L_H^\beta \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$J_{L,\ell}^\alpha(a, y(\cdot)) := \inf_{x(\cdot) \in A_H^\alpha} \left\{ \int_0^T L(t, x(t) + y(t), \dot{x}(t)) dt + \ell(x(0) + a, x(T)) \right\}.$$

The key to the proof is the following proposition

Proposition 2.1 *Assume that R is an automorphism of H , that $L(t, \cdot, \cdot) : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ is R -antiselfdual for each $t \in [0, T]$ and that ℓ is R -selfdual. Then*

1. *The functional $J_{L,\ell}^\alpha$ is convex on $H \times L_H^\beta$ and its Legendre transform in the duality $(H \times L_H^\beta, A_H^\alpha)$ satisfies for any $x \in A_H^\alpha$,*

$$(J_{L,\ell}^\alpha)^*(x) = \int_0^T L(t, -Rx(t), -R\dot{x}(t)) dt + \ell(-Rx(0), -Rx(T)) = I_{\ell,L}(-Rx). \quad (6)$$

2. *If $J_{L,\ell}^\alpha$ is subdifferentiable at $(0, 0)$ on the space $H \times L_H^\beta$, then there exists $\hat{x} \in A_H^\alpha$ such that $I_{L,\ell}(\hat{x}) = \inf_{x \in A_H^\alpha} I_{L,\ell}(x) = 0$.*

Proof: 1) The convexity of $J_{\ell,L}$ is easy to establish. Fix now $p \in A_H^\alpha$ and write:

$$J_{\ell,L}^*(p) = \sup_{a \in H} \sup_{y \in L_H^\beta} \sup_{u \in A_H^\alpha} \left\{ \langle a, p(0) \rangle + \int_0^T [\langle y(t), \dot{p}(t) \rangle - L(t, u(t) + y(t), \dot{u})] dt - \ell(u(0) + a, u(T)) \right\}.$$

Make a substitution $u(0) + a = a' \in H$ and $u + y = y' \in L_H^\beta$, we obtain

$$J_{\ell,L}^*(p) = \sup_{a' \in H} \sup_{y' \in L_H^\beta} \sup_{u \in A_H^\alpha} \left\{ \langle a' - u(0), p(0) \rangle - \ell(a', u(T)) + \int_0^T [\langle y'(t) - u(t), \dot{p}(t) \rangle - L(t, y'(t), \dot{u}(t))] dt \right\}.$$

Since $\dot{u} \in L_H^\alpha$ and $u \in L_H^\beta$, we have $\int_0^T \langle u, \dot{p} \rangle = -\int_0^T \langle \dot{u}, p \rangle + \langle p(T), u(T) \rangle - \langle p(0), u(0) \rangle$, which implies

$$\begin{aligned} J_{\ell,L}^*(p) &= \sup_{a' \in H} \sup_{y' \in L_H^\beta} \sup_{u \in A_H^\alpha} \left\{ \langle a', p(0) \rangle + \int_0^T \{ \langle y', \dot{p} \rangle + \langle \dot{u}, p \rangle - L(t, y'(t), \dot{u}(t)) \} dt \right. \\ &\quad \left. - \langle u(T), p(T) \rangle - \ell(a', u(T)) \right\}. \end{aligned}$$

It is now convenient to identify A_H^α with $H \times L_H^\alpha$ via the correspondence: $(c, v) \in H \times L_H^\alpha \mapsto c + \int_t^T v(s) ds \in A_H^\alpha$ and $u \in A_H^\alpha \mapsto (u(T), -\dot{u}(t)) \in H \times L_H^\alpha$. We finally obtain

$$\begin{aligned} J_{\ell,L}^*(p) &= \sup_{a' \in H} \sup_{c \in H} \left\{ \langle a', p(0) \rangle + \langle -c, p(T) \rangle - \ell(a', c) \right\} \\ &\quad + \sup_{y' \in L_H^\beta} \sup_{v \in L_H^\alpha} \left\{ \int_0^T [\langle y', \dot{p} \rangle + \langle v, p \rangle - L(t, y'(t), v(t))] dt \right\} \\ &= \int_0^T L^*(t, \dot{p}(t), p(t)) dt + \ell^*(p(0), -p(T)) \\ &= \int_0^T L(t, -Rp(t), -R\dot{p}(t)) dt + \ell(-Rp(0), -Rp(T)) \\ &= I_{\ell,L}(-Rp). \end{aligned}$$

2) Since R is an automorphism, weak duality gives

$$\inf_{u \in A_H^2} I_{\ell,L}(u) \geq \sup_{A_H^2} -J_{\ell,L}^*(u) = \sup_{A_H^2} -I_{\ell,L}(-Ru) = \sup_{A_H^2} -I_{\ell,L}(u) = - \inf_{u \in A_H^2} I_{\ell,L}(u)$$

and $\inf_{u \in A_H^2} I_{\ell,L}(u)$ is therefore non negative.

On the other hand, if we pick

$\hat{x} \in \partial J_{L,\ell}^\alpha(0, 0)$, we get

$$- \inf_{A_H^2} I_{\ell,L}(u) = -J_{L,\ell}^\alpha(0) = (J_{L,\ell}^\alpha)^*(\hat{x}) = I_{\ell,L}(-R\hat{x}) \geq \inf_{A_H^2} I_{\ell,L}(u)$$

which means that $\inf_{A_H^2} I_{\ell,L}(u) \leq 0$. It follows that $\inf_{A_H^2} I_{\ell,L}(u) = I_{\ell,L}(-R\hat{x}) = 0$.

Proof of Theorem 2.1: It remains to show that the convex functional $J_{\ell,L}$ is sub-differentiable at $(0, 0)$ on the space $H \times L_H^\beta$ so as to conclude using Proposition 2.1. But the boundedness assumptions (4) on L and ℓ immediately give

$$J_{\ell,L}(a, y) \leq \int_0^T L(t, y(t), 0)dt + \ell(a, 0) \leq \int_0^T C(1 + \|y(t)\|_H^\beta)dt + \ell(a, 0)$$

which means that $J_{\ell,L}$ is bounded on the bounded sets of $H \times L_H^\beta$ and since it is convex, it is therefore subdifferentiable at $(0, 0)$.

Theorem 2.2 *Let ψ_1 and ψ_2 be two convex and lower semi-continuous functions on a Hilbert space E , let $A_1, A_2 : E \rightarrow E$ be bounded positive operators and consider the manifolds*

$$\mathcal{M}_1 := \mathcal{M}_{+, \psi_1, A_1} = \{(x_1, x_2) \in E \times E; -x_2 - A_1 x_1 \in \partial\psi_1(x_1)\}$$

and

$$\mathcal{M}_2 := \mathcal{M}_{-, \psi_2, A_2} = \{(x_1, x_2) \in E \times E; x_2 - A_2 x_1 \in \partial\psi_2(x_1)\}.$$

Let $\Phi : [0, T] \times K \times K \rightarrow \mathbb{R}$ be such that $\Phi(t, \cdot, \cdot)$ is convex and lower semi-continuous for each $t \in [0, T]$ and consider the evolving manifold

$$\mathcal{M}_3(t) := \mathcal{M}_{S, \Phi}(t) = \{((x_1, x_2), (p_1, p_2)) \in E^2 \times E^2; (-p_2, -p_1) \in (\partial_1 \Phi(t, x_1, x_2), \partial_2 \Phi(t, x_1, x_2))\}$$

Now assume that ψ_1 is coercive and bounded on bounded sets of E , ψ_2 is bounded below with 0 in its domain, while for every $t \in [0, T]$ we have

$$\Phi(t, x_1, x_2) \leq C(1 + \|x_1\|_E^\beta + \|x_2\|_E^\beta). \quad (7)$$

Then there exists $x \in A_{E \times E}^\alpha$ such that:

$$x(0) \in \mathcal{M}_1, \quad x(T) \in \mathcal{M}_2 \quad \text{and} \quad (x(t), \dot{x}(t)) \in \mathcal{M}_3(t) \quad \text{for a.e. } t \in [0, T].$$

We shall need the following easy but interesting lemma.

Lemma 2.3 *Suppose ℓ_1 (resp., ℓ_2) is an $(+I)$ -anti-selfdual Lagrangian (resp., an $(+I)$ -anti-selfdual Lagrangian on the Hilbert space $E \times E$, then the Lagrangian $\ell : E^2 \times E^2 \rightarrow \mathbb{R}$ defined by*

$$\ell((a_1, a_2), (b_1, b_2)) = \ell_1(a_1, a_2) + \ell_2(b_1, b_2)$$

is S -selfdual on $E^2 \times E^2$ where S is the automorphism on $E \times E$ defined by $S(x_1, x_2) = (x_2, x_1)$. In particular, if ψ_1 and ψ_2 are convex lower semi-continuous on E and if A_1, A_2 are bounded skew-adjoint operators on E , then the Lagrangian $\ell(\cdot, \cdot) : H \times H$ defined by

$$\ell(a, b) := \psi_1(a_1) + \psi_1^*(-A_1 a_1 - a_2) + \psi_2(b_1) + \psi_2^*(-A_2 b_1 + b_2)$$

is S -selfdual.

The proof is left to the interested reader (See also [4]).

Proof of Theorem 2.2: In view of Remark 1.2, we can assume without loss that A_1 and A_2 are skew adjoint operators. Let $H = E \times E$ and consider the S -anti-selfdual Lagrangian on $H \times H$ defined by

$$L(t, x, p) := \Phi(t, x) + \Phi^*(t, -Sx)$$

as well as the S -selfdual boundary Lagrangian $\ell : H \times H$ defined by

$$\ell(a, b) := \psi_1(a_1) + \psi_1^*(-A_1 a_1 - a_2) + \psi_2(b_1) + \psi_2^*(-A_2 b_1 + b_2).$$

Since $\Phi(t, x_1, x_2) \leq C(1 + \|x_1\|_K^\beta + \|x_2\|_K^\beta) \forall t \in [0, T]$ and since φ is coercive and bounded on bounded sets, the functional defined on $A_{E \times E}^\alpha[0, T]$ by

$$I_{L, \ell}(u) = \int_0^T (\Phi(t, u(t)) + \Phi^*(t, -S\dot{u}(t))) dt + \psi_1(u_1(0)) + \psi_1^*(-A_1 u_1(0) - u_2(0)) + \psi_2(u_1(T)) + \psi_2^*(-A_2 u_1(T) + u_2(T))$$

satisfies all the hypothesis of Theorem 2.1. Hence there exists $x(\cdot) \in A_{E \times E}^\alpha$ such that $I_{L, \ell}(x) = 0$. Therefore,

$$\begin{aligned} 0 &= \int_0^T \Phi(t, x(t)) + \Phi^*(t, -S\dot{x}(t)) dt \\ &\quad + \psi_1(x_1(0)) + \psi_1^*(-A_1 x_1(0) - x_2(0)) + \psi_2(x_1(T)) + \psi_2^*(-A_2 x_1(T) + x_2(T)) \\ &\geq \int_0^T \langle x(t), -S\dot{x}(t) \rangle dt + \psi_1(x_1(0)) + \psi_1^*(-A_1 x_1(0) - x_2(0)) + \psi_2(x_1(T)) + \psi_2^*(-A_2 x_1(T) + x_2(T)) \\ &= - \int_0^T \frac{d\langle x_1(t), x_2(t) \rangle}{dt} dt + \psi_1(x_1(0)) + \psi_1^*(-A_1 x_1(0) - x_2(0)) + \psi_2(x_1(T)) + \psi_2^*(-A_2 x_1(T) + x_2(T)) \\ &= \langle x_1(0), x_2(0) \rangle + \psi_1(x_1(0)) + \psi_1^*(-A_1 x_1(0) - x_2(0)) \\ &\quad - \langle x_1(T), x_2(T) \rangle + \psi_2(x_1(T)) + \psi_2^*(-A_2 x_1(T) + x_2(T)) \\ &\geq 0. \end{aligned}$$

This means that every inequality in this chain is an equality, hence three applications of the limiting case in Legendre-Fenchel duality gives:

$$\begin{aligned} -(\dot{x}_2(t), \dot{x}_1(t)) &\in (\partial_1 \Phi(x_1(t), x_2(t)), \partial_2 \Phi(x_1(t), x_2(t))) \text{ a.e. } t \in [0, T] \\ -A_1 x_1(0) - x_2(0) &\in \partial \psi_1(x_1(0)) \\ -A_2 x_1(T) + x_2(T) &\in \partial \psi_2(x_1(T)). \end{aligned}$$

In other words,

$x(\cdot) \in A_{E \times E}^2$ is such that $x(0) \in \mathcal{M}_1$, $x(T) \in \mathcal{M}_1$ and $-(x(t), \dot{x}(t)) \in \mathcal{M}_3(t)$ for a.e. $t \in [0, T]$

Corollary 2.4 *Let E be a Hilbert space and $\mathcal{H}(\cdot, \cdot) : E \times E \rightarrow \mathbb{R}$ be a Hamiltonian of the form $\mathcal{H}(x_1, x_2) = \varphi_1(x_1) - \varphi_2(x_2)$ where φ_1, φ_2 are convex lower semi-continuous functions satisfying*

$$\varphi_1(x_1) + \varphi_2(x_2) \leq C(1 + \|x_1\|_K^\beta + \|x_2\|_K^\beta).$$

Furthermore, let $\psi_1, \psi_2, A_1,$ and A_2 be as in Theorem 2.2. Then there exists $(x_1, x_2) \in A_{E \times E}^\alpha([0, T])$ such that for almost all $t \in [0, T]$,

$$-\dot{x}_2(t) \in \partial_1 \mathcal{H}(x_1(t), x_2(t))$$

$$\dot{x}_1(t) \in \partial_2 \mathcal{H}(x_1(t), x_2(t))$$

and satisfying the boundary conditions

$$-A_1 x_1(0) - x_2(0) \in \partial \psi_1(x_1(0))$$

$$-A_2 x_1(T) + x_2(T) \in \partial \psi_2(x_1(T))$$

Proof: This is a restatement of Theorem 2.2 for $\Phi(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$.

Corollary 2.5 *Let E be a Hilbert space and let φ be a convex lower semi-continuous function on E satisfying $\varphi(x) \leq C(1 + \|x\|_H^\beta)$. Let $\psi_1, \psi_2, A_1,$ and A_2 be as in Theorem 2.2. Then there exists $x \in A_E^\alpha([0, T])$ such that for almost all $t \in [0, T]$,*

$$\begin{aligned} \ddot{x}(t) &\in \partial \varphi(x(t)) \\ -\dot{x}(0) &\in \partial \psi_1(x(0)) + A_1 x(0) \\ \dot{x}(T) &\in \partial \psi_2(x(T)) + A_2 x(T). \end{aligned}$$

Proof: It is enough to apply the above to $\varphi_2 = \varphi$ and $\varphi_1(x_1) = \frac{1}{2}\|x_1\|_H^2$.

2.2 The non-Hilbertian case

In the infinite dimensional setting –more suitable for applications to PDEs– we need the framework of an evolution triple $X \subset H \subset X^*$, where H is a Hilbert space with $\langle \cdot, \cdot \rangle$ as scalar product, and X is a dense vector subspace of H , that is a reflexive Banach space once equipped with its norm $\|\cdot\|$. Assuming the canonical injection $X \rightarrow H$, continuous, we identify the Hilbert space H with its dual H^* and we “inject” H in X^* in such a way that $\langle h, u \rangle_{X^*, X} = \langle h, u \rangle_H$ for all $h \in H$ and all $u \in X$. This injection is continuous, one-to-one, and H is also dense in X^* . In other words, the dual X^* of X is represented as the completion of H for the dual norm $\|h\| = \sup\{\langle h, u \rangle_H; \|u\|_X \leq 1\}$.

We shall consider here evolution equations with two types of initial conditions. The first ones are those involving bounded operators in the initial conditions, or boundary Lagrangians on the ambient Hilbert space H such as Hamiltonian systems of the form:

$$\begin{aligned} \dot{p}(t) &\in \partial_2 \mathcal{H}(p(t), q(t)) \\ -\dot{q}(t) &\in \partial_1 \mathcal{H}(p(t), q(t)) \\ p(0) = -q(0) &\quad \& \quad p(T) = q(T). \end{aligned}$$

We would also like to consider more complex initial conditions:

$$\begin{aligned} \dot{p}(t) &\in \partial_2 \mathcal{H}(p(t), q(t)) \\ -\dot{q}(t) &\in \partial_1 \mathcal{H}(p(t), q(t)) \\ -A_1 p(0) - q(0) &\in \partial \psi_1(p(0)) \\ -A_2 p(T) + q(T) &\in \partial \psi_2(p(T)) \end{aligned}$$

where ψ_1, ψ_2 may only be finite on the space X .

For the first system the spaces to consider are

$$A_{H, X^*}^\alpha = \{u : [0, T] \rightarrow X^*; u(0) \in H, \dot{u} \in L_{X^*}^\alpha\}$$

equipped with the norm $\|u\|_{A_{H, X^*}^\alpha} = \left(\int_0^T \|\dot{u}(t)\|_{X^*}^\alpha dt\right)^{1/\alpha} + \|u(0)\|_H$ for $1 < \alpha < \infty$.

For the second system we will need the space

$$A_{X^*}^\alpha = \{u : [0, T] \rightarrow X^*; u \& \dot{u} \in L_{X^*}^\alpha\}$$

equipped with the norm $\|u\|_{A_{X^*}^\alpha} = \|u(0)\|_{X^*} + \left(\int_0^T \|\dot{u}\|_{X^*}^\alpha dt\right)^{\frac{1}{\alpha}}$.

Since the proof of existence for both equations is similar in spirit, we will only show the detailed proof for the second initial value problem. The other case is left to the interested reader.

It is clear that $A_{X^*}^\alpha$ is a reflexive Banach spaces that can be identified with the product space $X^* \times L_{X^*}^\alpha$, while its dual $(A_{X^*}^\alpha)^* \simeq X \times L_X^\beta$ where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The duality is then given by the formula:

$$\langle u, (a, p) \rangle_{A_{X^*}^\alpha, X \times L_X^\beta} = \langle u(0), a \rangle + \int_0^T \langle \dot{u}(t), p(t) \rangle dt$$

where $\langle \cdot, \cdot \rangle$ is the duality on X, X^* and (\cdot, \cdot) is the inner product on H .

Let $\ell : X^* \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and weak*-lower semi-continuous on $X^* \times X^*$, and let $L : [0, T] \times X^* \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be measurable with respect to the σ -field in $[0, T] \times X^* \times X^*$ generated by the products of Lebesgue sets in $[0, T]$ and Borel sets in $X^* \times X^*$, in such a way that for each $t \in [0, T]$, $L(t, \cdot, \cdot)$ is convex and weak*-lower semi-continuous on $X^* \times X^*$.

Definition 2.6 *Let $R : X^* \rightarrow X^*$ be any map. We say that L is R -anti-self-dual and ℓ is R -selfdual on X if for all $(p, s) \in X^* \times X^*$, we have*

$$(\ell|_{X \times X})^*(p, s) = \ell(-Rp, Rs) \quad \text{and} \quad (L_t|_{X \times X})^*(t, p, s) = L(t, -Rs, -Rp).$$

where $(L_t|_{X \times X})^*$ and $(\ell|_{X \times X})^*$ denote the Legendre duals of the restrictions of $L_t = L(t, \cdot, \cdot)$ and ℓ to $X \times X$.

To any such a pair, we associate the action functional on $A_{X^*}^\alpha$ by:

$$I_{\ell, L}(u) = \int_0^T L(t, u(t), \dot{u}(t)) dt + \ell(u(0), u(T))$$

as well as the corresponding ‘‘variation function’’ $J_{\ell,L}^\alpha$ defined on $(A_{X^*}^\alpha)^* = X \times L_X^\beta$ by

$$J_{\ell,L}^\alpha(a, y) = \inf \left\{ \int_0^T L(t, u + y, \dot{u}) dt + \ell(u(0) + a, u(T)) ; u \in A_{X^*}^\alpha \right\}$$

Theorem 2.7 *Suppose that $R : X^* \rightarrow X^*$ is an automorphism whose restriction to H and X is also an automorphism on these spaces. Suppose that for each $t \in [0, T]$, the Lagrangians $L(t, \cdot)$ and l are two proper convex and weak*-lower semi-continuous functions on $X^* \times X^*$ such that L is R -anti-self-dual and l is R -selfdual on X . Suppose that for some $\alpha \in (1, 2]$, $J_{\ell,L}^\alpha : X \times L_X^\beta \rightarrow \mathbb{R} \cup \{+\infty\}$ is sub-differentiable at $(0, 0)$, then there exists $v \in A_{X^*}^\alpha$ such that: $(v(t), \dot{v}(t)) \in \text{Dom}(L)$ for almost all $t \in [0, T]$, and $I_{\ell,L}(v) = \inf_{A_{X^*}^\alpha} I_{\ell,L}(u) = 0$.*

Theorem 2.7 can be proved just like Theorem 2.1 above. The only serious change occurs in the following lemma whose proof we include.

Lemma 2.8 *Under the above conditions, we have*

$$J_{\ell,L}^*(p) \geq I_{\ell,L}(-Rp) \text{ for all } p \in A_{X^*}^\alpha.$$

Proof: For $p \in A_{X^*}^\alpha$, write:

$$J_{\ell,L}^*(p) = \sup_{a \in X} \sup_{y \in L_X^\beta} \sup_{u \in A_{X^*}^\alpha} \left\{ (a, p(0)) + \int_0^T [\langle y, \dot{p} \rangle - L(t, u + y, \dot{u})] dt - \ell(u(0) + a, u(T)) \right\}.$$

Set $F \stackrel{\text{def}}{=} \{u \in A_{X^*}^\alpha ; u \in L_X^\beta\} \subseteq A_{X^*}^\alpha$. Then

$$J_{\ell,L}^*(p) \geq \sup_{a \in X} \sup_{y \in L_X^\beta} \sup_{u \in F} \left\{ (a, p(0)) + \int_0^T [-L(t, u + y, \dot{u}) + \langle y, \dot{p} \rangle] dt - \ell(u(0) + a, u(T)) \right\}$$

Make a substitution $u + y = y' \in L_X^\beta$ to obtain

$$J_{\ell,L}^*(p) \geq \sup_{a \in X} \sup_{y' \in L_X^\beta} \sup_{u \in F} \left\{ (a, p(0)) - \ell(a + u(0), u(T)) + \int_0^T [\langle y' - u, \dot{p} \rangle - L(t, y', \dot{u})] dt \right\}.$$

Set now $S = \{u : [0, T] \rightarrow X ; u \in L_X^\beta, \dot{u} \in L_X^\beta, u(0) \in X\}$. Since $\beta \geq 2 \geq \alpha$ and $\|\cdot\|_{X^*} \leq C\|\cdot\|_X$, we have $S \subseteq A_{X^*}^\alpha \cap L_X^\beta = F$ and

$$J_{\ell,L}^*(p) \geq \sup_{a \in X} \sup_{y' \in L_X^\beta} \sup_{u \in S} \left\{ (a, p(0)) + \int_0^T [\langle y', \dot{p} \rangle - \langle u, \dot{p} \rangle - L(t, y', \dot{u})] dt - \ell(a + u(0), u(T)) \right\}$$

substitute $u(0) + a = a' \in X$ and write

$$J_{\ell,L}^*(p) \geq \sup_{a' \in X} \sup_{y' \in L_X^\beta} \sup_{u \in S} \left\{ (a' - u(0), p(0)) + \int_0^T [\langle y', \dot{p} \rangle - \langle u, \dot{p} \rangle - L(t, y', \dot{u})] dt - \ell(a', u(T)) \right\}$$

Since $\dot{u} \in L_X^\beta$ and $u \in L_X^\beta$, we have $\int_0^T \langle u, \dot{p} \rangle dt = -\int_0^T \langle \dot{u}, p \rangle dt + \langle p(T), u(T) \rangle - \langle p(0), u(0) \rangle$. which implies

$$J_{\ell,L}^*(p) \geq \sup_{a' \in H} \sup_{y' \in L_X^\beta} \sup_{u \in S} \left\{ (a', p(0)) + \int_0^T \{ \langle y', \dot{p} \rangle + \langle \dot{u}, p \rangle - L(t, y', \dot{u}) \} dt - \langle u(T), p(T) \rangle - \ell(a', u(T)) \right\}.$$

It is now convenient to identify $S = \{u : [0, T] \rightarrow X; u \in L_X^\beta, \dot{u} \in L_X^\beta, u(0) \in X\}$ with $X \times L_X^\beta$ via the correspondence $(c, v) \in X \times L_X^\beta \mapsto c + \int_t^T v(s) ds \in S$ and $u \in S \mapsto (u(T), -\dot{u}(t)) \in X \times L_X^\beta$. We finally obtain

$$\begin{aligned}
J_{\ell, L}^*(p) &\geq \sup_{a' \in X} \sup_{c \in X} \{ \langle a', p(0) \rangle + \langle -c, p(T) \rangle - \ell(a', c) \} \\
&\quad + \sup_{y' \in L_X^\beta} \sup_{v \in L_X^\beta} \left\{ \int_0^T [\langle y', \dot{p} \rangle + \langle v, p \rangle - L(t, y', v)] dt \right\} \\
&= \int_0^T L^*(t, \dot{p}(t), p(t)) dt + \ell^*(p(0), -p(T)) \\
&= \int_0^T L(t, -Rp(t), -R\dot{p}(t)) dt + \ell(-Rp(0), -Rp(T)) \\
&= I_{\ell, L}(-Rp).
\end{aligned}$$

An application to infinite dimensional Hamiltonian systems: Let now Y be a reflexive Banach space that is densely embedded in a Hilbert space E . Then the product $X := Y \times Y$ is clearly a reflexive Banach space that is densely embedded in the Hilbert space $H = E \times E$. Therefore we have an evolution triple $X \subset H \subset X^*$.

We shall consider a simple but illustrative example. Let φ_1, φ_2 be convex lower semi-continuous functions on E whose domain is Y and is coercive on Y .

Define the convex function $\Phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\Phi(x) = \varphi(y_1, y_2) := \varphi_1(y_1) + \varphi_2(y_2)$.

Finally, define the linear automorphism $S : X^* \rightarrow X^*$ by $Sx^* = E(y_1^*, y_2^*) := (y_2^*, y_1^*)$. Clearly S is an automorphism whose restriction to H and X are also automorphisms.

Consider now the Lagrangians $L : X^* \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as:

$$L(x, v) = \Phi(x) + (\Phi|_X)^*(-Sv) \quad (8)$$

Now for the boundary, consider convex, lower semi-continuous functions $\psi_1, \psi_2 : Y^* \rightarrow \mathbb{R} \cup \{\infty\}$ assuming that both are coercive on Y . To these functions we associate the boundary Lagrangian $\ell : X^* \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ by:

$$\ell((a_1, a_2), (b_1, b_2)) = \psi_1(a_1) + (\psi_1|_X)^*(-a_2) + \psi_2(b_1) + (\psi_2|_X)^*(b_2) \quad (9)$$

It is then easy to show that L is S -anti-selfdual on $X^* \times X^*$ since the convex function Φ is coercive on X and that ℓ is S -selfdual.

Proposition 2.2 *Suppose that $\varphi_j(y) \leq C(\|y\|_Y^\beta + 1)$ for $j = 1, 2$, that ψ_1 is bounded on the bounded sets of Y and consider the Hamiltonian $\mathcal{H}(p, q) = \varphi_1(p) - \varphi_2(q)$. Then for any $T > 0$, there exists solutions $(\bar{p}, \bar{q}) \in A_{H, X^*}^\alpha$ for the following Hamiltonian system:*

$$\begin{aligned}
\dot{p}(t) &\in \partial_2 \mathcal{H}(p(t), q(t)) \\
-\dot{q}(t) &\in \partial_1 \mathcal{H}(p(t), q(t)) \\
-p(0) &\in \partial \psi_1(q(0)) \quad \& \quad p(T) \in \partial \psi_2(q(T)).
\end{aligned}$$

It can be obtained by minimizing the following functional on the space A_{H, X^}^α*

$$I(p, q) = \int_0^T \varphi((p(t), q(t)) + (\varphi|_X)^*(-\dot{q}(t), -\dot{p}(t)) dt + \ell((p(0), q(0)), (p(T), q(T)))$$

where φ is the convex function $\varphi(p, q) = \varphi_1(p) + \varphi_2(q)$ and ℓ is as in (9).

Proof: We wish to apply Theorem 2.7 to the S-anti-selfdual Lagrangian pair (L, ℓ) defined above, so we must check that $J_{\ell, L}^\alpha : H \times L_X^\beta \rightarrow \mathbb{R} \cup \{+\infty\}$ is sub-differentiable at $(0, 0)$. To do this we use the assumption on φ_j to obtain the inequality:

$$\begin{aligned} J_{\ell, L}^\alpha(a, v) &= \inf \left\{ \int_0^T L(t, u + v, \dot{u}) dt + \ell(u(0) + a, u(T)) ; u \in A_{X^*}^\alpha \right\} \\ &\leq \int_0^T L(t, v, 0) dt + \ell(a, 0) \\ &\leq C \left(\frac{\|a\|_H^2}{2} + \int_0^T \|v\|_X^\beta dt + 1 \right). \end{aligned}$$

Again, since $J_{\ell, L}^\alpha$ is bounded on bounded sets of $H \times L_X^\beta$, we conclude that it is subdifferentiable at $(0, 0)$. Thus there exists $\bar{x}(\cdot) = (\bar{p}(\cdot), \bar{q}(\cdot)) \in A_{X^*}^\alpha$ such that $I_{L, \ell}(\bar{x}(\cdot)) = 0$. Therefore,

$$\begin{aligned} 0 &= \int_0^T L(t, \bar{x}, \dot{\bar{x}}) dt + \ell(\bar{x}(0), \bar{x}(T)) \\ &= \int_0^T \varphi(\bar{x}) + (\varphi|_X)^*(-S\dot{\bar{x}}) dt + \ell(\bar{x}(0), \bar{x}(T)) \\ &\geq - \int_0^T \langle \bar{x}, S\dot{\bar{x}} \rangle dt + \ell(\bar{x}(0), \bar{x}(T)) \\ &= - \int_0^T \frac{d\langle \bar{p}, \bar{q} \rangle}{dt} dt + \psi_1(\bar{p}(0)) + (\psi_1|_X)^*(-\bar{q}(0)) + \psi_2(\bar{p}(T)) + (\psi_2|_X)^*(\bar{q}(T)) \\ &= \langle \bar{p}(0), \bar{q}(0) \rangle - \langle \bar{p}(T), \bar{q}(T) \rangle + \psi_1(\bar{p}(0)) + (\psi_1|_X)^*(-\bar{q}(0)) + \psi_2(\bar{p}(T)) + (\psi_2|_X)^*(\bar{q}(T)) \\ &\geq 0. \end{aligned}$$

Therefore every inequality in this chain is actually an equality. We conclude that $-S\dot{\bar{x}}(t) \in \partial\Phi(\bar{x}(t))$ for almost all $t \in [0, T]$ and that

$$\langle \bar{p}(0), \bar{q}(0) \rangle + \psi_1(\bar{p}(0)) + (\psi_1|_X)^*(-\bar{q}(0)) = -\langle \bar{p}(T), \bar{q}(T) \rangle + \psi_2(\bar{p}(T)) + (\psi_2|_X)^*(\bar{q}(T)) = 0$$

By the definition of S and Φ and Fenchel inequality, this is precisely a solution of the equation above.

3 Two-parameter gradient flows

Behind the results of the previous section is the fact that an R-antiselfdual Lagrangian on a Hilbert space H lifts to an R-antiselfdual Lagrangian on path space. So far, we only needed anti-selfduality on the elements of $A_H^2 \times \{0\}$. However, we have the following stronger stability result announced in [3] and proved in [5]. For clarity we shall restrict ourselves to ASD-Lagrangians (i.e., $R(x) = x$).

Lemma 3.1 *Let H be a Hilbert space and let $L : [0, T] \times H \times H \rightarrow \mathbb{R}$ be an anti-selfdual Lagrangian such that for every $p \in H$ and $t \in [0, T]$ the map*

$$x \mapsto L(t, x, p) \quad \text{is bounded on the bounded sets of } H. \quad (10)$$

Then for every $x_0 \in H$, the Lagrangian defined on $L_H^2([0, T]) \times L_H^2([0, T])$ by

$$\mathcal{L}(x, p) := \begin{cases} \int_0^T L(t, x(t), \frac{dx}{dt}(t) + p(t))dt + \frac{1}{2}\|x(0)\|_H^2 + 2\langle x_0, x(0) \rangle + \|x_0\|_H^2 + \frac{1}{2}\|x(T)\|_H^2 & \text{if } x(\cdot) \in A_{H'}^2 \\ \infty & \text{otherwise} \end{cases}$$

is also an ASD Lagrangian on $L_H^2([0, T]) \times L_H^2([0, T])$.

Proof: Note that this also follows from a more general result established in [8]. Indeed, since $L(t, x, p)$ is an anti-self-dual Lagrangian on H , the map

$$(x(\cdot), p(\cdot)) \mapsto \int_0^T L(t, x(t), p(t))dt$$

is an ASD Lagrangian on the path space $L_H^2([0, T]) \times L_H^2([0, T])$ (See [5]). Now, using the terminology of [8], the map $x \mapsto \frac{dx}{dt}$ (with domain $A_H^2([0, T])$) is skew-adjoint modulo the boundary operator $x \mapsto (x(0), x(T))$ on the Hilbert space $L_H^2([0, T])$. Therefore \mathcal{L} is also an ASD Lagrangian.

Setting $H' := L_H^2([0, T])$ as a state space, and since $\mathcal{L}(\cdot, \cdot) : H' \times H' \rightarrow \mathbb{R}$ is now an anti-selfdual Lagrangian on H' , we can then lift it to a new path space $L_{H'}^2([0, S])$ and obtain a new action functional

$$\mathcal{I}(x(\cdot)) := \int_0^S \mathcal{L}(x(s), \frac{dx}{ds}(s))ds + \ell'(x(0), x(S))$$

that we can minimize on $A_{H'}^2([0, S])$. Here is the main result of this section. We recall from [8] that the partial domain $\text{Dom}_1(\partial L)$ of a Lagrangian L is defined as:

$$\text{Dom}_1(\partial L) = \{x \in H; \text{There exists } p \in H \text{ such that } -(p, x) \in \partial L(x, p)\}.$$

Theorem 3.2 *Let H be a Hilbert space and $L : H \times H \rightarrow \mathbb{R}$ be an ASD Lagrangian that is uniformly convex in the first variable. If $x_0 \in \text{Dom}_1(\partial L)$, then there exists $\hat{x}(\cdot, \cdot) \in A^2([0, S]; L_H^2([0, T])$ such that $\hat{x}(s, \cdot) \in A_H^2([0, T])$ for almost all $s \in [0, S]$ and*

$$\begin{aligned} 0 &= \int_0^S \int_0^T L(\hat{x}(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t))dtds \\ &+ \int_0^S \left(\frac{1}{2}\|\hat{x}(s, 0)\|_H^2 - 2\langle \hat{x}(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2}\|\hat{x}(s, T)\|_H^2 \right) ds \\ &+ \int_0^T \left(\frac{1}{2}\|\hat{x}(0, t)\|_H^2 - 2\langle \hat{x}(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2}\|\hat{x}(S, t)\|_H^2 \right) dt. \end{aligned} \quad (11)$$

Furthermore, for almost all $(s, t) \in [0, S] \times [0, T]$, we have

$$-\frac{\partial \hat{x}}{\partial t}(s, t) - \frac{\partial \hat{x}}{\partial s}(s, t) \in \partial_1 L(\hat{x}(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t)) \quad (12)$$

$$-\hat{x}(s, t) \in \partial_2 L(\hat{x}(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t)) \quad (13)$$

$$\hat{x}(0, t) = x_0 \text{ a.e. } t \in [0, T] \quad (14)$$

$$\hat{x}(s, 0) = x_0 \text{ a.e. } s \in [0, S]. \quad (15)$$

We first note that if L satisfies the boundedness condition (10) then the conclusions of the theorem are easy to establish as shown in the following Lemma. The main difficulty of the proof is to get rid of this condition.

Lemma 3.3 *Let $L : [0, T] \times H \times H \rightarrow \mathbb{R}$ be an ASD Lagrangian on a Hilbert space H such that $L(t, \cdot, \cdot)$ is uniformly convex in the first variable for each $t \in [0, T]$ while verifying condition (10). If $x_0 \in \text{Dom}_1(\partial L)$ then there exists $\hat{x}(\cdot, \cdot) \in A^2([0, S]; L^2_H([0, T]))$ such that $\hat{x}(s, \cdot) \in A^2_H([0, T])$ for almost all $s \in [0, S]$ and satisfying properties (12)-(15) above.*

Proof:

According to Lemma 3.1, \mathcal{L} is a uniformly convex ASD Lagrangian on $H' := L^2_H([0, T])$. Since $0 \in \text{Dom}_1(\partial L)$ we have that $0 \in \text{Dom}_1(\partial \mathcal{L})$. Therefore by Theorem 4.1 of [8], we can find an $\hat{x}(\cdot) \in A^2_{H'}([0, S]) = A^2([0, S]; L^2_H([0, T]))$ such that

$$0 = \int_0^T \mathcal{L}(\hat{x}(t), \dot{\hat{x}}(t)) dt + \frac{1}{2} \|\hat{x}(0)\|_{H'}^2 + \frac{1}{2} \|\hat{x}(T)\|_{H'}^2.$$

From the definition of \mathcal{L} , we get that $\hat{x}(s, \cdot) \in A^2_H([0, T])$ for almost all $s \in [0, S]$ while satisfying (11). We therefore get the following chain of inequalities:

$$\begin{aligned} 0 &= \int_0^S \int_0^T L(t, \hat{x}(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t)) dt ds \\ &\quad + \int_0^S \left(\frac{1}{2} \|\hat{x}(s, 0)\|_H^2 - 2\langle \hat{x}(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(s, T)\|_H^2 \right) ds \\ &\quad + \int_0^T \left(\frac{1}{2} \|\hat{x}(0, t)\|_H^2 - 2\langle \hat{x}(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(S, t)\|_H^2 \right) dt \\ &\geq \int_0^S \int_0^T -\langle x(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t) \rangle dt ds \\ &\quad + \int_0^S \left(\frac{1}{2} \|\hat{x}(s, 0)\|_H^2 - 2\langle \hat{x}(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(s, T)\|_H^2 \right) ds \\ &\quad + \int_0^T \left(\frac{1}{2} \|\hat{x}(0, t)\|_H^2 - 2\langle \hat{x}(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(S, t)\|_H^2 \right) dt \\ &\geq \int_0^T \|\hat{x}(0, t) - x_0\|_H^2 dt + \int_0^S \|\hat{x}(s, 0) - x_0\|_H^2 ds \geq 0. \end{aligned}$$

This means that for almost all $(s, t) \in [0, S] \times [0, T]$

$$-\frac{\partial \hat{x}}{\partial t}(s, t) - \frac{\partial \hat{x}}{\partial s}(s, t) \in \partial_1 L(t, \hat{x}(t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t))$$

$$\hat{x}(s, t) \in \partial_2 L(t, \hat{x}(t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t))$$

$$\hat{x}(0, t) = x_0 \text{ a.e. } t \in [0, T]$$

$$\hat{x}(s, 0) = x_0 \text{ a.e. } s \in [0, S]$$

In the next proposition we do away with the assumption of boundeness of the ASD Lagrangian L that was used in Lemma 3.3. The argument we use is similar to that in [8]. We first λ -regularize the Lagrangian L then derive some uniform bounds to ensure convergence in the proper topology when λ goes to 0. To do this we need to first state some precise estimates on approximate solutions obtained using inf-convolution. Recall first from [4] that the Lagrangian

$$L_\lambda(x, p) := \inf_{z \in H} \left\{ L(z, p) + \frac{1}{2\lambda} \|x - z\|_H^2 \right\} + \frac{\lambda}{2} \|p\|_H^2$$

is anti-selfdual for each $\lambda > 0$.

Lemma 3.4 *For a given convex functional $L : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\lambda > 0$, denote for each $(p, x) \in H \times H$, by $J_\lambda(x, p)$ the minimizer of the following optimization problem:*

$$\inf_z \left\{ L(z, p) + \frac{\|x - z\|_H^2}{2\lambda} \right\}.$$

Then for each $(x, p) \in H \times H$, we have

$$\partial_1 L_\lambda(x, p) = \frac{x - J_\lambda(x, p)}{\lambda} \in \partial_1 L(J_\lambda(x, p), p). \quad (16)$$

Proof: This is left to the reader.

Lemma 3.5 *Assume $L : H \times H \rightarrow \overline{\mathbb{R}}$ is an anti-selfdual Lagrangian and let L_λ be its λ -regularization, then the following hold:*

1. *If $-(y, x) = \partial L_\lambda(x, y)$, then necessarily $-(y, J_\lambda(x, y)) \in \partial L(J_\lambda(x, y), y)$.*
2. *If $x_0 \in \text{Dom}_1(\partial L)$, then $\|y_\lambda\|_H \leq \|\hat{p}\|_H$ whenever y_λ solves $-(y_\lambda, x_0) = \partial L_\lambda(x_0, y_\lambda)$ and \hat{p} solves $-(\hat{p}, x_0) = \partial L(x_0, \hat{p})$.*

Proof: (1) If $-(y, x) = \partial L_\lambda(x, y)$ then $L_\lambda(x, y) + L_\lambda^*(-y, -x) = -2\langle x, y \rangle$ and since L is an ASD Lagrangian, we have $L_\lambda(x, y) + L_\lambda(x, y) = -2\langle x, y \rangle$, hence

$$\begin{aligned} -2\langle x, y \rangle &= L_\lambda(x, y) + L_\lambda(x, y) \\ &= 2 \left(L(J_\lambda(x, y), y) + \frac{\|x - J_\lambda(x, y)\|_H^2}{2\lambda} + \frac{\lambda\|y\|_H^2}{2} \right) \\ &= L^*(-y, -J_\lambda(x, y)) + L(J_\lambda(x, y), y) + 2 \left(\frac{\| -x + J_\lambda(x, y) \|_H^2}{2\lambda} + \frac{\lambda\|y\|_H^2}{2} \right) \\ &\geq -2\langle y, J_\lambda(x, y) \rangle + 2\langle -x + J_\lambda(x, y), y \rangle \\ &= -2\langle x, y \rangle. \end{aligned}$$

The second last inequality is deduced by applying Fenchel's inequality to the first two terms and the last two terms. The above chain of inequality shows that all inequalities are equalities. This implies, again by Fenchel's inequality that $-(y, J_\lambda(x, y)) \in \partial L(J_\lambda(x, y), y)$.

(2) If $-(y_\lambda, x_0) = \partial L_\lambda(x_0, y_\lambda)$, we get from the previous lemma that

$$-y_\lambda = \frac{x_0 - J_\lambda(x_0, y_\lambda)}{\lambda} \in \partial_1 L(J_\lambda(x_0, y_\lambda), y_\lambda),$$

and by the first part of this lemma, that

$$-(y_\lambda, J_\lambda(x_0, y_\lambda)) \in \partial L(J_\lambda(x_0, y_\lambda), y_\lambda).$$

Now since $x_0 \in \text{Dom}_1(\partial L)$, there exists \hat{p} such that $(-\hat{p}, -x_0) \in \partial L(x_0, \hat{p})$. Setting $v_\lambda = J_\lambda(x_0, y_\lambda)$, and since $-(y_\lambda, v_\lambda) \in \partial L(v_\lambda, y_\lambda)$, we get from monotonicity and by the fact that $y_\lambda = \frac{v_\lambda - x_0}{\lambda}$,

$$\begin{aligned} 0 &\leq \langle (x_0, \hat{p}) - (v_\lambda, y_\lambda), (\partial_1 L(x_0, \hat{p}), \partial_2 L(x_0, \hat{p})) - (-y_\lambda, -v_\lambda) \rangle \\ &= \langle (x_0, \hat{p}) - (v_\lambda, y_\lambda), (-\hat{p}, -x_0) - \left(\frac{x_0 - v_\lambda}{\lambda}, -v_\lambda\right) \rangle \\ &= -\frac{\|x_0 - v_\lambda\|_H^2}{\lambda} + \langle v_\lambda - x_0, \hat{p} \rangle + \langle \hat{p}, v_\lambda - x_0 \rangle - \langle y_\lambda, v_\lambda - x_0 \rangle \\ &= -2\frac{\|x_0 - v_\lambda\|_H^2}{\lambda} + 2\langle v_\lambda - x_0, \hat{p} \rangle \end{aligned}$$

which yields that $\frac{\|x_0 - v_\lambda\|_H}{\lambda} \leq \|\hat{p}\|_H$ and finally the desired bound $\|y_\lambda\| \leq \|\hat{p}\|$ for all $\lambda > 0$.

Lemma 3.6 *Let $L : H \times H$ be an anti-self-dual Lagrangian that is uniformly convex in the first variable. If $x_0 \in \text{Dom}_1(\partial L)$ and if $\hat{x}(\cdot) \in A_H^2([0, T])$ satisfies*

$$\int_0^T L(\hat{x}(t), \dot{\hat{x}}(t)) dt + \frac{1}{2} \|\hat{x}(0)\|_H^2 + \|x_0\|_H^2 + \langle \hat{x}(0), x_0 \rangle + \frac{1}{2} \|\hat{x}(T)\|_H^2 = 0,$$

then we have the estimate

$$\int_0^T \|\dot{\hat{x}}(t)\|_H^2 dt \leq T \|p_0\|_H^2, \quad (17)$$

where p_0 is the point that satisfies $-(p_0, x_0) \in \partial L(x_0, p_0)$

Proof:

By the uniqueness of the minimizer established in [8], $\hat{x}(\cdot)$ is the weak limit in $A_H^2([0, T])$ of $\{x_\lambda(\cdot) \in C^{1,1}([0, T])\}$ where $(-\dot{x}_\lambda(t), -x_\lambda(t)) \in \partial L_\lambda(x_\lambda(t), \dot{x}_\lambda(t))$, $x_\lambda(0) = x_0$.

Standard arguments using monotonicity shows that $\|\dot{x}_\lambda(t)\|_H \leq \|\dot{x}_\lambda(0)\|_H$ for all $t \in [0, T]$. Since $(-\dot{x}_\lambda(0), -x_0) \in \partial L_\lambda(x_0, \dot{x}_\lambda(0))$, Lemma 3.5 shows that $\|x_\lambda(0)\|_H \leq \|p_0\|_H$ for all $\lambda > 0$. Therefore, letting $\lambda \rightarrow 0$ and by weak lower semi-continuity of the norm we get that $\int_0^T \|\dot{\hat{x}}(t)\|_H^2 dt \leq T \|p_0\|_H^2$.

Proof of Theorem 3.2:

Apply Lemma 3.3 to L_λ we obtain an $\hat{x}_\lambda(\cdot) \in A^2([0, S]; L_H^2([0, T])$ satisfying for all $(s, t) \in [0, S] \times [0, T]$

$$\begin{aligned} -\frac{d\hat{x}_\lambda}{dt}(s, t) - \frac{d\hat{x}_\lambda}{ds}(s, t) &\in \partial_1 L_\lambda(\hat{x}_\lambda(t), \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t)) \\ -\hat{x}_\lambda(s, t) &\in \partial_2 L_\lambda(\hat{x}_\lambda(t), \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t)) \end{aligned} \quad (18)$$

$$\hat{x}_\lambda(0, t) = x_0 \quad \forall t \in [0, T]$$

$$\hat{x}_\lambda(s, 0) = x_0 \quad \forall s \in [0, S]$$

and

$$\begin{aligned} 0 &= \int_0^S \int_0^T L_\lambda(\hat{x}_\lambda(s, t), \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t)) dt ds \\ &+ \int_0^S \left(\frac{1}{2} \|\hat{x}_\lambda(s, 0)\|_H^2 - 2\langle x_\lambda(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}_\lambda(s, T)\|_H^2 \right) ds \\ &+ \int_0^T \left(\frac{1}{2} \|\hat{x}_\lambda(0, t)\|_H^2 - 2\langle x_\lambda(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}_\lambda(S, t)\|_H^2 \right) dt. \end{aligned} \quad (19)$$

Now consider the ASD Lagrangian \mathcal{L}_λ on $L_H^2([0, T])$ defined by:

$$\mathcal{L}_\lambda(x, p) := \begin{cases} \int_0^T L_\lambda(x(t), \frac{dx}{dt}(t) + p(t)) dt + \frac{1}{2} \|x(0)\|_H^2 + \frac{1}{2} \|x(T)\|_H^2 - 2\langle x_0, x(0) \rangle + \|x_0\|_H^2 & \text{if } x \in A_H^2([0, T]) \\ \infty & \text{else} \end{cases}$$

Let $\hat{\mathcal{X}}_\lambda(\cdot) : [0, S] \rightarrow L_H^2([0, T])$ be the map $s \mapsto \hat{x}_\lambda(s, \cdot) \in L_H^2([0, T])$ and denote by $\mathcal{X}_0 \in L_H^2([0, T])$ the constant map $t \mapsto x_0$. Then by (19) $\hat{\mathcal{X}}_\lambda(\cdot)$ is the arc in $A^2([0, S]; L_H^2([0, T]))$ satisfying

$$0 = \int_0^S \mathcal{L}_\lambda(\hat{\mathcal{X}}_\lambda(s), \frac{d\hat{\mathcal{X}}_\lambda}{ds}(s)) ds + \frac{\|\hat{\mathcal{X}}_\lambda(0)\|_{L_H^2([0, T])}^2}{2} + \frac{\|\hat{\mathcal{X}}_\lambda(S)\|_{L_H^2([0, T])}^2}{2} - 2\langle \mathcal{X}_0, \hat{\mathcal{X}}_\lambda(S) \rangle_{L_H^2([0, T])} + \|\mathcal{X}_0\|_{L_H^2([0, T])}^2$$

with $\mathcal{X}_0 \in \text{Dom}_1(\partial \mathcal{L}_\lambda)$. Apply Lemma 3.6 to the ASD Lagrangian \mathcal{L}_λ and the Hilbert space $L_H^2([0, T])$ we get that

$$\int_0^S \int_0^T \left\| \frac{d\hat{x}_\lambda(s, t)}{ds} \right\|_H^2 dt ds \leq S \int_0^T \|\mathcal{P}_\lambda(t)\|_H^2 dt$$

where $\mathcal{P}_\lambda \in L_H^2([0, T])$ is any arc that satisfies $(-\mathcal{P}_\lambda, -\mathcal{X}_0) \in \partial \mathcal{L}_\lambda(\mathcal{X}_0, \mathcal{P}_\lambda)$. Observe that if the point $p_\lambda \in H$ satisfies the equation $-(p_\lambda, x_0) \in \partial L_\lambda(x_0, p_\lambda)$, then we can just take \mathcal{P}_λ to be the constant arc $t \mapsto p_\lambda$. Combining this fact with Lemma 3.5, we obtain that for all $s \in [0, S]$ and all $\lambda > 0$,

$$\int_0^S \int_0^T \left\| \frac{d\hat{x}_\lambda(s, t)}{ds} \right\|_H^2 dt ds \leq ST \|p_0\|_H^2$$

In deriving the above estimates, we interpreted $\hat{x}_\lambda(s, t)$ as a map $\hat{\mathcal{X}}_\lambda(\cdot) : [0, S] \rightarrow L_H^2([0, T])$. However, we can also view it as a map from $[0, T] \rightarrow L_H^2([0, S])$ and run the above argument in this new setting. By doing this we obtain that for all $\lambda > 0$:

$$\int_0^S \int_0^T \left\| \frac{d\hat{x}_\lambda(s, t)}{ds} \right\|_H^2 dt ds + \int_0^S \int_0^T \left\| \frac{d\hat{x}_\lambda(s, t)}{dt} \right\|_H^2 dt ds \leq 2TS \|p_0\|_H^2. \quad (20)$$

Now for any $(v_1(s, t), v_2(s, t))$ satisfying equation (18) we can use monotonicity to derive the bound:

$$\frac{d}{dt} \|v_1(s, t) - v_2(s, t)\|_H^2 + \frac{d}{ds} \|v_1(s, t) - v_2(s, t)\|_H^2 \leq 0.$$

So we obtain

$$\begin{aligned} & \int_0^S \|v_1(s, t) - v_2(s, t)\|_H^2 ds + \int_0^T \|v_1(s, t) - v_2(s, t)\|_H^2 dt \\ & \leq \int_0^S \|v_1(s, 0) - v_2(s, 0)\|_H^2 ds + \int_0^T \|v_1(0, t) - v_2(0, t)\|_H^2 dt. \end{aligned}$$

Now pick $v_1(s, t) = \hat{x}_\lambda(s, t)$ and $v_2(s, t) = \hat{x}_\lambda(s + h, t)$ we get that

$$\int_0^S \left\| \frac{d}{ds} \hat{x}_\lambda(s, t) \right\|_H^2 ds + \int_0^T \left\| \frac{d}{ds} \hat{x}_\lambda(s, t) \right\|_H^2 dt \leq \int_0^S \left\| \frac{d}{ds} \hat{x}_\lambda(s, 0) \right\|_H^2 ds + \int_0^T \left\| \frac{d}{ds} \hat{x}_\lambda(0, t) \right\|_H^2 dt. \quad (21)$$

Setting $s = 0$ in equation (18) we get that for all $t \in [0, T]$

$$-\left(\frac{d\hat{x}_\lambda}{dt}(0, t) + \frac{d\hat{x}_\lambda}{ds}(0, t), x_0 \right) \in \left(\partial_1 L_\lambda(x_0, \frac{d\hat{x}_\lambda}{dt}(0, t) + \frac{d\hat{x}_\lambda}{ds}(0, t)), \partial_2 L_\lambda(x_0, \frac{d\hat{x}_\lambda}{dt}(0, t) + \frac{d\hat{x}_\lambda}{ds}(0, t)) \right)$$

Therefore by Lemma 3.5 we have that for all $t \in [0, T]$ and $\lambda > 0$,

$$\left\| \frac{d\hat{x}_\lambda}{dt}(0, t) + \frac{d\hat{x}_\lambda}{ds}(0, t) \right\|_H \leq \|p_0\|_H.$$

Observe that if we take $v_2(s, t) = \hat{x}_\lambda(s, t + h)$ we can use the same argument as above to get that for all $s \in [0, S]$,

$$\left\| \frac{d\hat{x}_\lambda}{dt}(s, 0) + \frac{d\hat{x}_\lambda}{ds}(s, 0) \right\|_H \leq \|p_0\|_H.$$

Therefore, for all $s \in [0, S]$, $t \in [0, T]$, and $\lambda > 0$:

$$\left\| \frac{d\hat{x}_\lambda}{dt}(0, t) + \frac{d\hat{x}_\lambda}{ds}(0, t) \right\|_H + \left\| \frac{d\hat{x}_\lambda}{dt}(s, 0) + \frac{d\hat{x}_\lambda}{ds}(s, 0) \right\|_H \leq 2\|p_0\|_H. \quad (22)$$

Combining (22), (21), and (20) we get that

$$\int_0^S \int_0^T \left\| \frac{d\hat{x}_\lambda}{ds}(s, t) \right\|_H^2 + \left\| \frac{d\hat{x}_\lambda}{dt}(s, t) \right\|_H^2 dt ds \leq C, \quad (23)$$

for some constant independent of λ . If we denote by $J_\lambda(x, p)$ the point that satisfies $L_\lambda(x, p) = L(J_\lambda(x, p), p) + \frac{\lambda}{2} \|p\|_H^2$ and $v_\lambda(s, t)$ to be $J_\lambda(\hat{x}_\lambda(s, t), \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t))$, then we can deduce from equation (18) that

$$-\frac{d\hat{x}_\lambda}{dt}(s, t) - \frac{d\hat{x}_\lambda}{ds}(s, t) = \frac{\hat{x}_\lambda(s, t) - v_\lambda(s, t)}{\lambda}.$$

The estimate given by equation (23) then implies

$$\lim_{\lambda \rightarrow 0} \int_0^T \int_0^S \|\hat{x}_\lambda(s, t) - v_\lambda(s, t)\|_H^2 ds dt = 0$$

Therefore, combining this with (23) we obtain the following convergence result:

$$\hat{x}_\lambda(\cdot, \cdot) \rightharpoonup \hat{x}(\cdot, \cdot) \text{ in } A^2([0, S]; L_H^2([0, T])) \quad (24)$$

$$\hat{x}_\lambda(\cdot, \cdot) \rightharpoonup \hat{x}(\cdot, \cdot) \text{ in } A^2([0, T]; L_H^2([0, S])) \quad (25)$$

$$v_\lambda(\cdot, \cdot) \rightharpoonup \hat{x}(\cdot, \cdot) \text{ in } L_H^2([0, S] \times [0, T]). \quad (26)$$

Write (19) in the form

$$\begin{aligned} 0 &= \int_0^S \int_0^T L(v_\lambda(s, t), \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t)) + \frac{\lambda}{2} \left\| \frac{d\hat{x}_\lambda}{dt}(s, t) + \frac{d\hat{x}_\lambda}{ds}(s, t) \right\|_H^2 dt ds \\ &+ \int_0^S \frac{1}{2} \|\hat{x}_\lambda(s, 0)\|_H^2 - 2\langle x_\lambda(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}_\lambda(s, T)\|_H^2 ds \\ &+ \int_0^T \frac{1}{2} \|\hat{x}_\lambda(0, t)\|_H^2 - 2\langle x_\lambda(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}_\lambda(S, t)\|_H^2 dt \end{aligned}$$

and take $\lambda \rightarrow 0$ using the convergence results in (24) in conjunction with lower-semi-continuity we get

$$\begin{aligned} 0 &\geq \int_0^S \int_0^T L(\hat{x}(s, t), \frac{\partial \hat{x}}{\partial t}(s, t) + \frac{\partial \hat{x}}{\partial s}(s, t)) dt ds \\ &+ \int_0^S \frac{1}{2} \|\hat{x}(s, 0)\|_H^2 - 2\langle \hat{x}(s, 0), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(s, T)\|_H^2 ds \\ &+ \int_0^T \frac{1}{2} \|\hat{x}(0, t)\|_H^2 - 2\langle \hat{x}(0, t), x_0 \rangle + \|x_0\|_H^2 + \frac{1}{2} \|\hat{x}(S, t)\|_H^2 dt \geq 0 \end{aligned}$$

Standard arguments then give the desired result.

Clearly, this argument can be extended to obtain N-parameter gradient flow. We state the result without proof.

Corollary 3.7 *Let $L(\cdot, \cdot) : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ be an ASD Lagrangian that is uniformly convex in the first variable and let $u_0 \in \text{Dom}_1(\partial L)$. Then for all $T_1 \geq T_2 \dots \geq T_N > 0$, there exists $u \in L_H^2(\prod_{j=0}^N [0, T_j])$ such that $\frac{\partial u}{\partial t_j} \in L_H^2(\prod_{j=0}^N [0, T_j])$ for all $j = 1, \dots, N$ and which satisfies the differential equation*

$$-\sum_{j=1}^N \frac{\partial u}{\partial t_j}(t_1, \dots, t_N) \in \partial_1 L(u(t_1, \dots, t_N), \sum_{j=1}^N \frac{\partial u}{\partial t_j}(t_1, \dots, t_N))$$

with boundary data

$u(t_1, \dots, t_N) = u_0$ if one of the $t_j = 0$

We conclude this paper with some remarks.

Remark 3.8 Let $u : [0, T] \rightarrow H$ be the 1-parameter gradient flow associated to an ASD Lagrangian L (See [8]). Namely,

$$-\frac{du}{dt}(t) \in \partial_1 L(u(t), \frac{du}{dt}(t))$$

$$u(0) = u_0$$

If we make the change of variables $v(s', t') = u(s' + t')$, then $v(\cdot, \cdot)$ obviously solves (12), with however the boundary condition $v(s', t') = u_0$ on the hyperplane $s' = -t'$. In comparison, Theorem 3.2 above yields a solution $u(\cdot, \cdot)$ for (12) with a boundary condition that is prescribed on two hyperplanes, namely $u(0, t) = u(s, 0) = u_0$ for all $(s, t) \in [0, S] \times [0, T]$.

Remark 3.9 Suppose now $u(\cdot, \cdot) : [0, \infty) \times [0, \infty) \rightarrow H$ solve (12) with initial boundary condition $u(0, t) = u(s, 0) = u_0$ for all $(s, t) \in [0, \infty) \times [0, \infty)$, and consider the change of variable

$$v(s', t') = u(s', (1 - C)s' + Ct')$$

for some $C > 0$. Then $v(s', t')$ again solves (12) on the domain

$$D = \{(s', t') \in \mathbb{R} \times \mathbb{R}; s' \geq 0, t' \geq (1 - \frac{1}{C})s'\}.$$

The boundary condition for $v(s', t')$ is

$$v(0, t') = v(s', \frac{1-C}{-C}s') = u_0 \text{ for all } t' \geq 0 \text{ and } s' \geq 0.$$

This is essentially a two-parameter ASD flow on the wedge D .

Remark 3.10 Let now $u(\cdot, \cdot, \cdot) : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow H$ be a solution for the three-parameter ASD flow.

$$-\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}(r, s, t) \in \partial_1 L(u(r, s, t), \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}(r, s, t))$$

$$u(0, s, t) = u(r, 0, t) = u(r, s, 0) = u_0$$

With the change of variable $v(r', s', t') = u(\frac{s'+r'}{2}, \frac{t'+r'}{2}, \frac{s'+t'}{2})$, $v(r', s', t')$ again solves the differential equation

$$-\frac{\partial v}{\partial r'} - \frac{\partial v}{\partial s'} - \frac{\partial v}{\partial t'} \in \partial_1 L(u, \frac{\partial v}{\partial r'} + \frac{\partial v}{\partial s'} + \frac{\partial v}{\partial t'})$$

on the domain

$$D = \{(r', s', t') \mid s' \geq -r', r' \geq -t', s' \geq -t'\}$$

with boundary conditions

$$v(r', s', t') = u_0 \text{ if } s' = -r' \text{ or } r' = -t' \text{ or } s' = -t'.$$

Looking now at (r', s') as "state" variables and t' as the time variable, we see that at any given time t' , $v(r', s', t')$ solves the equation on $\{(r', s') \mid s' \geq -r', r' \geq -t', s' \geq -t'\}$ with $v = u_0$ on the boundary of this domain. This essentially describes a simple PDE with a time evolving boundary.

References

- [1] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines équations paraboliques. Le cas indépendant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 971–974.
- [2] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines équations paraboliques. Le cas dépendant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 1197–1198.
- [3] N. Ghoussoub, *A theory of anti-selfdual Lagrangians: Dynamical case*, C.R. Acad. Sci., Paris, Ser. I 340 (2005) p. 325-330
- [4] N. Ghoussoub, *Anti-selfdual Lagrangians: Variational resolutions of non self-adjoint equations and dissipative evolutions*, (2005). Submitted.
- [5] N. Ghoussoub, *Anti-self dual Hamiltonians: Variational resolution for Navier-Stokes equations and other nonlinear evolutions*, Submitted (2005)
- [6] N. Ghoussoub, *R-antiselfduality and the variational resolution of partial differential systems*, In preparation (2005)
- [7] N. Ghoussoub, L. Tzou. *A variational principle for gradient flows*, Math. Annalen, Vol 30, 3 (2004) p. 519-549.
- [8] N. Ghoussoub, L. Tzou. *Anti-self dual Lagrangians II: Unbounded non self-adjoint operators and evolution equations*, (2005). Submitted