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THE MOORE-PENROSE INVERSE OF MATRICES WITH AN ACYCLIC BIPARTITE GRAPH

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Abstract. The Moore-Penrose inverse of a real matrix having no square submatrix with two or more diagonals is described in terms of bipartite graphs. For such a matrix, the sign of every entry of the Moore-Penrose inverse is shown to be determined uniquely by the signs of the matrix entries; i.e., the matrix has a signed generalized inverse. Necessary and sufficient conditions on an acyclic bipartite graph are given so that each nonnegative matrix with this graph has a nonnegative Moore-Penrose inverse. Nearly reducible matrices are proved to contain no square submatrix having two or more diagonals, implying that a nearly reducible matrix has a signed generalized inverse. Furthermore, it is proved that the term rank and rank are equal for each submatrix of a nearly reducible matrix.

1. Introduction

For any real $m \times n$ matrix A, the *Moore-Penrose inverse* $A^\dagger$ is the unique matrix that satisfies the following four properties [11, 12]:

$$A^\dagger A A^\dagger = A^\dagger \qquad A A^\dagger A = A \qquad (A^\dagger A)^T = A^\dagger A \qquad (AA^\dagger)^T = AA^\dagger.$$

If A is a square, nonsingular matrix, then $A^\dagger = A^{-1}$. Thus, Moore-Penrose inversion generalizes standard matrix inversion. For more information on the Moore-Penrose inverse, see [3] and its extensive bibliography. For several decades, standard matrix inversion has been described in terms of graph and digraph properties; see for example [6, 7, 10, 16] and references therein. As introduced in [6, p. 273], a matrix A has a *signed generalized inverse* if the sign pattern of $A^\dagger$ is uniquely determined by the sign pattern of A. Signed generalized inverses have been considered, for example, by Shader [13] and Shao et al [14, 15]. Here we use graph-theoretical techniques to describe a special family of such matrices.

The main result of Section 2 below, Theorem 2.6, describes how bipartite graphs may be used to provide a relatively simple description of Moore-Penrose inversion for the special class $\mathcal{A}$ of $m \times n$ matrices having no square submatrix with two or more diagonals. In terms of graphs, the bipartite

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graph of such a matrix is acyclic. Section 3 focuses on the sign pattern of the Moore-Penrose inverse of a matrix $A \in \mathcal{A}$. The main result, Theorem 3.1, states that each submatrix of $A \in \mathcal{A}$ has a signed generalized inverse. Also, necessary and sufficient conditions on an acyclic bipartite graph are given so that each matrix $A \geq 0$ with this graph has $A^\dagger \geq 0$. For fixed arbitrary $A \geq 0$, necessary and sufficient conditions for $A^\dagger \geq 0$ are given in [1] and in [4, p. 123].

An irreducible $n \times n$ matrix is *nearly reducible* if it becomes reducible whenever any nonzero entry is replaced by zero. Such matrices have been shown to have interesting properties, many of which are described by Brualdi and Ryser [5, Section 3.3]. Section 4 is devoted to the study of such matrices and their Moore-Penrose inverses. By applying a structural result of Hartfiel [8], it is proved that every nearly reducible matrix is a member of $\mathcal{A}$; see Theorem 4.3. This generalizes the result of Hedrick and Sinkhorn [9] that each nearly reducible matrix has at most one diagonal, and shows that the bipartite graph of a nearly reducible matrix is acyclic. Furthermore, it implies that most of the results in Sections 2 and 3 are valid for nearly reducible matrices. One such result, Theorem 4.5, states that the rank and the term rank are equal for each submatrix of a nearly reducible matrix. In particular, the rank of a nearly reducible matrix is equal to its term rank. Another result, Theorem 4.6, states that the submatrices of a nearly reducible matrix each has a signed generalized inverse.

2. Bipartite graphs and the Moore-Penrose inverse

Let $m, n \geq 1$ be given, and let $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ be disjoint sets. For any $m \times n$ matrix $A = [a_{ij}]$, let $B(A)$ be the bipartite graph with vertices $U \cup V$ and edges $\{\{u_i, v_j\} \mid u_i \in U,\ v_j \in V,\ a_{ij} \neq 0\}$. Let $\mathcal{B}$ denote the family of finite acyclic bipartite graphs. A *matching* in a (bipartite) graph is a subset of its edges no two of which are adjacent. If a matching E covers all the vertices, then E is said to be a *perfect matching* (or *factor*). The fact that a bipartite graph B contains a cycle if and only if some subgraph of B contains two perfect matchings is restated in the following lemma.

Lemma 2.1. *A bipartite graph B is acyclic if and only if each subgraph of B contains at most one perfect matching.*

A *diagonal* in a $k \times k$ matrix is a set of k nonzero entries, no two of which occur in the same row or column. The *term rank* of a matrix is the maximum number of nonzero entries, no two of which lie in the same row or column. By Lemma 2.1, if A is an $m \times n$ matrix, then $B(A) \in \mathcal{B}$ if and only if each square submatrix of A has at most one diagonal. We denote the family of all such $m \times n$ matrices by $\mathcal{A}$. The next result follows immediately from Lemma 2.1.

Lemma 2.2. *The rank of each submatrix of $A \in \mathcal{A}$ equals its term rank.*

For all integers k, l with $1 \leq k \leq l$, let $Q_{k,l}$ denote the family of strictly increasing sequences of k of the integers $1, \ldots, l$. If $\gamma = (\gamma_1, \ldots, \gamma_k) \in Q_{k,l}$ and $i \notin \gamma$, then let $(i;\gamma)$ denote the ordered set $(i, \gamma_1, \ldots, \gamma_k)$. If $A = [a_{ij}]$ is an $m \times n$ matrix with rows and columns labelled by the integers $1, \ldots, m$ and $1, \ldots, n$, respectively, and γ, δ are ordered subsets of $(1, \ldots, m)$ and $(1, \ldots, n)$, respectively, then let $A[\gamma|\delta]$ denote the $|\gamma| \times |\delta|$ matrix whose (i,j)th entry equals $a_{\gamma_i \delta_j}$. Note that $A[\gamma|\delta]$ is the submatrix of A with rows γ and columns δ, whereas $A[i;\gamma|j;\delta]$ has rows and columns ordered $(i;\gamma)$ and $(j;\delta)$, respectively. The following theorem is due to Moore [11]; see also [2] and [3, Appendix A] for recent accounts.

Theorem 2.3. *Let A be an $m \times n$ matrix with rank $r \geq 2$, and let $A^\dagger = [\alpha_{ij}]$ denote the Moore-Penrose inverse of A. Then*

$$\alpha_{ji} = \frac{\displaystyle\sum_{\gamma \in Q_{r-1,m}, i \notin \gamma} \sum_{\delta \in Q_{r-1,n}, j \notin \delta} \det A[\gamma|\delta] \det A[i;\gamma|j;\delta]}{\displaystyle\sum_{\rho \in Q_{r,m}} \sum_{\tau \in Q_{r,n}} (\det A[\rho|\tau])^2}.$$

In the following two lemmas, let $A = [a_{ij}] \in \mathcal{A}$ be a matrix with rank $r \geq 2$, let $1 \leq i \leq m$ and $1 \leq j \leq n$ be given, and suppose $\gamma \in Q_{r-1,m}$ and $\delta \in Q_{r-1,n}$ such that $i \notin \gamma$ and $j \notin \delta$.

Lemma 2.4. *If $\det A[\gamma|\delta] \det A[i;\gamma|j;\delta] \neq 0$, then $B(A[i;\gamma|j;\delta])$ contains a path from u_i to v_j and a (possibly empty) matching that together cover all vertices of $B(A[i;\gamma|j;\delta])$ and that are vertex-disjoint.*

Proof. If $\det A[\gamma|\delta] \det A[i;\gamma|j;\delta] \neq 0$, then $A[\gamma|\delta]$ and $A[i;\gamma|j;\delta]$ each contains a diagonal. Thus, $B(A[\gamma|\delta])$ and $B(A[i;\gamma|j;\delta])$ each contains a matching, say E_1 and E_2, respectively. Let G denote the bipartite graph with vertex set $V(B(A[i;\gamma|j;\delta]))$ and edge set $E_1 \cup E_2$, and let $\widehat{G}$ denote the subgraph of G obtained by deleting each edge not containing u_i or v_j that is not adjacent to any other edge of G. In $\widehat{G}$, vertices u_i and v_j have degree 1 and all other vertices have degree 2. Since $B(A) \in \mathcal{B}$, $\widehat{G}$ cannot contain any cycles. Thus, $\widehat{G}$ consists of a path from u_i to v_j. Hence, $B(A[i;\gamma|j;\delta])$ contains this path from u_i to v_j, as well as a (possibly empty) matching that covers all of the vertices not on this path. □

Lemma 2.5. *If $B(A[i;\gamma|j;\delta])$ contains a path from u_i to v_j,*

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \to \cdots \to v_{j_s} \to u_{i_s} \to v_j,$$

and a perfect matching E of $B(A[\gamma - \{i_1, \ldots, i_s\}|\delta - \{j_1, \ldots, j_s\}])$, then

$$\det A[\gamma|\delta] \det A[i;\gamma|j;\delta] = (-1)^s a_{ij_1} a_{i_1 j_1} a_{i_1 j_2} \cdots a_{i_s j_s} a_{i_s j} \prod_{\{u_k, v_l\} \in E} (a_{kl})^2.$$

Proof. Re-order the numbers $(i_1, \dots, i_s)$ to form an increasing sequence, denoted by $\mathbf{i}$, and let π be the permutation on the set $(1, \dots, s)$ that effects this re-ordering. Similarly, re-order the numbers $(j_1, \dots, j_s)$ to form an increasing sequence, denoted by $\mathbf{j}$, and let σ be the permutation on the set $(1, \dots, s)$ that effects this re-ordering.

The entries $a_{i_1j_1}, a_{i_2j_2}, \dots, a_{i_sj_s}$ form a diagonal of $A[\mathbf{i}|\mathbf{j}]$, the entries a_{ij_1}, $a_{i_1j_2}, \dots, a_{i_{s-1}j_s}, a_{i_sj}$ form a diagonal of $A[i;\mathbf{i}|j;\mathbf{j}]$, and the entries $\{a_{kl} \mid (k,l) \in E\}$ form a diagonal of $A[\gamma - \mathbf{i}|\delta - \mathbf{j}]$. By Lemma 2.1, these are the only diagonals of these matrices, and the matrices $A[\gamma|\delta]$, $A[i;\gamma|j;\delta]$ each has precisely one diagonal. Thus, taking into account the signs of the determinants,

$$\det A[\gamma|\delta] = \det A[\mathbf{i}|\mathbf{j}] \det A[\gamma - \mathbf{i}|\delta - \mathbf{j}] \qquad \text{and}$$
$$\det A[i;\gamma|j;\delta] = \det A[i;\mathbf{i}|j;\mathbf{j}] \det A[\gamma - \mathbf{i}|\delta - \mathbf{j}].$$

In the matrix $A[i_1, \dots, i_s|j_1, \dots, j_s]$, the entries $a_{i_1j_1}, a_{i_2j_2}, \dots, a_{i_sj_s}$ lie on the main diagonal, so

$$\begin{aligned}\det A[\mathbf{i}|\mathbf{j}] &= \operatorname{sgn}(\pi) \det A[i_1, \dots, i_s|j_1, \dots, j_s] \operatorname{sgn}(\sigma)\\ &= \operatorname{sgn}(\pi) a_{i_1j_1} a_{i_2j_2} \cdots a_{i_sj_s} \operatorname{sgn}(\sigma).\end{aligned}$$

In the matrix $A[i, i_1, \dots, i_s|j, j_1, \dots, j_s]$, the permutation corresponding to the diagonal consisting of the entries $a_{ij_1}, a_{i_1j_2}, \dots, a_{i_{s-1}j_s}, a_{i_sj}$ is a cycle of length $s+1$. Thus,

$$\begin{aligned}\det A[i;\mathbf{i}|j;\mathbf{j}] &= \operatorname{sgn}(\pi) \det A[i, i_1, \dots, i_s|j, j_1, \dots, j_s] \operatorname{sgn}(\sigma)\\ &= \operatorname{sgn}(\pi)(-1)^{(s+1)-1} a_{ij_1} a_{i_1j_2} \cdots a_{i_{s-1}j_s} a_{i_sj} \operatorname{sgn}(\sigma).\end{aligned}$$

It follows that

$$\begin{aligned}&\det A[\gamma|\delta] \det A[i;\gamma|j;\delta]\\ &= \det A[\mathbf{i}|\mathbf{j}] \det A[i;\mathbf{i}|j;\mathbf{j}] (\det A[\gamma - \mathbf{i}|\delta - \mathbf{j}])^2\\ &= \operatorname{sgn}(\pi) a_{i_1j_1} \cdots a_{i_sj_s} \operatorname{sgn}(\sigma) \operatorname{sgn}(\pi)(-1)^s a_{ij_1} a_{i_1j_2} \cdots a_{i_sj} \operatorname{sgn}(\sigma) \prod_{\{u_k,v_l\} \in E} (a_{kl})^2\\ &= (-1)^s a_{ij_1} a_{i_1j_1} a_{i_1j_2} \cdots a_{i_sj_s} a_{i_sj} \prod_{\{u_k,v_l\} \in E} (a_{kl})^2.\end{aligned}$$

□

For $k \geq 1$ and any bipartite graph B, let $M_k(B)$ denote the family of matchings in B that contain k edges.

Theorem 2.6. *Let $A \in \mathcal{A}$ be an $m \times n$ matrix with rank $r \geq 2$, and let $A^\dagger = [\alpha_{ij}]$ denote the Moore-Penrose inverse of A. If $B(A)$ contains a path p from u_i to v_j*

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \to \cdots \to v_{j_s} \to u_{i_s} \to v_j$$

of length $2s+1$ *with* $s \geq 0$*, then*

$$\alpha_{ji} = (-1)^s a_{ij_1} a_{i_1j_1} a_{i_1j_2} \cdots a_{i_sj_s} a_{i_sj} \frac{\displaystyle\sum_{\substack{E \in M_{r-s-1}(B(A)),\\ V(E)\cap V(p)=\emptyset}} \prod_{\{u_k,v_l\}\in E} (a_{kl})^2}{\displaystyle\sum_{F \in M_r(B(A))} \prod_{\{u_k,v_l\}\in F} (a_{kl})^2} .$$

Otherwise, $\alpha_{ji} = 0$.

Proof. Consider first the numerator

$$N = \sum_{\gamma \in Q_{r-1,m}, i\notin\gamma} \sum_{\delta\in Q_{r-1,n}, j\notin\delta} \det A[\gamma|\delta] \det A[i;\gamma|j;\delta]$$

in the expression for α_{ji} in Theorem 2.3. By Lemmas 2.4 and 2.5, the nonzero terms of N correspond precisely to the instances in which $B(A)$ contains a path p of length $2s+1$ from u_i to v_j and a matching with $r-s-1$ edges that are not adjacent to p. If there are no paths from i to j, then $N=0$, so by Theorem 2.3, $\alpha_{ji}=0$. If there is a path p from u_i to v_j

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \to \cdots \to v_{j_s} \to u_{i_s} \to v_j,$$

then this is the only such path, since $B(A)$ is acyclic. Thus by Lemma 2.5,

$$N = (-1)^s a_{ij_1} a_{i_1j_1} a_{i_1j_2} \cdots a_{i_sj_s} a_{i_sj} \sum_{\substack{E \in M_{r-s-1}(B(A)),\\ V(E)\cap V(p)=\emptyset}} \prod_{\{u_k,v_l\}\in E} (a_{kl})^2 .$$

Now consider the denominator in the expression for α_{ji}, and let $\rho \in Q_{r,m}$ and $\tau \in Q_{r,n}$ be given. By Lemma 2.1, $\det A[\rho|\tau] \neq 0$ if and only if $B(A[\rho|\tau])$ has a matching F. If this is true, then

$$\det A[\rho|\tau] = c \prod_{\{u_k,v_l\}\in F} a_{kl} ,$$

where $c = \pm 1$. Thus,

$$\sum_{\rho\in Q_{r,m}} \sum_{\tau \in Q_{r,n}} (\det A[\rho|\tau])^2 = \sum_{F\in M_r(B(A))} \prod_{\{u_k,v_l\}\in F} (a_{kl})^2 ,$$

and Theorem 2.3 concludes the proof. □

The following corollary describes explicitly the bipartite graph $B(A^\dagger)$ as well as the sign of each nonzero entry of $A^\dagger$.

Corollary 2.7. *Let* $A \in \mathcal{A}$ *be an* $m \times n$ *matrix with rank* $r \geq 2$*, and let* $A^\dagger = [\alpha_{ij}]$*. Then for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*, the entry* α_{ji} *is nonzero if and only if* $\{u_j, v_i\}$ *is an edge of* $B(A^\dagger)$ *if and only if* $B(A)$ *contains a path* p *from* u_i *to* v_j

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \to \cdots \to v_{j_s} \to u_{i_s} \to v_j$$

of length $2s+1$ *with* $s \geq 0$*, and at least one matching* E *with* $r-s-1$ *edges, none of which are adjacent to* p*. If such a path exists, then* α_{ji} *has the* *same sign as* $(-1)^s a_{ij_1} a_{i_1 j_1} a_{i_1 j_2} \cdots a_{i_s j_s} a_{i_s j}$ *, and* $A^\dagger[j_1, \ldots, j_s, j | i, i_1, \ldots, i_s]$ *has no zero entries on or below the main diagonal.*

Proof. The equivalences follow immediately from Theorem 2.6. To prove the second part, suppose that $B(A)$ contains a path p and a matching E as given in the statement of the corollary. By Theorem 2.6, α_{ji} has the same sign as

$$(-1)^s a_{ij_1} a_{i_1 j_1} a_{i_1 j_2} \cdots a_{i_s j_s} a_{i_s j} \neq 0\,.$$

If $s = 0$, then the submatrix $A^\dagger[j_1, \ldots, j_s, j | i, i_1, \ldots, i_s]$ consists of the single nonzero entry α_{ji}. So considering $s \geq 1$, suppose that k is an integer such that $1 \leq k < l \leq s$. Then

$$u_{i_k} \to v_{j_{k+1}} \to u_{i_{k+1}} \to v_{j_{k+2}} \to u_{i_{k+2}} \to \cdots \to v_{j_{l-1}} \to u_{i_{l-1}} \to v_{j_l}$$

is a path p' from u_{i_k} to v_{j_l} of length $2s' + 1$ with $s' = l - k - 1$, and

$$\begin{array}{rcl} E & \cup & \{\{u_i, v_{j_1}\}, \{u_{i_1}, v_{j_2}\}, \ldots, \{u_{i_{k-1}}, v_{j_k}\}, \\ & & \{u_{i_l}, v_{j_{l+1}}\}, \ldots, \{u_{i_{s-1}}, v_{j_s}\}, \{u_{i_s}, v_j\}\} \end{array}$$

is a matching with $r - s' - 1$ edges, none of which are adjacent to p'. The equivalences already established give that $\alpha_{j_l i_k} \neq 0$. By similar arguments for all $1 \leq l \leq s$, $\alpha_{j_l i} \neq 0$ and $\alpha_{j i_l} \neq 0$. $\square$

Some consequences of an edge $\{u_i, v_j\}$ being contained in no matching with r edges, in at least one matching with r edges, or in every matching with r edges in $B(A)$ are now considered.

Proposition 2.8. *Let* $A \in \mathcal{A}$ *be a matrix with rank* $r \geq 2$*, and let* $A^\dagger = [\alpha_{ij}]$*. Let* $\{u_i, v_j\}$ *be an edge of* $B(A)$*. Then* $\{u_j, v_i\}$ *is an edge of* $B(A^\dagger)$ *if and only if* $\{u_i, v_j\}$ *is contained in some matching in* $M_r(B(A))$*. If* $\{u_i, v_j\}$ *is contained in every matching in* $M_r(B(A))$*, then* $\alpha_{ji} = \frac{1}{a_{ij}}$*.*

Proof. If $\{u_i, v_j\}$ is contained in some matching $E \in M_r(B(A))$, then $u_i \to v_j$ is a path that together with the matching $E - \{u_i, v_j\}$ satisfies the conditions in Corollary 2.7, so $\{u_j, v_i\}$ is an edge of $B(A^\dagger)$. Conversely, suppose that $\{u_j, v_i\}$ is an edge of $B(A^\dagger)$. By Corollary 2.7, there exists a path p from u_i to v_j in $B(A)$ of length $2s + 1$ and a matching $E \in M_{r-s-1}(B(A))$ that is vertex disjoint from p. Since $B(A)$ is acyclic and contains the edge $\{u_i, v_j\}$, p must be the path $u_i \to v_j$, and $s = 0$. Thus, $E \cup \{u_i, v_j\}$ is a matching in $M_r(B(A))$ that contains the edge $\{u_i, v_j\}$.

To prove the last statement of the proposition, suppose that $\{u_i, v_j\}$ is contained in every matching in $M_r(B(A))$. By Theorem 2.6,

$$\alpha_{ji} = a_{ij} \frac{\displaystyle\sum_{E \in M_r(B(A))} \prod_{\{u_k, v_l\} \in E - \{u_i, v_j\}} (a_{kl})^2}{\displaystyle\sum_{F \in M_r(B(A))} \prod_{\{u_k, v_l\} \in F} (a_{kl})^2} = \frac{a_{ij}}{a_{ij}^2} = \frac{1}{a_{ij}}\,.$$

$\square$

If an edge $\{u_i, v_j\}$ of $B(A)$ is not contained in any matching in $M_r(B(A))$, then by Proposition 2.8 $\{u_j, v_i\}$ is not an edge of $B(A^\dagger)$. However, other edges owe their presence in $B(A^\dagger)$ to the edge $\{u_i, v_j\}$. Such edges are described in the following proposition.

Proposition 2.9. *Let $A \in \mathcal{A}$ be a matrix with rank $r \geq 2$, and let $A^\dagger = [\alpha_{ij}]$. Let $\{u_i, v_j\}$ be an edge of $B(A)$ that is not contained in any matching in $M_r(B(A))$. If $E \in M_r(B(A))$, then there exist (unique) edges $\{u_i, v_{i_E}\}$ and $\{u_{j_E}, v_j\}$ of E such that $\{u_{i_E}, v_{j_E}\}$ is an edge of $B(A^\dagger)$, and in this case, $\{u_{j_E}, v_{i_E}\}$ is not an edge of $B(A)$.*

Proof. If E does not contain an edge $\{u_i, v_{i_E}\}$ or $\{u_{j_E}, v_j\}$, then $E \cup \{u_i, v_j\} \in M_{r+1}(B(A))$, which contradicts the maximality of r. Thus, E contains at least one such edge, say $\{u_i, v_{i_E}\}$. If E does not also contain an edge $\{u_{j_E}, v_j\}$, then $(E - \{u_i, v_{i_E}\}) \cup \{u_i, v_j\} \in M_r(B(A))$ contains the edge $\{u_i, v_j\}$, a contradiction. Thus, E contains (unique) edges $\{u_i, v_{i_E}\}, \{u_{j_E}, v_j\}$. Now $u_{j_E} \to v_j \to u_i \to v_{i_E}$ is a path from u_{j_E} to v_{i_E} of length $2s+1$ with $s = 1$, and $E - \{u_i, v_{i_E}\} - \{u_{j_E}, v_j\}$ is a matching in $B(A)$ with $r - s - 1 = r - 2$ edges, none of which contain u_{j_E}, u_i, v_j, or v_{i_E}. By Corollary 2.7, $\{u_{i_E}, v_{j_E}\}$ is an edge of $B(A^\dagger)$. The edge $\{u_{j_E}, v_{i_E}\}$ cannot be contained in $B(A)$ since this would imply that

$$(E - \{u_{j_E}, v_j\} - \{u_i, v_{i_E}\}) \cup \{u_i, v_j\} \cup \{u_{j_E}, v_{i_E}\}$$

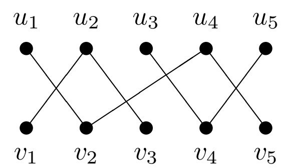
is a matching in $M_r(B(A))$ that contains $\{u_i, v_j\}$. □

Remark 2.10. The results of this section on $A^\dagger$ apply to a matrix $A \in \mathcal{A}$ with rank at least two. For completeness, the rank zero and rank one cases are now considered. If the rank of A is zero, then $A = 0$, and $A^\dagger = 0$. If the rank of A is one, then by Lemma 2.2, A is permutation similar to a matrix either of the form $[v\, 0]$ or of the form $[v\, 0]^T$, where v is a nonzero vector (and the 0 submatrix is vacuous if either m or n is 1). Hence, $A^\dagger$ is either of the form $[\hat{v}\, 0]^T$ or of the form $[\hat{v}\, 0]$, where $\hat{v} = v/||v||^2$. Note that in both cases, the sign of each entry α_{ji} in $A^\dagger$ is equal to that of a_{ij} in A.

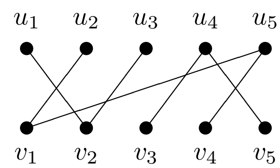
Example 2.11. The above results are illustrated with a 5×5 matrix $A \in \mathcal{A}$, i.e., $B(A) \in \mathcal{B}$. The matrix A, the Moore-Penrose inverse $A^\dagger$, and the associated bipartite graphs $B(A)$ and $B(A^\dagger)$ are displayed in Figure 1. The maximal cardinality of a matching in $B(A)$ is four, thus the rank of A is $r = 4$ by Lemma 2.2. The matchings in $M_4(B(A))$ are the sets of edges in each of the subgraphs B_1, B_2, B_3, and B_4 of $B(A)$; see Figure 1. Note that the edge $\{u_1, v_2\}$ is contained in all four of these, whereas the edge $\{u_4, v_2\}$ is contained in none of these. By Proposition 2.8, the entry α_{21} of $A^\dagger$ equals $\frac{1}{a_{12}}$, and the entry α_{24} of $A^\dagger$ is zero. To compute the entry α_{51} of $A^\dagger$, note that $u_1 \to v_2 \to u_4 \to v_5$ is a path p in $B(A)$ from u_1 to v_5 of length $2s+1 = 3$ with $s = 1$. The matchings $\{\{u_2, v_3\}, \{u_3, v_4\}\}$, $\{\{u_2, v_1\}, \{u_3, v_4\}\}$, $\{\{u_2, v_3\}, \{u_5, v_4\}\}$, and $\{\{u_2, v_1\}, \{u_5, v_4\}\}$ in $B(A)$ each contains $r - s -$

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & a_{34} & 0 \\ 0 & a_{42} & 0 & 0 & a_{45} \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix} \qquad A^\dagger = \begin{bmatrix} 0 & \frac{a_{21}}{a_{21}^2+a_{23}^2} & 0 & 0 & 0 \\ \frac{1}{a_{12}} & 0 & 0 & 0 & 0 \\ 0 & \frac{a_{23}}{a_{21}^2+a_{23}^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{a_{34}}{a_{34}^2+a_{54}^2} & 0 & \frac{a_{54}}{a_{34}^2+a_{54}^2} \\ -\frac{a_{42}}{a_{12}a_{45}} & 0 & 0 & \frac{1}{a_{45}} & 0 \end{bmatrix}$$

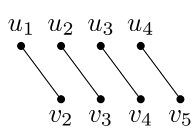
A $\qquad$ $A^\dagger$



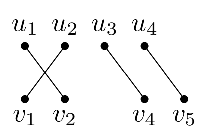
$B(A)$



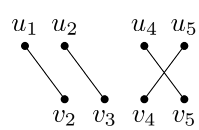
$B(A^\dagger)$



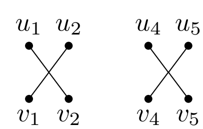
B_1



B_2



B_3



B_4

FIGURE 1

$1 = 2$ edges, none of which are adjacent to p, and these are the only such matchings. Thus by Theorem 2.6,

$$\alpha_{51} = (-1)^s a_{12}a_{42}a_{45}\frac{a_{23}^2a_{34}^2 + a_{21}^2a_{34}^2 + a_{23}^2a_{54}^2 + a_{21}^2a_{54}^2}{S},$$

where S is the sum

$$(a_{12}a_{23}a_{34}a_{45})^2 + (a_{12}a_{21}a_{34}a_{45})^2 + (a_{12}a_{23}a_{54}a_{45})^2 + (a_{12}a_{21}a_{54}a_{45})^2$$

corresponding to the four matchings in $M_4(B(A))$. Hence,

$$\alpha_{51} = -\frac{a_{12}a_{42}a_{45}}{(a_{12}a_{45})^2} = -\frac{a_{42}}{a_{12}a_{45}}.$$

Since the path p has length $2s+1$ with $s = 1$, and $B(A)$ contains at least one matching with $r - s - 1 = 2$ edges, none of which are adjacent to p, (for instance the matching $\{\{u_2, v_3\}, \{u_3, v_4\}\}$), Corollary 2.7 implies that the submatrix $A^\dagger[2,5|1,4]$ has only nonzero entries on and below the main diagonal. Indeed this is the case:

$$A^\dagger[2,5|1,4] = \begin{bmatrix} \frac{1}{a_{12}} & 0 \\ -\frac{a_{42}}{a_{12}a_{45}} & \frac{1}{a_{45}} \end{bmatrix}.$$

Finally, since the edge $\{u_4, v_2\}$ is not contained in any matching in $M_4(B(A))$, and there exists a matching in $M_4(B(A))$ that contains edges $\{u_4, v_5\}$ and $\{u_1, v_2\}$, the edge $\{u_5, v_1\}$ is in $B(A^\dagger)$ by Proposition 2.9.

Example 2.12. The example given in Figure 2 illustrates that $A \in \mathcal{A}$ and $A^\dagger$ do not necessarily contain the same number of nonzero entries, and that $A^\dagger$ need not be a member of $\mathcal{A}$ (cf. Example 2.11).

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ \frac{-a_{21}a_{22}}{a_{11}(a_{22}^2+a_{23}^2)} & \frac{a_{22}}{a_{22}^2+a_{23}^2} \\ \frac{-a_{21}a_{23}}{a_{11}(a_{22}^2+a_{23}^2)} & \frac{a_{23}}{a_{22}^2+a_{23}^2} \end{bmatrix}$$

A $\qquad$ $A^\dagger$

$B(A)$ $\qquad$ $B(A^\dagger)$

FIGURE 2

3. THE SIGN PATTERN OF THE MOORE-PENROSE INVERSE

A matrix A is said to have a *signed generalized inverse* [6, p. 273] if the sign pattern of the Moore-Penrose inverse $A^\dagger$ is uniquely determined by the sign pattern of A. In other words, A has a signed generalized inverse if for each matrix B with the same sign pattern as A, the sign pattern of $B^\dagger$ is the same as the sign pattern of $A^\dagger$.

Theorem 3.1. *Each submatrix of a matrix $A \in \mathcal{A}$ has a signed generalized inverse.*

Proof. If the rank of $A \in \mathcal{A}$ is at most one, then A has a signed generalized inverse by Remark 2.10. If the rank of $A \in \mathcal{A}$ is two or greater, then A has a signed generalized inverse by Corollary 2.7. Since $\widehat{A} \in \mathcal{A}$ for each submatrix $\widehat{A}$ of A, the result now follows. □

Note that in each of the Examples 2.11 and 2.12, the sign of each entry of $A^\dagger$ is determined by the signs of entries of A. Thus as claimed in Theorem 3.1, each matrix A has a signed generalized inverse.

If $\{u_i, v_j\}$ is an edge of $B(A)$, then Proposition 2.8 specifies the circumstances under which $\{u_j, v_i\}$ is an edge of $B(A^\dagger)$. The next result gives a characterization of the condition $B(A^\dagger) = B(A^T)$. Quantitative results of this nature are given in [1] and in [4, p. 123].

Theorem 3.2. *Let $B \in \mathcal{B}$ and let r be the maximal cardinality of a matching in B. The following statements are equivalent:*

(i) *for each matrix A with $B(A) = B$, $B(A^\dagger) = B(A^T)$;*
(ii) *for each nonnegative matrix A with $B(A) = B$, $A^\dagger$ is nonnegative;*
(iii) *B does not contain a path p*

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \to \cdots \to v_{j_s} \to u_{i_s} \to v_j$$

of length $2s+1$ with $s \geq 1$, and a matching with $r-s-1$ edges, none of which are adjacent to p.

Proof. If $r \leq 1$, then all three statements are true and thus equivalent. For the remainder of the proof, assume that $r \geq 2$.

Let A be a matrix with $B(A) = B$ and assume that statement (i) is true. By Proposition 2.8, each edge in B is contained in some matching in $M_r(B(A))$. Assume that statement (iii) is false. Then $B(A)$ contains a path p

$$u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \cdots \to v_{j_s} \to u_{i_s} \to v_j$$

of length $2s+1$ with $s \geq 1$, and a matching E with $r-s-1$ edges, none of which contain $u_i, v_{j_1}, u_{i_1}, v_{j_2}, u_{i_2}, \ldots, v_{j_s}, u_{i_s}, v_j$. By Corollary 2.7, the entry α_{ji} in $A^\dagger$ is nonzero. Since $B(A^T) = B(A^\dagger)$, the entry a_{ij} of A is nonzero. Thus, $B(A)$ contains the cycle $u_i \to v_{j_1} \to u_{i_1} \to v_{j_2} \to u_{i_2} \cdots \to v_{j_s} \to u_{i_s} \to v_j \to u_i$, a contradiction since $B(A) \in \mathcal{B}$. Hence, statement (i) implies statement (iii). To prove that statement (iii) implies statement (i), suppose that statement (i) is not true, i.e., $B(A^\dagger) \neq B(A^T)$. Then $B(A^T)$ contains an edge that is not an edge of $B(A^\dagger)$, or $B(A^\dagger)$ contains an edge that is not an edge of $B(A^T)$. If the former is true, then by Proposition 2.8 this edge is not contained in any matching in B with r edges. By the proof of Proposition 2.9, statement (iii) is false. On the other hand, if $\{u_j, v_i\}$ is an edge of $B(A^\dagger)$ but not an edge of $B(A^T)$, then $\{u_i, v_j\}$ is not an edge of $B(A)$. By Corollary 2.7, there is a path of length $2s+1$ with $s \geq 1$ from u_i to v_j and a corresponding matching, contradicting statement (iii). Thus, in either case, statement (iii) is false. By taking the contrapositive, statement (iii) implies statement (i).

To prove that statements (ii) and (iii) are equivalent, let A be a nonnegative matrix with $B(A) = B$. If statement (iii) is true, then by Corollary 2.7 each nonzero entry of $A^\dagger$ corresponds to a path in B of length $2s+1$ with $s = 0$ and at least one corresponding matching in B. By Corollary 2.7, the nonzero entries of $A^\dagger$ are positive. Thus, statement (iii) implies statement (ii). To prove the converse, suppose that statement (iii) is not true. In this case, B contains a path of length $2s+1$ with $s \geq 1$ and a matching E with the properties described in statement (iii). If $s = 1$, then by Corollary 2.7, $\alpha_{ji} < 0$ and statement (ii) is false. If $s \geq 2$, then B contains the path $u_i \to v_{j_1} \to u_{i_1} \to v_{j_2}$ of length $2s'+1$ with $s'=1$ and the matching

$$E \cup \{\{u_{i_2}, v_{j_3}\}, \ldots, \{u_{i_{s-1}}, v_{j_s}\}, \{u_{i_s}, v_j\}\},$$

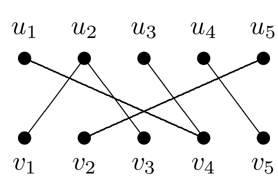
which by Corollary 2.7 implies that the entry $\alpha_{j_2 i}$ of $A^\dagger$ is negative. Thus, statement *(ii)* is false. By taking the contraposition, statement *(ii)* implies statement *(iii)*. All three statements are thus equivalent. $\square$

Note that the conditions of Theorem 3.2 are not satisfied by $B(A)$ in Examples 2.11 or 2.12, but are satisfied in the following example.

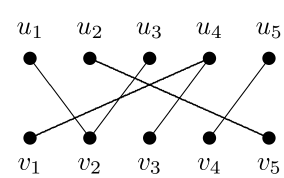
Example 3.3. A 5×5 matrix $A \in \mathcal{A}$, the Moore-Penrose inverse $A^\dagger$, and the associated bipartite graphs $B(A)$ and $B(A^\dagger)$ are displayed in Figure 3. Note that A has a signed generalized inverse, as asserted by Theorem 3.1. If A is nonnegative, then since $B(A)$ contains no path of length $2s+1$ with $s \geq 1$, Theorem 3.2 implies that the entries of $A^\dagger$ are nonnegative and $B(A^\dagger) = B(A^T)$.

$$\begin{bmatrix} 0 & 0 & 0 & a_{14} & 0 \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & a_{34} & 0 \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & a_{52} & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \frac{a_{21}}{a_{21}^2+a_{23}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_{52}} \\ 0 & \frac{a_{23}}{a_{21}^2+a_{23}^2} & 0 & 0 & 0 \\ \frac{a_{14}}{a_{14}^2+a_{34}^2} & 0 & \frac{a_{34}}{a_{14}^2+a_{34}^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_{45}} & 0 \end{bmatrix}$$

A $\qquad\qquad$ $A^\dagger$



$B(A)$



$B(A^\dagger)$

FIGURE 3

4. Nearly reducible matrices

In this section, all matrices are $n \times n$ with $n \geq 2$. An irreducible matrix is *nearly reducible* if it is reducible whenever any nonzero entry is set to zero [5, Section 3.3]. For each $n \times n$ matrix $A = [a_{ij}]$, let $D(A)$ be the directed graph with vertices $W = \{w_1, \ldots, w_n\}$ and edges $\{(w_i, w_j) \in W \times W \mid a_{ij} \neq 0\}$. In terms of digraphs, A is nearly reducible if and only if $D(A)$ is *minimally strongly connected*, i.e., $D(A)$ is strongly connected but the removal of any arc of $D(A)$ causes the digraph to no longer be strongly connected. Note that the matrices A in Figures 1 and 3 are nearly reducible.

Hedrick and Sinkhorn [9] proved that the permanent of a nearly reducible matrix contains at most one term. Restated in terms of bipartite graphs, this result may be expressed as follows.

Theorem 4.1. [9] *The bipartite graph of a nearly reducible matrix contains at most one perfect matching.*

The following theorem of Hartfiel [8] (which was stated for $(0,1)$-matrices) provides a structural characterization of nearly reducible matrices.

Theorem 4.2. [8] *Let A be an $n \times n$ nearly reducible matrix. There exists a permutation matrix P and an integer m such that $1 \leq m \leq n-1$ and*

$$P^T AP = \begin{bmatrix} 0 & 0 & E_1 \\ F & 0 & 0 \\ 0 & E_2 & A_1 \end{bmatrix}, \tag{1}$$

where A_1 is an $m \times m$ nearly reducible matrix, F is an $(n-m-1) \times (n-m-1)$ diagonal matrix with nonzero diagonal, and E_1 and E_2 are row and column vectors, respectively, that each contains precisely one nonzero entry.

The matrix A in Figure 3 is in the form (1) with $D(A_1)$ a 4-cycle and F vacuous. Using Theorem 4.2, Hedrick and Sinkhorn's result may be generalized as follows.

Theorem 4.3. *Each square submatrix of a nearly reducible matrix contains at most one diagonal.*

Proof. The result is trivially true for matrices of order 2, so (proceeding by induction) let $n \geq 3$ and assume that the result is true for all nearly reducible matrices of order less than n. Let A be an $n \times n$ nearly reducible matrix. By Theorem 4.2, it may be assumed that A is in the form (1). Let H be any square submatrix of A. If H does not contain any entry of A_1, then clearly H has at most one diagonal. Assume that H contains at least one entry of A_1. If H contains a nonzero entry that is not contained in A_1, then this entry is contained in all of the possible diagonals of H. Thus, the number of diagonals is the same in H as in the submatrix $\widehat{H}$ of H obtained by deleting the rows and columns of H containing all such entries. Either $\widehat{H}$ contains zero rows or columns and thus contains no diagonal, or $\widehat{H}$ is a square submatrix of A and contains at most one diagonal, by the induction assumption. Thus, H contains at most one diagonal. □

The next corollary follows immediately from Lemma 2.1 and Theorem 4.3.

Corollary 4.4. *Each nearly reducible matrix A is a member of $\mathcal{A}$; equivalently, $B(A) \in \mathcal{B}$.*

The matrix $A = \left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right]$ demonstrates that the converse of Corollary 4.4 is not true. By Corollary 4.4, most of the results of Sections 2 and 3 concerning a matrix $A \in \mathcal{A}$ are valid for any nearly reducible matrix A. Such results include Lemma 2.2, Propositions 2.8 and 2.9, Theorems 2.6, 3.1, and 3.2, and Corollary 2.7. Each of these results may be restated as a new result for nearly reducible matrices. To highlight this, Lemma 2.2 and Theorem 3.1 are now restated in this context.

Theorem 4.5. *The rank of each submatrix of a nearly reducible matrix A equals its term rank. In particular, the rank of A equals its term rank.*

Theorem 4.6. *Each submatrix of a nearly reducible matrix has a signed generalized inverse.*

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