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A Finite Element Approximation of the Navier-Stokes-Alpha Model

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Abstract

We consider the Navier-Stokes-Alpha model as an approximation of turbulent flows under realistic, non-periodic, boundary conditions. We derive that the variational formulation of Navier-Stokes-Alpha model under non-periodic boundary conditions, and prove that it has a unique weak solution. Next we consider finite element approximation of the model. We give semi discretization of the model and prove convergence of the method.

Keywords. Navier-Stokes α model, Kelvin's circulation theorem.
AMS Subject classifications. 65N30.

1 Introduction

Both laminar and turbulent flows are described in great detail by the Navier-Stokes equations which have been known for over a century. However, when a flow is turbulent they do not provide a tractable mathematical model for them. Scientists have aimed at developing tractable mathematical models that can accurately predict properties of turbulent flows since there are very small number of solutions to these equations are known. One of the most promising model is the Navier-Stokes Alpha NS-α model of or Cammasa-Holm model of turbulent flow is introduced in [3], [4], [5], [7], [6], recently. In [8] it is proven that NS-α equations with periodic boundary conditions have a global regularity in $3D$, and there exits a subsequence of solutions to the NS-α equations converge as $\alpha \to 0$ to a weak solution of the $3D$ NSE.

The aim of this paper is to begin the development of disceretizations for the NS-α equations with a non-periodic boundary conditions. We approximate the NS-α equations by a finite element method. Here, we prove solution to weak formulation of the NS-α equations have a unique solution. We also show that the method is convergent and give an error estimate.

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In this section we give some background information and prove some steps toward the development of the NS-α equations. These were stated without the proof (see [8]).

Definition: The circulation contained within a closed contour in a body of fluid is defined as the integral around the contour of the component of the velocity vector which is locally tangent to the contour. That is, the circulation Γ is defined as

$$\Gamma = \oint \mathbf{u} \cdot dl, \tag{1.1}$$

where dl represents an element of the contour.

Definition: The vorticity ω of an element of fluid is defined as the curl of its velocity. That is,

$$\omega = \nabla \times \mathbf{u}. \tag{1.2}$$

Applying Stokes theorem to the definition of circulation we obtain:

$$\Gamma = \oint \mathbf{u} \cdot dl = \int_A (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dA, \tag{1.3}$$

where A is the area defined by the closed contour around which is calculated and $\mathbf{n}$ is the unit normal to the surface.

Invoking the definition of the vorticity vector, (1.1) becomes:

$$\Gamma = \int_A \omega \cdot \mathbf{n} dA. \tag{1.4}$$

Theorem (Kelvin's Theorem): For an inviscid fluid in which the density is constant, or in which the pressure depends on the density alone, and for which any body forces which exist are conservative, the vorticity of each fluid will be preserved.

$$\frac{D\Gamma}{Dt} = 0, \tag{1.5}$$

where $\frac{D\Gamma}{Dt}$ is the material derivative of the circulation.

Proof See [9].

Flows of a viscous incompressible fluid are described by NSE. We consider these equations in a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$ $d = 2, 3$, equipped with homogeneous boundary conditions:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p & = -\nu \Delta \mathbf{u} + \mathbf{f} \quad \text{in} \quad \Omega \times (0, T), \\ \mathbf{u} & = 0 \quad \text{on} \quad \Gamma = \partial\Omega \times (0, T), \\ \nabla \cdot \mathbf{u} & = 0 \quad \text{in} \quad \Omega \times [0, T] = Q, \\ \mathbf{u}(x, 0) & = \mathbf{u}_0 \quad \text{in} \quad \Omega \end{cases} \tag{1.6}$$

where $\mathbf{u}(x, t)$ is the velocity of the fluid and $p(x, t)$ is the corresponding pressure function, ν is the constant kinematic viscosity, and $\mathbf{f}$ and $\mathbf{u}_0$ are given.

Theorem 1.1 Let us suppose $\mathbf{u}$ and p are classical solutions of (1.6), and say $\mathbf{u} \in C^2(\bar{Q}),\ p \in C^1(\bar{Q})$. Then $\mathbf{u}$ satisfies the Kelvin's Circulation theorem:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}\cdot dx = \oint_{\gamma(\mathbf{u})} (\nu\Delta\mathbf{u} + \mathbf{f})\cdot dx$$

where $\gamma(\mathbf{u})$ is a fluid loop that moves with the Eulerian velocity $\mathbf{u}(x,t)$.

Proof 1.1 We must show that the material derivative of the circulation Γ is equal to $\oint_{\gamma(\mathbf{u})}(\nu\Delta\mathbf{u} + \mathbf{f})\cdot dx$. From (1.1)

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}\cdot dx = \frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}_j dx_j = \oint_{\gamma(\mathbf{u})} \frac{D}{Dt}(\mathbf{u}_j dx_j)$$

$\frac{D(dx_j)}{Dt}$ is material derivative of an element dx_j of the contour around which the circulation to be calculated. We can find $\frac{d(Dx_j)}{Dt}$ as follows:

$$\frac{D(dx_j)}{Dt} = d(\frac{Dx_j}{Dt}) = d(\frac{\partial x_j}{\partial t} + \mathbf{u}_k\frac{\partial x_j}{\partial x_k})$$

Since $\mathbf{u}(x,t)$ is the Eulerian fluid that is, t and the spatial coordinates x_j are independent variables then $\frac{\partial x_j}{\partial t} = 0$ and $\frac{\partial x_j}{\partial x_k} = \delta_{jk}$ which is zero unless $k = j$. Hence

$$\frac{D(dx_j)}{Dt} = d\mathbf{u}_j.$$

So the expression for the rate of change of circulation becomes:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}_j\cdot dx = \frac{D\mathbf{u}_j}{Dt}dx_j + \oint_{\gamma(\mathbf{u})} \mathbf{u}_j d\mathbf{u}_j$$

The quantity $\frac{D\mathbf{u}_j}{Dt}$ can be eliminated by using the momentum equation. That is,

$$\frac{D\mathbf{u}_j}{Dt} = \frac{\partial\mathbf{u}_j}{\partial t} + \mathbf{u}_k\frac{\partial\mathbf{u}_j}{\partial x_k} = -\frac{\partial p}{\partial x_k} + \nu\frac{\partial^2\mathbf{u}_j}{\partial^2 x_k} + f_j.$$

Then we obtain

$$\begin{aligned}\frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}_j dx_j &= -\oint_{\gamma(\mathbf{u})} \frac{\partial p}{\partial x_j} + \left[\nu\frac{\partial^2\mathbf{u}_j}{\partial^2 x_k} + f_j\right] dx_j + \oint_{\gamma(\mathbf{u})} \mathbf{u}_j d\mathbf{u}_j \\ &= \oint_{\gamma(\mathbf{u})} \frac{\partial}{\partial x_j}(-p + \frac{1}{2}\mathbf{u}_j\mathbf{u}_j) + \nu\frac{\partial^2\mathbf{u}_j}{\partial^2 x_k} + f_j\end{aligned} \quad dx_j = \oint_{\gamma(\mathbf{u})} (dq + \nu\Delta\mathbf{u} + \mathbf{f})\cdot dx$$

where the $q = -p + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$. The integral of dq will vanish since we integrate around a closed loop. Thus rate of change of circulation becomes:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})} \mathbf{u}\cdot dx = \oint_{\gamma(\mathbf{u})} [\nu\Delta\mathbf{u} + \mathbf{f}]\cdot dx. \qquad (1.7)$$

We denote

$$H_{DIV}(\Omega) = \{\mathbf{u} \in L^2(\Omega) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma\}$$

P_H is the orthogonal projector in $L^2(\Omega)$ onto $H_{DIV}(\Omega)$.
A is a positive selfadjoint operator from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$.
$O = I + \alpha^2 \Delta$ with α a constant. Note that with this choice of the operator O $\nabla \cdot \mathbf{w} = 0$.

Definition: The "spatially filtered fluid velocity $\mathbf{w}(x,t)$ " is defined by $\mathbf{w} = g{*}\mathbf{u}$ where $*$ denotes the convolution

$$\mathbf{w} \tilde{=} g * \mathbf{u} = \int_\Omega g(x-y)\mathbf{u}(y) d^3y \equiv L_g \mathbf{u}(x).$$

where g is the Green's function of operator O and $\mathbf{u}$ is the classical solution of NSE.
Definition: The "inverse" operation is denoted by

$$\mathbf{u} = O\mathbf{w} = L_g^{-1}\mathbf{w}.$$

Theorem 1.2 Let $\mathbf{u}$ be a solution of the Navier-Stokes equations with homogeneous boundary condition and $\mathbf{w}$ be the spatially filtered fluid velocity as described above. Then $\mathbf{u}$ also satisfy the Kelvin's circulation theorem even when we replace the loop $\gamma(\mathbf{u})$ by the loop $\gamma(\mathbf{w})$. That is,

$$\frac{D}{Dt} \oint_{\gamma(\mathbf{w})} \mathbf{u} \cdot dx = \oint_{\gamma(\mathbf{w})} (\nu \Delta \mathbf{u} + \mathbf{f}) \cdot dx$$

Proof 1.2

$$\begin{aligned} \frac{D}{Dt} \oint_{\gamma(\mathbf{w})} \mathbf{u} \cdot dx &= \oint_{\gamma(\mathbf{w})} \frac{D\mathbf{u}_j}{Dt} dx_j + \oint_{\gamma(\mathbf{w})} \mathbf{u}_j \frac{D}{Dt}(dx_j) \\ &= \oint_{\gamma(\mathbf{w})} \left[\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{w}_k \frac{\partial \mathbf{u}_j}{\partial x_k} \right] dx_j + \oint_{\gamma(\mathbf{w})} \mathbf{u}_j \left[d(\frac{Dx_j}{Dt}) \right] \end{aligned}$$

$\frac{D\mathbf{u}_j}{Dt}$ is material derivative of $\mathbf{u}_j$ of the contour $\gamma(\mathbf{w})$ around which the circulation to be calculated. We can find $\frac{d(Dx_j)}{Dt}$ as follows:

$$\frac{D(dx_j)}{Dt} = d(\frac{Dx_j}{Dt}) = d(\frac{\partial x_j}{\partial t} + \mathbf{w}_k \frac{\partial x_j}{\partial x_k})$$

Since $\mathbf{u}(x,t)$ is the Eulerian fluid $\frac{\partial x_j}{\partial t} = 0$ and $\frac{\partial x_j}{\partial x_k} = \delta_{jk}$ which is zero unless $k = j$. Hence

$$\frac{D(dx_j)}{Dt} = d\mathbf{w}_j.$$

We obtain:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{w})} \mathbf{u}\cdot dx = \oint_{\gamma(\mathbf{w})}\left[\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{w}_k \frac{\partial \mathbf{u}_j}{\partial x_k}\right] dx_j + \oint_{\gamma(\mathbf{w})} \mathbf{u}_j d\mathbf{w}_j$$

Since $d\mathbf{w}_j = \frac{\partial \mathbf{w}_k}{\partial x_j} dx_j$ the second integrand on the right hand side can be written as follows.

$$\oint_{\gamma(\mathbf{w})} \mathbf{u}_j d\mathbf{w}_j = \oint_{\gamma(\mathbf{w})} \mathbf{u}_j \frac{\partial \mathbf{w}_k}{\partial x_j} dx_j$$

Then we get:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{w})} \mathbf{u}\cdot dx = \oint_{\gamma(\mathbf{w})}\left[\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{w}_k \frac{\partial \mathbf{u}_j}{\partial x_k} + \mathbf{u}_j \frac{\partial \mathbf{w}_k}{\partial x_j}\right] dx_j$$

Along with the momentum equation we obtain:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{w})} \mathbf{u}\cdot dx = \oint_{\gamma(\mathbf{w})}\left[\frac{\partial \mathbf{u}_j}{\partial t} + \mathbf{w}_k \frac{\partial \mathbf{u}_j}{\partial x_k} + \mathbf{u}_k \frac{\partial \mathbf{w}_k}{\partial x_j}\right] dx_j - \left[\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 \mathbf{u}_j}{\partial^2 x_k} + \mathbf{f}_j\right]$$

for all $j, k = 1, 2, 3$.

The latter equation with $\nabla \cdot \mathbf{w} = 0$ are called NS-Alpha model (or the Cammasa-Holm model) with the homogeneous boundary conditions. They can be written as follow:

$$\begin{cases} \mathbf{u}_t + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} + \nabla p & = -\nu\Delta\mathbf{u} + \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} & = 0 \quad \text{on} \quad \Gamma = \partial\Omega \times (0.T), \\ \nabla\cdot\mathbf{w} & = 0 \quad \text{in} \quad \Omega\times[0,T], \end{cases} \tag{1.8}$$

Lemma: The operator O is positive, symmetric, and time-independent. The kinetic energy type quantity:

$$|||\mathbf{w}||| := \frac{1}{2}\int \mathbf{w}\cdot O\mathbf{w}\ d^3x$$

defines a norm.

Proof: We need to show that $|||\mathbf{w}||| := \frac{1}{2}\int_\Omega \mathbf{w}\cdot O\mathbf{w}\ d^3x$ satisfies the following properties.

$N1)$ $|||\mathbf{w}||| = (\mathbf{w}, O\mathbf{w}) \geq 0$ since O is a positive operator .

$N2)$ $0 = |||\mathbf{w}|||^2 = (\mathbf{w}, O\mathbf{w}) = \frac{1}{2}(\mathbf{w}, \mathbf{w} + \alpha^2 A\mathbf{w}) =$
$\frac{1}{2}(\mathbf{w},\mathbf{w}) + \frac{\alpha^2}{2}(\mathbf{w}, A\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|^2 + \alpha^2\|P_H\nabla\mathbf{w}\|^2 \geq \frac{C(1+\alpha^2)}{2}\|\mathbf{w}\|^2.$
The last inequality implies that $0 = |||\mathbf{w}||| \iff |\mathbf{w}| = 0 \iff \mathbf{w} = 0.$

$N3)$ $|||\mathbf{w}+\mathbf{u}|||^2 = (\mathbf{w}+\mathbf{u}, O(\mathbf{w}+\mathbf{u})) = (\mathbf{u}, O\mathbf{u}) + (\mathbf{w}, O\mathbf{u}) + (\mathbf{u}, O\mathbf{w}) + (\mathbf{w}, O\mathbf{w})$ since O is symmetric we get

$$|||\mathbf{w}+\mathbf{u}|||^2 = (\mathbf{w}+\mathbf{u}, O(\mathbf{w}+\mathbf{u})) = (\mathbf{u}, O\mathbf{u}) + 2(\mathbf{u}, O\mathbf{w}) + (\mathbf{w}, O\mathbf{w})$$

Applying Cauchy-Schawartz inequality and definition of $|||.|||$ to the right hand side of the last equation respectively gives:

$$|||\mathbf{w}+\mathbf{u}|||^2 = |||\mathbf{u}|||^2 + 2|\mathbf{u}||O\mathbf{w}| + |||\mathbf{w}|||^2 \leq |||\mathbf{u}|||^2 + 2|||\mathbf{u}||||||O\mathbf{w}||| + |||\mathbf{w}|||^2$$

So the triangle inequality follows after taking square root of both sides.
Lemma: Let $\mathbf{u}$ be a classical solution of the NS-α model. Then $\mathbf{w}$ satisfies

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{u})}\mathbf{f}\cdot\mathbf{w}\ d^3x - \nu\oint_{\gamma(\mathbf{u})}\|\nabla\mathbf{w}\|^2 + \alpha^2|P_H\Delta\mathbf{w}|^2\ d^3x.$$

Proof: By the definition of material derivative we have:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{u})}\frac{D(\mathbf{u}_j\mathbf{w}_j dx_j)}{Dt}.$$

Applying the product rule for derivatives we obtain the following:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{u})}\frac{D\mathbf{w}_j}{Dt}\mathbf{u}_j d^3x_j + \oint_{\gamma(\mathbf{u})}\mathbf{w}_j\frac{D\mathbf{u}_j}{Dt}d^3x_j + \oint_{\gamma(\mathbf{u})}\mathbf{u}_j\mathbf{w}_j\frac{D(d^3x_j)}{Dt}.$$

As in the proof of the previous theorem $\frac{D(d^3x_j)}{Dt} = d^3(\frac{Dx_j}{Dt}) = d^3\mathbf{u}_j$. Hence

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{w})}\mathbf{u}_j\frac{\partial\mathbf{w}_j}{\partial t} + \mathbf{u}_k\frac{\partial\mathbf{w}_j}{\partial x_k}\mathbf{u}_j + \mathbf{w}_j\frac{\partial\mathbf{u}_j}{\partial t} + \mathbf{w}_j\mathbf{u}_k\frac{\partial\mathbf{u}_j}{\partial x_k}d^3x_j + \oint_{\gamma(\mathbf{w})}\mathbf{w}_j\mathbf{u}_j d^3\mathbf{u}_j.$$

By the product for derivatives we have:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{u})}\frac{\partial(\mathbf{u}_j\mathbf{w}_j)}{\partial t} + \mathbf{u}_k\frac{\partial(\mathbf{u}_j\mathbf{w}_j)}{\partial x_j}d^3x_j.$$

Using the momentum equation we get:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}d^3x = \oint_{\gamma(\mathbf{u})}\left[\frac{\partial p}{\partial x_j}\mathbf{w} + \nu\frac{\partial^2(\mathbf{u}_j\mathbf{w}_j)}{\partial^2 x_k} + \mathbf{f}_j\mathbf{w}_j\right]d^3x_j$$

The first term in the last equation will vanish since we integrate over a closed loop. Applying Green's theorem to the remaining terms on the right hand side ,using the fact that $\nabla\cdot\mathbf{w}=0$, and $\mathbf{w}=0$ on the boundary as we integrate by parts, we obtain:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}\ d^3x = \oint_{\gamma(\mathbf{u})}\mathbf{f}\cdot\mathbf{w}\ d^3x - \nu\oint_{\gamma(\mathbf{u})}\nabla\mathbf{u}\cdot\nabla\mathbf{w}\ d^3x$$

Replacing $\mathbf{u}$ by $O\mathbf{w}$ gives:

$$\frac{D}{Dt}\oint_{\gamma(\mathbf{u})}\frac{1}{2}\mathbf{u}\cdot\mathbf{w}\ d^3x = \oint_{\gamma(\mathbf{u})}\mathbf{f}\cdot\mathbf{w}\ d^3x - \nu\oint_{\gamma(\mathbf{u})}\nabla O\mathbf{w}\nabla\mathbf{w}\ d^3x.$$

The result follows from replacing O by $I + \alpha^2 A$. $\square$

Using the relation $\mathbf{u} = (I + \alpha^2 A)\mathbf{w}$ in the $NS - \alpha$ equations give us the relation between pressure q and the hydrodynamic pressure p as

$$q = p - \frac{\|\mathbf{w}\|^2}{2} + \frac{\alpha^2 |P_H \nabla \mathbf{w}|^2}{2}$$

and the kinetic energy for NS-α model becomes:

$$E_\alpha = \int (\frac{\|\mathbf{w}\|^2}{2} + \frac{\alpha^2 |P_H \nabla \mathbf{w}|^2}{2}) d^3 x.$$

2 Notation and Mathematical Preliminaries

For our mathematical formulation we introduce the following spaces:

$$\begin{aligned}
&X = H_0^1(\Omega)^d \cap H^2(\Omega)^d, \\
&Y = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \quad (q, 1) = 0 \right\}, \\
&V = \{ \mathbf{v} \in X \mid \quad (\nabla \cdot \mathbf{v}, q) = 0, \quad \forall\, q \in L_0^2(\Omega) \},
\end{aligned}$$

and $H^{-2}(\Omega)^d$ is the dual space of X. Through out the paper $L^p(\Omega)^d$ denotes the lebesgue space of n-vector functions with components being to the $p-th$ power integrable functions over Ω. $H^m(\Omega)^d$ is the $m-th$ order Sobolev space of L^2-functions having generalized derivatives up to order m in $L^2(\Omega)^d$. The corresponding norms are

$$\|\mathbf{u}\|_{L^p} = \int_\Omega |\mathbf{u}|^p$$

${}^{1/p}\|\mathbf{u}\|_m = \sum_{k=0}^m \|\nabla^k \mathbf{u}\|_L^{2\,2\,1/2} where \nabla^k \mathbf{u}$ is the tensor of all $k-th$ order derivatives of $\mathbf{u}$. In the case $p = 2$ we set for convenience

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \; dx$$

$$\|\mathbf{u}\|_{L^2} = \|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{1/2}.$$

$$|\mathbf{u}|^2 = (B\mathbf{u}, \mathbf{u})$$

where B is a linear operator. $H_0^1(\Omega)^d$ denotes the closure of $C_0^\infty(\Omega)^d$ vector functions having compact support in Ω. For time dependent functions into some Banach space X we use the

$$L^p(0, T; X) = \{\mathbf{u} |\; \mathbf{u}(t) : (0, T) \to X \text{ measurable} : \int_0^T \|\mathbf{u}(s)\|_X^p \leq \infty\}$$

Finally we introduce the trilinear forms

$$\tilde{b}(\mathbf{u}; \mathbf{v}, \mathbf{w}) := 2(\mathbf{u} \times \nabla \times \mathbf{v}, \mathbf{w}), \text{and} \quad b(\mathbf{u}; \mathbf{v}, \mathbf{w}) := (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}).$$

Lemma (**Vector Identity I**) :

$$(b \cdot \nabla a) + \sum_{j=1}^{3} a_j \nabla b_j = -b \times \nabla \times a + \nabla (a \cdot b).$$

Lemma (**Vector Identity II**):

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - (\nabla \times \mathbf{u}) \times \mathbf{u}.$$

Proof:
Proof of the last two lemmas are direct calculations.

Lemma:

$$\tilde{b}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = b(\mathbf{u}; \mathbf{v}, \mathbf{w}) + b(\mathbf{w}; \mathbf{v}, \mathbf{u})$$

Proof:
It follows from the vector identities.
In the coming sections we will be frequently using the following Sobolev inequalities for $d \leq 3$.
Lemma: Let Ω be a bounded, convex domain and $\mathbf{w} \in H^1(\Omega)^d$ then the following inequalities hold

$$\|\mathbf{w}\|_{L^6} \leq C\|\mathbf{w}\|_1.$$

$$\|\mathbf{w}\|_{L^4} \leq C\|\mathbf{w}\|_{L^2}^{1/4}\|\mathbf{w}\|_1^{3/4}.$$

$$\|\mathbf{w}\|_{L^3} \leq C\|\mathbf{w}\|_{L^2}^{1/2}\|\mathbf{w}\|_1^{1/2}.$$

Proof:
See [1].
Lemma: Let Ω be a bounded, convex domain then Poincaré inequality

$$\|\mathbf{w}\|_1 \leq C\|\nabla \mathbf{w}\|, \qquad \mathbf{w} \in H_0^1(\Omega)^d \text{ holds.}$$

Proof: See [14].
Lemma: Let Ω be a bounded, convex domain then the priori estimate

$$\|\mathbf{w}\|_2 \leq C\|\Delta \mathbf{w}\|, \qquad \mathbf{w} \in H_0^1(\Omega)^d \cap H^2(\Omega)^d.$$

Proof: See [14].
The last inequality leads together with the Hölder's inequality

$$\|\mathbf{w}\|_{L^3} \leq C\|\mathbf{w}\|^{1/2}\|\nabla \mathbf{w}\|_{L^6}^{1/2}$$

to the estimates

$$\|\mathbf{w}\|_{L^3} \leq C\|\nabla \mathbf{w}\|^{1/2}\|\Delta \mathbf{w}\|^{1/2}, \qquad \mathbf{w} \in H_0^1(\Omega)^d \cap H^2(\Omega)^d,$$

$$\|\mathbf{w}\|_{L^3} \leq C\|\nabla \mathbf{w}\|^{1/2}\|\Delta \mathbf{w}\|^{1/2}, \qquad \mathbf{w} \in H_0^1(\Omega)^d.$$

Lemma: Let Ω be a bounded, convex domain then The Agmon's inequality

$$\|\mathbf{w}\|_{L^\infty} \leq C\|\mathbf{w}\|_1\|\mathbf{w}\|_2, \qquad \mathbf{w} \in H_0^1(\Omega)^d \cap H^2(\Omega)^d \text{ holds.}$$

Proof: See [14].
Lemma: The differential form of the Gronwall Inequality:
Let $g(\cdot)$ be nonnegative, absolutely continuous on $[0,T]$, which satisfies for a.e the differential inequality

$$g'(t) \leq f(t)g(t) + h(t),$$

where $f(t)$ and $h(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$g(t) \leq \exp^{\int_0^T f(s)}[g(0) + \int_0^T h(s)]$$

for all $0 \leq t \leq T$.
Proof: See [13].

3 The Continuous Problem

In the first section we have shown how to to obtain the NS-α model in detail. In this section we will obtain variational formulation of the model for finite element analysis of it. Before we find the variational formulation of the model we would like to put the model in the form of NSE. For the reason we substitute $\mathbf{u} = \mathbf{w} + \alpha^2 A\mathbf{w}$ and $\mathbf{u}_t = \mathbf{w}_t + \alpha^2 A\mathbf{w}_t$ in (1.8) this leads the following problem:
Find $(\mathbf{w}, q) \in (X, Y)$ *such that*

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{w} + \alpha^2 A\mathbf{w}) - \nu\Delta(\mathbf{w} + \alpha^2 A\mathbf{w}) + 2\mathbf{w}\cdot\nabla\mathbf{w} + \\ \qquad \alpha^2\mathbf{w}\cdot\nabla A\mathbf{w} + \alpha^2 A\mathbf{w}\cdot\nabla\mathbf{w} - \nabla q = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T), \\ \qquad \Delta\mathbf{w} = \mathbf{w} = 0 \quad \text{on} \quad \Gamma, \\ \qquad \nabla\cdot\mathbf{w} = 0 \quad \text{in} \quad \Omega\times(0,T). \end{cases} \tag{3.1}$$

Remark: The boundary condition $\Delta\mathbf{w} = 0$ is sensical since $\alpha^2\Delta\mathbf{w} = \mathbf{u} - \mathbf{w}$ and $\mathbf{u} = \mathbf{w} = 0$ on Γ.

Using the vector identities I and II we can rewrite (4.1) as

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{w} + \alpha^2 A\mathbf{w}) - \nu\Delta(\mathbf{w} + \alpha^2 A\mathbf{w}) - 2(\mathbf{w}\times\nabla)\times\mathbf{w} + \\ \qquad \alpha^2(\mathbf{w}\times\nabla)\times A\mathbf{w} + \nabla r = \mathbf{f} \text{ in } \quad \Omega\times(0,T) \\ \qquad \Delta\mathbf{w} = \mathbf{w} = 0 \text{ on } \quad \Gamma, \\ \qquad \nabla\cdot\mathbf{w} = 0 \text{ in } \quad \Omega \end{cases} \tag{3.2}$$

.
where $r = -p + |\mathbf{w}|^2 + \alpha^2(\mathbf{w}\cdot A\mathbf{w})$. In order to obtain the variational formulation of the model we multiply (3.2) by a test function in X and integrate by parts. This yields the variational formulation of (3.2) as follows :

Find $(\mathbf{w}, p) \in (X, Y)$ *satisfying*

$$\begin{cases} (\mathbf{w}_t, \phi) + \alpha^2(P_H \nabla \mathbf{w}_t \cdot \nabla \phi) + \nu(\nabla \mathbf{w} : \nabla \phi) + \nu\alpha^2(A\mathbf{w} : A\phi) - (r, \nabla \cdot \phi) \\ 2(\mathbf{w} \times \nabla \times \mathbf{w}, \phi) + \alpha^2(\mathbf{w} \times \nabla \times A\mathbf{w}, \phi) = (\mathbf{f}, \phi) \qquad \forall \phi \in X. \end{cases} \tag{3.3}$$

.

Theorem 3.1 *(***Energy Estimate***)* Let $\mathbf{f} \in L^2(0, T; L^2(\Omega)^d)$ and $\mathbf{u}_0 \in V$. Then for any $T \geq 0$ the equation (3.3) has a solution $\mathbf{w}$ in the interval $[0, T)$, and it satisfies

$$\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2 + \frac{\nu}{2} \int_0^t (\|\nabla \mathbf{w}(s)\|^2 + \alpha^2 |A\mathbf{w}(s)|^2)\, ds \leq e^{-Ct}(\|\mathbf{w}_0\|^2 + \alpha^2 |A\mathbf{w}_0|^2) + C(1 - \exp^{-Ct})$$

where

$$C = 2\min\{\frac{\|\mathbf{f}\|^2}{2\nu}, \frac{\|\mathbf{f}\|^2_{-2}}{\nu\alpha^2}\}.$$

Proof 3.1 Existence of solution $\mathbf{w}$ of (3.3) with the homogeneous boundary condition can be obtain following exactly same analysis as in Theorem 3 in [8]. In order to prove the inequality we set $\phi = \mathbf{w}$ in (3.3). This gives

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2) + \nu(\|\nabla \mathbf{w}\|^2 + \alpha^2 |A\mathbf{w}|^2) + \tilde{b}(\mathbf{w}; \mathbf{w}, \mathbf{w}) + \tilde{b}(\mathbf{w}; A\mathbf{w}, \mathbf{w}) = (\mathbf{f}, \mathbf{w})$$

$$\begin{aligned} \frac{d}{dt}(\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2) + \nu(\|\nabla \mathbf{w}\|^2 + \alpha^2 \|A\mathbf{w}\|^2) &\leq |\mathbf{f}||\mathbf{w}| \\ &\leq \frac{\|\mathbf{f}\|^2}{2\nu} + \frac{\nu}{2}\|\mathbf{w}\|^2 \\ &\leq \frac{\|\mathbf{f}\|^2}{2\nu} + \frac{c\nu}{2}\|\nabla \mathbf{w}\|^2, \end{aligned}$$

or

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2) + \nu(\|\nabla \mathbf{w}\|^2 + \alpha^2 |A\mathbf{w}|^2) \leq \|\mathbf{f}\|_{-2}\|\mathbf{w}\|_2.$$

By Young's inequality

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2) + \nu(\|\nabla \mathbf{w}\|^2 + \alpha^2 |A\mathbf{w}|^2) \leq \frac{\|\mathbf{f}\|^2_{-2}}{\nu\alpha^2} + \frac{\nu\alpha^2}{2}\|\mathbf{w}\|^2_2.$$

Via the priori estimate, we have

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \alpha^2 |P_H \nabla \mathbf{w}|^2) + \frac{\nu}{2}(|\nabla \mathbf{w}|^2 + \alpha^2 |A\mathbf{w}|^2) \leq C$$

where

$$C = 2\min\{\frac{\|\mathbf{f}\|^2}{2\nu}, \frac{\|\mathbf{f}\|^2_{-2}}{\nu\alpha^2}\}$$

Multiplying both sides of the last inequality by $\exp^{Ct}$ and integrating from 0 to t yields the following inequality

$$\|\mathbf{w}\|^2+\alpha^2|P_H\nabla\mathbf{w}|^2+\frac{\nu}{2}\int_0^t(\|\nabla\mathbf{w}(s)\|^2+\alpha^2|A\mathbf{w}(s)|^2)\,ds \le \exp^{-Ct}(\|\mathbf{w}_0\|^2+\alpha^2|A\mathbf{w}_0|^2)+C(1-e^{-Ct}).\Box$$

Taking the maximum over $0 \le t \le T$ of the latter inequality implies that $\mathbf{w} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Theorem 3.2 *(***Uniqueness***)* Let $\mathbf{f} \in L^2(0,T;L^2(\Omega)^d)$ and $\mathbf{u}_0 \in V$. Then for any $T \ge 0$ the equation (3.3) has a unique solution $\mathbf{w}(t)$ in the interval $[0,T)$.

Proof 3.2 Let $\mathbf{w}$ and $\mathbf{v}$ be two solutions of (3.3) in V then

$$(\mathbf{w}_t,\phi)+\alpha^2(P_H\nabla\mathbf{w}_t:\nabla\phi)+\nu(\nabla\mathbf{w}:\nabla\phi)+\nu\alpha^2(A\mathbf{w}:A\phi)- \\ \tilde{b}(\mathbf{w};\mathbf{w},\phi)+\tilde{b}(\mathbf{w};A\mathbf{w},\phi)=(\mathbf{f},\phi) \qquad \forall\phi\in V. \tag{3.4}$$

$$(\mathbf{v}_t,\phi)+\alpha^2(P_H\nabla\mathbf{v}_t:\nabla\phi)+\nu(\nabla\mathbf{v}:\nabla\phi)+\nu\alpha^2(A\mathbf{v}:A\phi)- \\ \tilde{b}(\mathbf{v};\mathbf{v},\phi)+\tilde{b}(\mathbf{v};A\mathbf{v},\phi)=(\mathbf{f},\phi) \qquad \forall\phi\in V. \tag{3.5}$$

Subtracting (3.5) from (3.4) gives:

$$(\mathbf{w}_t-\mathbf{v}_t,\phi)+\alpha^2(P_H\nabla(\mathbf{w}_t-\mathbf{v}_t):\nabla\phi)+\nu(\nabla(\mathbf{w}-\mathbf{v}):\nabla\phi) \\ +\nu\alpha^2(A(\mathbf{w}-\mathbf{v}):A\phi)+\tilde{b}(\mathbf{w};\mathbf{w},\phi)- \\ \tilde{b}(\mathbf{v};\mathbf{v},\phi)+\tilde{b}(\mathbf{w};A\mathbf{w},\phi)-\tilde{b}(\mathbf{v};A\mathbf{v},\phi)=0 \quad \forall\phi\in V. \tag{3.6}$$

Let $\Phi=\mathbf{w}-\mathbf{v}$ and set $\phi=\Phi$ in (3.6) we obtain

$$\frac{d}{dt}(\|\Phi\|^2+\alpha^2|P_H\nabla\Phi|^2)+\nu(\|\nabla\Phi\|^2+\alpha^2|A\Phi|^2)-\tilde{b}(\mathbf{w};\Phi,\Phi)+\tilde{b}(\mathbf{w};A\Phi,\Phi)=0$$

$$\frac{d}{dt}(|\Phi|^2+\alpha^2|\nabla\Phi|^2)+\nu(|\nabla\Phi|^2+\alpha^2|A\Phi|^2)=\tilde{b}(\mathbf{w};\Phi,\Phi)-\tilde{b}(\mathbf{w};A\Phi,\Phi)$$

Using Holder's inequality we get

$$\le C\|\mathbf{w}\|_{L^3}\|P_H\nabla\Phi\|_{L^2}\|\Phi\|_{L^6}+C\|\mathbf{w}\|_{L^6}\|A\Phi\|_{L^2}\|\Phi\|_{L^3}$$

By Sobolev Embedding theorem we have

$$\le C(\|\mathbf{w}\|_{L^2}^{1/2}\|\mathbf{w}\|_1^{1/2}\|P_H\nabla\Phi\|_{L^2}\|\Phi\|_1+\|\mathbf{w}\|_{L^6}\|A\Phi\|_{L^2}\|\Phi\|_{L^2}^{1/2}\|\Phi\|_1^{1/2})$$

Using Young's inequality we obtain

$$\le \frac{C}{\nu}(\|\mathbf{w}\|_{L^2}\|\mathbf{w}\|_1\|\Phi\|_1^2+\|\mathbf{w}\|_{L^6}^2\|\Phi\|_{L^2}\|\Phi\|_1)+\frac{\nu}{2}(\alpha^2|P_H\nabla\Phi|^2+\alpha^2|A\Phi|^2)$$

$$\leq \frac{C(\alpha)\|\mathbf{w}\|_1^2}{\nu}\left(\|\Phi\|^2+\alpha^2|P_H\nabla\Phi|^2\right)+\frac{\nu}{2}(\|\nabla\Phi\|^2+\alpha^2|A\Phi|^2)$$

$$\frac{d}{dt}(\|\Phi\|^2+\alpha^2|P_H\nabla\Phi|^2)+\frac{\nu}{2}(\|\nabla\Phi\|^2+\alpha^2|A\Phi|^2)\leq \frac{C(\alpha)\|\mathbf{w}\|_1^2}{\nu}\left(\|\Phi\|^2+\alpha^2|P_H\nabla\Phi|^2\right)$$

applying the Gronwall's inequality gives :

$$(|\Phi(t)|^2+\alpha^2|\nabla\Phi(t)|^2)\leq(\|\Phi_0\|^2+\alpha^2|P_H\nabla\Phi_0|^2)\exp^{(\frac{C\alpha}{\nu}\int_0^t\|\mathbf{w}(s)\|_1^2\ ds)}.$$

□

4 Finite Element Method

Let $\mathcal{T}^h$ be the finite element rectangulations of the polyhedral domain Ω which satisfies the usual regularity conditions (see e.g. [2]) that is for mesh size h tending to zero namely that each $K\in\Omega_h$ contains a d-ball of diameter kh and is contained in a d-ball of diameter $k^{-1}h$.

Our purpose now is to construct an approximation of X by piecewise polynomial functions of degree $\leq k$ and to obtain an internal approximation of V in the following sense:
Since $X=H_0^1(\Omega)^d\cap H^2(\Omega)^d$ the discrete space $X^h\subset H_0^1(\Omega)^d\cap H^2(\Omega)^d$ requires the use of finite element functions that are continuously differentiable over Ω, and we impose boundary conditions by setting all degrees of freedom at the boundary nodes to be zero. Thus we let X^h be the finite element space associated with Bogner-Fox-Schmit rectangle. Then by the Theorem 2.2.15 in [2] $X^h\subset C^1(\Omega)\cap H^2(\Omega)$ holds. Also let $Y^h\subset Y$.

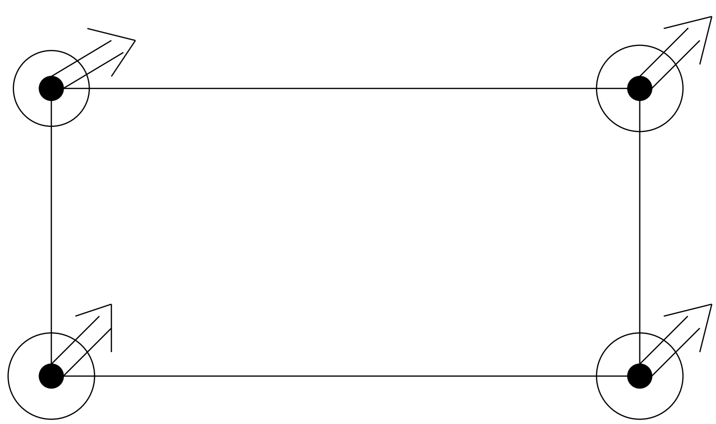

Figure 1: Velocity Space.

In the element $P_K=Q_3$; $\dim P_K=16$ (in 2 d) and

$$\sum_K=\{p(a_i),\partial_1 p(a_i),\partial_2 p(a_i),\partial_{12}p(a_i),1\leq i\leq 4\}.$$

For pressure space we consider P_2^{disc}. Q_3/P_2^{disc} pair form a stable pair (See [2]).

We assume the spaces X^h and Y^h satisfy the following properties:

I. There is a constant $\tilde{\beta} > 0$ independent of h for which:

$$\inf_{0\neq p^h\in Y^h} \sup_{0\neq \mathbf{u}^h\in X_0^h} \frac{\int_\Omega p^h \mathrm{div}\mathbf{u}^h\, dx}{\|p^h\|_{0,\Omega}\|\mathbf{u}^h\|_{1,\Omega}} \geq \tilde{\beta}, \tag{4.1}$$

II.

$$\inf_{p^h\in Y^h} \|p - p^h\|_{0,\Omega} \leq ch\|p\|_{1,\Omega}, \quad \forall\, p \in H^1(\Omega),$$

III. There exists a continuous linear operator $\Pi^h : H^1(\Omega)^d \to X^h$ for which:

$$\Pi^h(H_0^1(\Omega)^d) \subset X_0^h,$$

$$\|\mathbf{u} - \Pi^h\mathbf{u}\|_{s,\Omega} \leq ch^{t-s}\|\mathbf{u}\|_{t,\Omega}, \quad \forall\, \mathbf{u} \in H^t(\Omega) \quad \text{with } s = 0,1 \text{ and } t = 1,2,$$

$$\|\mathbf{u} - \Pi^h\mathbf{u}\|_{0,\Gamma} \leq ch^{1/2}\|\mathbf{u}\|_{1,\Omega},$$

where $\|.\|_{0,\Gamma} = (\sum_{j=1}^k \|.\|_{0,\Gamma_j})^{1/2}$. Assumption I balances the influence of the constraint $\mathrm{div}\mathbf{u} = 0$ and also implies that the space:

$$V^h = \left\{\mathbf{v}^h \in X^h \mid \quad \langle q^h, \nabla\cdot \mathbf{v}^h\rangle = 0, \quad \forall\, q^h \in Y^h\right\}$$

is also not empty.

5 Semi-Discretization in Space

Note that similar inequalities as stated in section three even hold for the functions in the discrete space V^h, for $n \leq 3$ especially for the conforming case.

Let V^h be a finite-dimensional subspace of V with basis $\{\phi_1, \phi_2, ..., \phi_M\}$. Replacing V by the finite-dimensional subspace V^h we get the following semi-discrete analogue of (3.3) as:

$$\begin{cases} (\mathbf{w}_t^h, \mathbf{v}) + \alpha^2(P_H\nabla\mathbf{w}_t^h, \nabla\mathbf{v}) + \nu(\nabla\mathbf{w}^h, \nabla\mathbf{v}) + \nu\alpha^2(A\mathbf{w}^h, A\mathbf{v}) \\ -\tilde{b}(\mathbf{w}^h; \mathbf{w}^h, \mathbf{v}) - \alpha^2\tilde{b}(\mathbf{w}^h; A\mathbf{w}^h, \mathbf{v}) - (r^h, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\mathbf{v} \in V^h,\ t \in [0,T) \\ (\mathbf{w}^h(0), \mathbf{v}) = (\mathbf{w}_0^h, \mathbf{v}) \qquad \forall\mathbf{v} \in V^h \end{cases} \tag{5.1}$$

where $\mathbf{w}_0^h \in V^h$ is an approximation to the initial data $\mathbf{w}_0 \in V$ satisfying uniformly for $h \to 0$:

$$|\mathbf{w}_0 - \mathbf{w}_0^h| \leq Ch^{k+1}\|\mathbf{w}_0\|_{k+1}, \quad k = 1,2,.. \qquad \forall\mathbf{w}_0 \in H_0^k(\Omega)^d \cap H^{k+2}(\Omega)^d \tag{5.2}$$

We can rewrite (5.1) using the representation

$$\mathbf{w}^h(t,x) = \sum_{i=1}^M c_i(t)\phi_i(x), \quad t \in [0,T]$$

with the time dependent coefficients $c_i(t) \in \mathbb{R}$, and setting $\mathbf{v} = \phi_j$, $j = 1, 2, ...M$ in (5.1) we obtain:

$$\sum_{i=1}^{M} c_i'(t) \left[(\phi_i^h, \phi_j^h) + \alpha^2 (P_H \nabla \phi_i^h, \nabla \phi_j^h)\right] + \sum_{i=1}^{M} c_i(t) \left[\nu(\nabla \phi_i^h, \nabla \phi_j^h) + \nu\alpha^2 (A\phi_i^h, A\phi_j^h)\right] +$$

$$\sum_{l,i=1}^{M} c_i(t) c_l(t) \left[\tilde{b}(\phi_i^h; \phi_i^h, \phi_j^h) + \alpha^2 \tilde{b}(\phi_i^h; A\phi_i^h, \phi_j^h)\right] = (\mathbf{f}(t), \phi_j^h) \tag{5.3}$$

$$\sum_{i=1}^{M} c_i(0)(\phi_i^h, \phi_j^h) = (\mathbf{w}_0, \phi_j^h), \ \ j = 1, 2, ...M.$$

These equations form a nonlinear differential system for the functions $(c_1(t), c_2(t), ..., c_M(t))$. Inverting the nonsingular matrix with elements

$$\left[(\phi_i^h, \phi_j^h) + \alpha^2 (P_H \nabla \phi_i^h, \nabla \phi_j^h)\right], \quad i, j = 1, ..., M.$$

We can write the differential equations in the usual form as:

$$c_i'(t) + \sum_{i=1}^{M} a_{ji} c_j(t) + \sum_{k,i=1}^{M} a_{jik} c_i(t) c_k(t) = \sum_{j=1}^{M} b_{ji} (\mathbf{f}(t), \phi_j^h), \quad \forall t \in [0, T) \tag{5.4}$$

where $a_{ji}, a_{jik}, b_{ji} \in \mathbb{R}$.

The last condition in (5.3) is equivalent to $c_i(0)$ is the $i - th$ component of $\mathbf{w}_0$. The nonlinear differential system (5.4) with the initial condition has a local solution on some interval $[0, t]$. If $t \leq T$, we will prove that this doesn't happen and therefore $t = T$.

Stability of (5.1) can be obtained taking $\mathbf{v} = \mathbf{w}^h(t)$ in (5.1)

$$\frac{d}{dt} \left(\|\mathbf{w}^h\|^2 + \alpha^2 |P_H \nabla \mathbf{w}^h|^2\right) + \nu(\|\nabla \mathbf{w}^h\|^2 + \alpha^2 |A\mathbf{w}^h|^2) \leq \|\mathbf{f}\|_{-2} \|\mathbf{w}^h\|_2$$

Similar to proof of the energy estimate and also recalling

$$(\mathbf{w}^h(0), \mathbf{w}_0) = (\mathbf{w}_0^h, \mathbf{w}_0)$$

we get

$$\|\mathbf{w}^h\|^2 + \alpha^2 |P_H \nabla \mathbf{w}^h|^2 + \int_0^t (|\mathbf{w}^h|^2 + \alpha^2 |A\mathbf{w}^h|^2) \, ds \leq e^{-ct}(\|\mathbf{w}_0^h\|^2 + \alpha^2 |A\mathbf{w}_0^h|^2) + c(1 - \exp^{-ct}).$$

This guarantees the existence of discrete solution of equations (5.1) which are $L^\infty(0, T; V^h) \cap L^2(0, T; V^h)$ for $T > 0$. To guarantee the existence and uniqueness of the pressure r^h we assume our discrete spaces satisfy the inf-sup condition:

There is a constant $\beta > 0$ independent of h for which:

$$\inf_{0 \neq p^h \in Y^h} \sup_{0 \neq \mathbf{w}^h \in X_0^h} \frac{(p^h \cdot \nabla \mathbf{u}^h)}{\|p^h\|_1 \|\mathbf{w}^h\|_{-1}} \geq \beta \tag{5.5}$$

, and

$$|\mathbf{f}|_2 := \sup_{v^h \in V^h} \frac{(\mathbf{f}, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h)\|_2}.$$

For the approximate solution $\mathbf{w}^h$ we have the following convergent result.

Theorem: Assume that the finite element spaces V^h are $k-th$ order approximation of the space V in the sense described above. Further, assume that $\mathbf{w}_0 \in V$, $\mathbf{f} \in L^\infty(0,T;L^2(\Omega))$, and let $\mathbf{w} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $\mathbf{w}^h \in L^\infty(0,T;V^h) \cap L^2(0,T;V^h)$ for some $T \neq 0$ be the corresponding unique solutions of equations (3.3) and (5.1), respectively then the error function $e = \mathbf{w} - \mathbf{w}^h$ satisfies the estimate

$$\begin{aligned}
&\|\mathbf{w} - \mathbf{w}^h\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha^2 |P_H \nabla(\mathbf{w} - \mathbf{w}^h)|^2_{L^\infty(0,T;L^2(\Omega))} \leq \\
&\ Ch^{2(k+1)} \left\{ \|\mathbf{w}\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} + C_1 \left(\|\mathbf{w}_0\|^2 + 1/\nu \|\mathbf{w}_t - \mathbf{v}^h{}_t\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} \right) \right\} + \\
&\ Ch^{2k} \left\{ \alpha^2 \|\nabla \mathbf{w}\|^2_{L^\infty(0,T;H^{k+1}(\Omega))} + C_1 \alpha^2 \|\nabla \mathbf{w}_0\|^2 + 1/\nu \|p\|^2_{L^2(0,T;H^k(\Omega))} \right\} + \\
&\ Ch^{2k}(1 + \nu\alpha^2) \|\Delta \mathbf{w}\|^2_{L^\infty(0,T;H^k(\Omega))} + C_1 \left(\|\mathbf{w}_0 - \mathbf{w}_0^h\|^2 + \alpha^2 |P_H \nabla(\mathbf{w}_0 - \mathbf{w}_0^h)|^2 \right)
\end{aligned}$$

where $C_1 = e^{(-C(\alpha)/\nu\alpha^2)} \|\mathbf{w}\|_{L^2(0,T;H^1(\Omega))}$.
Proof: Suppose V^h uses piecewise polynomials of degree $\leq k$, by approximation assumption $\|\nabla(\mathbf{w} - \mathbf{v}^h)\| \leq Ch^k \|\mathbf{w}\|_{k+1}$.
Since $\mathbf{w}$ is in $H^2(\Omega)$ so it is optimal

$$\|\nabla(\mathbf{w} - \mathbf{v}^h)\| \leq Ch^{k+1} \|\mathbf{w}\|_{k+1}.$$

Subtracting (5.1) from (3.3) and defining $\eta := \mathbf{w} - \mathbf{v}^h$ and $\phi^h := \mathbf{w}^h - \mathbf{v}^h$ then we obtain:

$$\begin{aligned}
&(\eta_t - \phi_t^h, \mathbf{v}) + \alpha^2 (P_H \nabla(\eta_t - \phi_t^h), \nabla \mathbf{v}) + \nu \left[(\nabla(\eta - \phi^h), \nabla \mathbf{v}) + \alpha^2 (A(\eta - \phi^h), A\mathbf{v}) \right] \\
&\quad - \tilde{b}(\mathbf{w}; \eta - \phi^h, \mathbf{v}) - \tilde{b}(\mathbf{w}; A(\eta - \phi^h), \mathbf{v}) - (p - q^h), \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in X^h.
\end{aligned} \tag{5.6}$$

Let elliptic projection $P_E : X \to V^h$ be defined by

$$(\nabla(\mathbf{w} - P_E \mathbf{w}), \nabla \mathbf{v}) = 0 \quad \forall \mathbf{v} \in X^h.$$

Note that the projection operator commute with time differentiation that is $P_E \mathbf{w}_t = (P_E \mathbf{w})_t$. Hence the following equation also holds

$$(P_H \nabla(\mathbf{w}_t - P_E \mathbf{w}_t), \nabla \mathbf{v}) = 0 \quad \forall \mathbf{v} \in V^h.$$

Then the last equation can be rewritten as :

$$
\begin{aligned}
(\phi^h_t, \mathbf{v}) + \alpha^2(P_H\nabla\phi^h_t, \nabla\mathbf{v}) + \nu(\nabla\phi^h, \nabla\mathbf{v}) + \alpha^2(A\phi^h, A\mathbf{v}) = \\
(\eta_t, \mathbf{v}) + \alpha^2(A\eta, A\mathbf{v}) - \tilde{b}(\mathbf{w}; \eta, \mathbf{v}) + \tilde{b}(\mathbf{w}; \phi^h, \mathbf{v}) + \\
\tilde{b}(\mathbf{w}; A\phi^h, \mathbf{v}) + \tilde{b}(\mathbf{w}; A\eta, \mathbf{v}) + (p - q^h, \nabla\cdot\mathbf{v}) \quad \forall \mathbf{v} \in X^h.
\end{aligned}
$$

Setting $\mathbf{v} = \phi^h$ and rearranging terms in the latter equation we obtain:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\phi^h\|^2 + \alpha^2|P_H\nabla\phi^h|^2) + \nu\left(\|\nabla\phi^h\|^2 + \alpha^2|A\phi^h|^2\right) = \\
(\eta_t, \phi^h) + \nu\alpha^2(A\eta, A\phi^h) + \tilde{b}(\mathbf{w}; \eta, \phi^h) + \tilde{b}(\mathbf{w}; \phi^h, \phi^h) + \\
\tilde{b}(\mathbf{w}; A\eta, \phi^h) + \tilde{b}(\mathbf{w}; A\phi^h, \phi^h) + (p - q^h, \nabla\cdot\phi^h). \qquad (5.7)
\end{aligned}
$$

We would like to bound ϕ^h by in terms of η.
Applying Hölder's inequality, Sobolev Imbedding, Poincaré inequality respectively we get that the right hand side of (5.7) as follows:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\phi^h\|^2 + \alpha^2|P_H\nabla\phi^h|^2) + \nu\left(\|\nabla\phi^h\|^2 + \alpha^2|A\phi^h|^2\right) \leq \qquad (5.8) \\
\|\eta_t\||P_H\nabla\phi^h| + \nu\alpha^2|A\eta||A\phi^h| + \\
\|\mathbf{w}\|^{1/2}|P_H\nabla\mathbf{w}|^{1/2}\|\eta\ |\|\phi^h\|_1 + \|\mathbf{w}\|^{1/2}\|\nabla\mathbf{w}\|^{1/2}|P_H\nabla\phi^h|\|\phi^h\|_1 \\
+ \|\mathbf{w}\|^{1/2}\|\nabla\mathbf{w}\|^{1/2}|A\eta|\|\phi^h\|_1 + \\
\|\mathbf{w}\|^{1/2}\|\nabla\mathbf{w}\|^{1/2}|A\phi^h|\|\phi^h\|_1 + \|p - q^h\|\|\nabla\phi^h\|
\end{aligned}
$$

Applying Young's inequality to (5.8) we get:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\phi^h\|^2 + \alpha^2|P_H\nabla\phi^h|^2) + \nu\left(\|\nabla\phi^h\|^2 + \alpha^2|A\phi^h|^2\right) \leq \\
(\frac{8}{\nu}\|\eta_t\ |^2 + \frac{\nu}{8}|P_H\nabla\phi^h|^2) + (\nu\alpha^2|A\eta|^2 + \frac{\alpha^2\nu}{4}|A\phi^h|^2) + (\frac{1}{4}\|\mathbf{w}\|_1^2\|\phi^h\|_1^2 + \|\eta\|^2) + \\
(\frac{8}{\nu}\|\mathbf{w}\|_1^2\|\phi^h\|_1^2 + \frac{\nu}{8}|P_H\nabla\phi^h|^) + (\frac{1}{4}\|\mathbf{w}\|_1^2\|\phi^h\|_1^2 + |A\eta|^2) + \\
(\frac{1}{\nu\alpha^2}\|\mathbf{w}\|_1^2\|\phi^h\|_1^2 + \frac{\nu\alpha^2}{4}|A\phi^h|^2|) + (\frac{1}{\nu}\|p - q^h\|^2 + \frac{\nu}{4}\|\nabla\phi^h\|^2)
\end{aligned}
$$

Grouping the similar terms together gives:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\phi^h\|^2 + \alpha^2|P_H\nabla\phi^h|^2\right) + \nu\left(\|\nabla\phi^h\|^2 + \alpha^2|A\phi^h|^2\right) \leq \\
\frac{8}{\nu}\|\eta_t\|^2 + (1 + \nu\alpha^2)|A\eta|^2 + \|\eta\|^2 + \frac{C(\alpha)}{\nu\alpha^2}\|\mathbf{w}\|^2\left(\|\phi^h\|^2 + \alpha^2|P_H\nabla\phi^h|^2\right) + \frac{1}{\nu}\|p - q^h\|^2
\end{aligned}
$$

Applying Gronwall's inequality:

$$\frac{1}{2}\frac{d}{dt}\left(\|\phi^h\|^2+\alpha^2|P_H\nabla\phi^h|^2\right)+\nu\left(\|\nabla\phi^h\|^2+\alpha^2|A\phi^h|^2\right)\leq$$
$$e^{\frac{C(\alpha)}{\nu\alpha^2}\int_0^t\|\mathbf{w}(s)\|_1^2\;ds}\left(\|\phi_0^h\|^2+\alpha^2|P_H\nabla\phi_0^h|^2\right)+\frac{8}{\nu}\int_0^t\|\eta_t\|^2\;ds$$
$$+(1+\nu\alpha^2)\int_0^t|A\eta|^2\;ds+\int_0^t\|\eta\|^2\;ds+\frac{1}{\nu}\int_0^t\|p-q^h\|^2\;ds.$$

Adding and subtracting $\mathbf{w}(t)$ on the right hand side of the latter inequality we obtain

$$\max_{0\leq t\leq T}(\|\mathbf{w}-\mathbf{w}^h\|^2+\alpha^2|P_H\nabla(\mathbf{w}-\mathbf{w}^h)|^2)\leq\max_{0\leq t\leq T}\left(2\|\mathbf{w}-\mathbf{v}^h\|^2+\alpha^2|\nabla(\mathbf{w}-\mathbf{v}^h)|^2\right)+$$
$$K[|\mathbf{w}_0-\mathbf{w}_0^h|^2+|\mathbf{w}_0-\mathbf{v}_0^h|^2+\alpha^2|P_H\nabla(\mathbf{w}_0-\mathbf{w}_0^h)|^2+\alpha^2|\nabla(\mathbf{w}_0-\mathbf{v}_0^h)|^2]+$$
$$+\frac{8}{\nu}\max_{0\leq t\leq T}\|\mathbf{w}_t-\mathbf{w}_t^h\|^2+(1+\nu\alpha^2)\max_{0\leq t\leq T}|A(\mathbf{w}-\mathbf{v}^h)|^2+\frac{1}{\nu}\|p-q^h\|^2$$

where $K=e^{\frac{C\alpha}{\nu\alpha^2}\|\mathbf{w}\|^2_L(0,T;H^1(\Omega))}$.
Applying the approximation assumption:

$$\max_{0\leq t\leq T}(\|\mathbf{w}-\mathbf{w}^h\|^2+\alpha^2|P_H\nabla(\mathbf{w}-\mathbf{w}^h)|^2)\leq$$
$$Ch^{2(k+1)}\max_{0\leq t\leq T}\|\mathbf{w}\|^2_{k+1}+Ch^{2k}\max_{0\leq t\leq T}\|\nabla\mathbf{w}\|^2_{k+1}+K[\|\mathbf{w}_0-\mathbf{v}_0^h\|^2$$
$$+Ch^{2(k+1)}\|\mathbf{w}_0\|^2_{k+1}+\alpha^2Ch^k\|\nabla\mathbf{w}_0\|^2_{k+1}+|P_H\nabla(\mathbf{w}_0-\mathbf{v}_0^h)|^2]+$$
$$\frac{Ch^{2(k+1)}}{\nu}\max_{0\leq t\leq T}\|\mathbf{w}_t\|^2_{k+1}+C(1+\nu\alpha^2)h^{2k}\max_{0\leq t\leq T}\|A\mathbf{w}\|^2_k+\frac{Ch^{2k}}{\nu}\|\nabla p\|^2_k$$

From the latter inequality the desired result follows.

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