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T. Britz
Dept of Math & Stat
University of Victoria
Victoria, BC, Canada

J. J. McDonald
Mathematics Dept
Washington State University
Pullman, WA, USA

D. D. Olesky
Dept of Computer Science
University of Victoria
Victoria, BC, Canada

P. van den Driessche
Dept of Math & Stat
University of Victoria
Victoria, BC, Canada

Preprint number: PIMS-04-11
Received on June 8, 2004

http://www.pims.math.ca/publications/preprints/

MINIMAL SPECTRALLY ARBITRARY SIGN PATTERNS*

T. BRITZ[†], J. J. MCDONALD[‡], D. D. OLESKY[§], AND P. VAN DEN DRIESSCHE[†]

Abstract. An $n \times n$ sign pattern $\mathcal{A}$ is spectrally arbitrary if given any self-conjugate spectrum there exists a matrix realization of $\mathcal{A}$ with that spectrum. If replacing any nonzero entry (or entries) of $\mathcal{A}$ by zero destroys this property, then $\mathcal{A}$ is a minimal spectrally arbitrary sign pattern. For $n \geq 3$, several families of $n \times n$ spectrally arbitrary sign patterns are presented, and their minimal spectrally arbitrary subpatterns are identified. These are the first known families of $n \times n$ minimal spectrally arbitrary sign patterns. Furthermore, all such 3×3 sign patterns are determined and it is proved that any irreducible $n \times n$ spectrally arbitrary sign pattern must have at least $2n-1$ nonzero entries, and conjectured that the minimum number of nonzero entries is $2n$.

Key words. Spectrum, sign pattern, nilpotent matrix

AMS subject classifications. 15A18, 15A48.

1. Introduction. A sign pattern is a square matrix with entries in $\{+, -, 0\}$. If $\mathcal{A}$ is a sign pattern and A is a real matrix for which each entry has the same sign as the corresponding entry of $\mathcal{A}$, then A is said to be a *realization* of $\mathcal{A}$, and we write $A \in \mathcal{A}$. This convention is also used for zero-nonzero patterns $\mathcal{A}$. A sign pattern $\mathcal{B} = [b_{ij}]$ is a *superpattern* of a sign pattern $\mathcal{A} = [a_{ij}]$ if $b_{ij} = a_{ij}$ whenever $a_{ij} \neq 0$. Similarly, $\mathcal{B}$ is a *subpattern* of $\mathcal{A}$ if $b_{ij} = 0$ whenever $a_{ij} = 0$. Note that each sign pattern is a superpattern and a subpattern of itself. An $n \times n$ sign pattern $\mathcal{A}$ is *spectrally arbitrary* if for each real monic polynomial $r(x)$ of degree n, there exists some $A \in \mathcal{A}$ with characteristic polynomial $p_A(x) = r(x)$. Thus, $\mathcal{A}$ is spectrally arbitrary if given any self-conjugate spectrum, there exists $A \in \mathcal{A}$ with that spectrum. A sign pattern $\mathcal{A}$ is *minimally spectrally arbitrary* if it is spectrally arbitrary but is not spectrally arbitrary if any nonzero entry (or entries) of $\mathcal{A}$ is replaced by zero. If $\mathcal{A}$ is an $n \times n$ sign pattern or zero-nonzero pattern, then $\mathcal{A}$ *allows nilpotency* if there exists some $A \in \mathcal{A}$ with characteristic polynomial $p_A(x) = x^n$. Note that each spectrally arbitrary sign pattern must allow nilpotency, must be inertially arbitrary (as explained below Theorem 2.5), and must also be potentially stable. These are three important sign pattern problems that are considered in the literature (see, for example, [1, 3, 4, 5, 7, 8, 9]).

In [8, Theorem 2.6], it is proved that a *p-striped sign pattern*, that is an $n \times n$ $(n \geq 2)$ sign pattern having p $(1 \leq p \leq n-1)$ columns all of whose entries are positive and $n-p$ columns all of whose entries are negative, is spectrally arbitrary. The proof is based on constructions using a Soules matrix, and gives (as far as we are aware) the first spectrally arbitrary sign pattern family for all $n \geq 2$.

Each p-striped sign pattern is full, and current interest is in determining minimal spectrally arbitrary patterns. In Section 2, an $n \times n$ $(n \geq 3)$ irreducible sign pattern $\mathcal{V}_n$ is presented and proved to be minimally spectrally arbitrary. To our knowledge, no

*This work was supported through Discovery Grants from the Natural Sciences and Engineering Research Council of Canada. T. Britz was also supported by a PIMS Postdoctoral Fellowship.

[†]Department of Mathematics and Statistics, University of Victoria, BC V8W 3P4, Canada (britz@math.uvic.ca, pvdd@math.uvic.ca).

[‡]Mathematics Department, Washington State University, Pullman, WA 99164-3113, U.S.A. (jmcdonald@math.wsu.edu).

[§]Department of Computer Science, University of Victoria, BC V8W 3P6, Canada (dolesky@cs.uvic.ca).

such family of minimal spectrally arbitrary sign patterns has been presented previously. Each of these sign patterns is a Hessenberg matrix, and all superpatterns of these sign patterns are shown to be spectrally arbitrary. This strengthens results in [5].

In Section 3, the family of sign patterns $\mathcal{V}_n$ is extended to a larger family of $n \times n$ irreducible sign patterns $\mathcal{W}_n(k)$ with each superpattern shown to be spectrally arbitrary. This provides an alternate proof that every p-striped pattern is spectrally arbitrary [8]. The sign pattern $\mathcal{W}_n(k)$ is not necessarily minimally spectrally arbitrary. However, the minimal spectrally arbitrary sign patterns that are contained in $\mathcal{W}_n(k)$ are characterized.

The family of sign patterns $\mathcal{V}_n$ is generalized in another way in Section 4 by introducing a family of zero-nonzero patterns $\mathcal{V}_n^*(I)$. It is shown that if $\mathcal{V}_n^*(I)$ allows nilpotency, then $\mathcal{V}_n^*(I)$ determines an $n \times n$ irreducible sign pattern $\mathcal{V}_n(I)$ that is minimally spectrally arbitrary with each superpattern being spectrally arbitrary. Two families of irreducible minimal spectrally arbitrary patterns that arise in this manner are described.

In this paper, two sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{B}$ may be obtained from $\mathcal{A}$ by some combination of negation, transposition, permutation similarity, and signature similarity. Note that if $\mathcal{A}$ and $\mathcal{B}$ are equivalent, then $\mathcal{A}$ is spectrally arbitrary if and only if $\mathcal{B}$ is spectrally arbitrary. In Section 5, the family of spectrally arbitrary 3×3 sign patterns is characterized explicitly (up to equivalence).

In the concluding Section 6, it is proved that any $n \times n$ irreducible spectrally arbitrary sign pattern must contain at least $2n-1$ nonzero entries. It is conjectured that it must in fact contain at least $2n$ nonzero entries.

2. Hessenberg sign patterns $\mathcal{V}_n$. Results throughout rely heavily upon the following lemma, which is stated as Observations 10 and 15 in [1] and is proved using the Implicit Function Theorem. Let $x_1, \ldots, x_n$ be real variables, and for each $i = 1, \ldots, n$, let $\alpha_i = \alpha_i(x_1, \ldots, x_n)$ be a real function of $(x_1, \ldots, x_n)$ that is continuous and differentiable in each x_j. The Jacobian $J = \frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(x_1, \ldots, x_n)}$ is the $n \times n$ matrix with (i,j) entry equal to $\frac{\partial \alpha_i}{\partial x_j}$ for $1 \le i, j \le n$.

LEMMA 2.1. *[1] Let $\mathcal{A}$ be an $n \times n$ sign pattern, and suppose that there exists some nilpotent $A \in \mathcal{A}$ with at least n nonzero entries, say $a_{i_1 j_1}, \ldots, a_{i_n j_n}$. Let X be the matrix obtained by replacing these entries in A by variables $x_1, \ldots, x_n$, and let*

$$p_X(x) = x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots + (-1)^{n-1}\alpha_{n-1}x + (-1)^n \alpha_n.$$

If $J = \frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(x_1, \ldots, x_n)}$ is nonsingular at $(x_1, \ldots, x_n) = (a_{i_1 j_1}, \ldots, a_{i_n j_n})$, then every superpattern of $\mathcal{A}$ is spectrally arbitrary.

EXAMPLE 2.2.
Let $\mathcal{A} = \begin{bmatrix} + & - \\ + & - \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \in \mathcal{A}$ is nilpotent. Let $X = \begin{bmatrix} x_1 & -1 \\ 1 & x_2 \end{bmatrix}$. Then

$$p_X(x) = x^2 - \alpha_1 x + \alpha_2,$$

where $\alpha_1 = x_1 + x_2$ and $\alpha_2 = x_1 x_2 + 1$. Thus

$$J = \frac{\partial(\alpha_1, \alpha_2)}{\partial(x_1, x_2)} = \begin{bmatrix} 1 & 1 \\ x_2 & x_1 \end{bmatrix}, \quad \text{and} \quad \det J = x_1 - x_2.$$

At $(x_1, x_2) = (1, -1)$, $\det J = 2 \neq 0$. By Lemma 2.1, $\mathcal{A}$ is spectrally arbitrary, and it is easily seen that it is minimal. Note that up to equivalence, $\mathcal{A}$ is the unique (minimal) spectrally arbitrary 2×2 sign pattern.

Given a sign pattern $\mathcal{A}$, let $D(\mathcal{A})$ be its associated digraph. For any digraph D, let $G(D)$ denote the underlying multigraph of D, i.e., the graph obtained from D by ignoring the direction of each arc. The following lemma is well known and can be proved by induction. We use this to normalize an $n \times n$ matrix $A \in \mathcal{A}$ by fixing up to $n-1$ entries to have magnitude 1.

LEMMA 2.3. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and let $A \in \mathcal{A}$. If T is a subdigraph of $D(\mathcal{A})$ such that $G(T)$ is a forest, then $\mathcal{A}$ has a realization that is positive diagonally similar to A such that each entry corresponding to an arc of T has magnitude 1. In particular, if $\mathcal{A}$ is irreducible, then $G(D(\mathcal{A}))$ contains a spanning tree, and $\mathcal{A}$ must therefore have a realization with at least $n-1$ off-diagonal entries in $\{-1, 1\}$ that is positive diagonally similar to A.*

Let $n \geq 3$, and consider the $n \times n$ Hessenberg sign pattern

$$\mathcal{V}_n = \begin{bmatrix} + & - & & & & \\ + & & - & & \text{\Large 0} & \\ \vdots & & & \ddots & & \\ + & & & & - & \\ + & & \text{\Large 0} & & & - \\ + & & & & & - \end{bmatrix}.$$

THEOREM 2.4. *For $n \geq 3$, the pattern $\mathcal{V}_n$ is a minimal spectrally arbitrary pattern.*

Proof. Let

$$r(x) = x^n - r_1 x^{n-1} + r_2 x^{n-2} - \cdots + (-1)^{n-1} r_{n-1} x + (-1)^n r_n$$

be a fixed but arbitrary real monic polynomial of degree n. Let

$$A = \begin{bmatrix} a_1 & -1 & & & & \\ a_2 & & -1 & & \text{\Large 0} & \\ \vdots & & & \ddots & & \\ a_{n-2} & & & & -1 & \\ a_{n-1} & & \text{\Large 0} & & & -1 \\ a_n & & & & & -t \end{bmatrix}.$$

The characteristic polynomial of A is

$$p_A(x) = x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots + (-1)^{n-1} \alpha_{n-1} x + (-1)^n \alpha_n,$$

where each coefficient α_i is the sum of the $i \times i$ principal minors of A; thus $\alpha_1 = a_1 - t$ and $\alpha_i = a_i - t a_{i-1}$ for $i = 2, \ldots, n$. Set $a_1 = r_1 + t$. For each $i = 2, \ldots, n$, set

$$a_i = t^i + \sum_{j=1}^{i} r_j t^{i-j}.$$

Then $\alpha_1 = a_1 - t = r_1 + t - t = r_1$, and for $i = 2, \ldots, n$,

$$\alpha_i = a_i - t a_{i-1} = \left(t^i + \sum_{j=1}^{i} r_j t^{i-j} \right) - t \left(t^{i-1} + \sum_{j=1}^{i-1} r_j t^{i-1-j} \right) = r_i.$$

Thus, $\alpha_i = r_i$ for all $i = 1, \ldots, n$, i.e., $p_A(x) = r(x)$. For all $t > 0$ sufficiently large, each $a_j > 0$ $(1 \le j \le n)$ and thus $A \in \mathcal{V}_n$. Hence, $\mathcal{V}_n$ is spectrally arbitrary.

By Lemma 2.3, each matrix with sign pattern $\mathcal{V}_n$ is positive diagonally similar to a matrix A in the above form. If one of the -1 entries in columns $2, \ldots, n-1$ of A is replaced by zero, then the resulting matrix is necessarily singular. Similarly, if $t = 0$ or the -1 entry in column n of A is replaced by zero, then the resulting matrix is necessarily nonsingular. If $a_i = 0$ for some $1 \le i \le n$, then $\alpha_i \le 0$. Thus, $\mathcal{V}_n$ is minimally spectrally arbitrary. □

Set $t = 1$ in the matrix A from the above proof. If $a_1 = \cdots = a_n = 1$, then A is nilpotent. The Jacobian $J = \frac{\partial(\alpha_1,\ldots,\alpha_n)}{\partial(a_1,\ldots,a_n)}$ has 1 in each diagonal position, $-t = -1$ in each subdiagonal position, and zeros elsewhere. Thus, $\det J = 1 \ne 0$. Hence, the theorem below follows from Lemma 2.1.

THEOREM 2.5. *For $n \ge 3$, any superpattern of $\mathcal{V}_n$ is a spectrally arbitrary pattern.*

An $n \times n$ sign pattern $\mathcal{A}$ is *inertially arbitrary* if given a nonnegative triple of integers (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = n$, there exists some $A \in \mathcal{A}$ that has n_1 eigenvalues with positive real part, n_2 eigenvalues with negative real part, and n_3 eigenvalues with zero real part. Note that if a sign pattern is spectrally arbitrary, then it is also inertially arbitrary. Recently, several families of sign patterns have been shown to be inertially arbitrary (see [5, 8, 9]). The sign patterns described in [5] are superpatterns of the pattern $\mathcal{V}_n$. It follows from Theorem 2.5 that these sign patterns are not only inertially arbitrary but are indeed spectrally arbitrary.

3. Non-Hessenberg sign patterns $\mathcal{W}_n(k)$. We now define a general class of $n \times n$ sign patterns that includes the Hessenberg patterns $\mathcal{V}_n$. Let $n \ge 3$ and $0 \le k \le n-2$ be given. Define $\mathcal{W}_n(k)$ to be the $n \times n$ sign pattern with positive signs throughout the first column and in the entries

$$\{(j, j+1) \;:\; j = 1, \ldots, k\};$$

negative signs in the entries

$$\{(j, j+1) \;:\; j = k+1, \ldots, n-1\},\ \{(j, n) \;:\; j = 1, \ldots, k\}, \quad \text{and} \quad (n, n);$$

and zeros elsewhere. For $k \ge 1$, let $W_n(k) \in \mathcal{W}_n(k)$ have values $a_1, \ldots, a_n$ in column 1; $-b_1, \ldots, -b_k$ in the first k entries of column n; $-b_n$ in the (n, n) entry; and all entries on the superdiagonal have magnitude 1. For example, the sign pattern $\mathcal{W}_7(3)$ and a realization $W_7(3)$ are

$$\begin{bmatrix} + & + & 0 & 0 & 0 & 0 & - \\ + & 0 & + & 0 & 0 & 0 & - \\ + & 0 & 0 & + & 0 & 0 & - \\ + & 0 & 0 & 0 & - & 0 & 0 \\ + & 0 & 0 & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & 0 & 0 & - \\ + & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & -b_1 \\ a_2 & 0 & 1 & 0 & 0 & 0 & -b_2 \\ a_3 & 0 & 0 & 1 & 0 & 0 & -b_3 \\ a_4 & 0 & 0 & 0 & -1 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & -1 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & -1 \\ a_7 & 0 & 0 & 0 & 0 & 0 & -b_7 \end{bmatrix},$$

respectively. Then matrix $W_n(k) \in \mathcal{W}_n(k)$ has characteristic polynomial

$$x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots + (-1)^{n-1}\alpha_{n-1} x + (-1)^n \alpha_n,$$

where

$$\begin{aligned}\alpha_1 &= a_1 - b_n\\ \alpha_i &= (-1)^{i-1}(a_i + a_{i-1}b_n - b_{i-1}a_n) && \text{for } i = 2, \ldots, k+1\\ \alpha_i &= (-1)^k(a_i - a_{i-1}b_n) && \text{for } i = k+2, \ldots, n.\end{aligned}$$

PROPOSITION 3.1. *For each pair $n \geq 3$ and $0 \leq k \leq n-2$, the pattern $\mathcal{W}_n(k)$ is a spectrally arbitrary pattern, and every superpattern of $\mathcal{W}_n(k)$ is spectrally arbitrary.*

Proof. Since the patterns $\mathcal{W}_n(0)$ are the Hessenberg patterns $\mathcal{V}_n$, the result for $k = 0$ follows from Theorem 2.5.

Let $1 \leq k \leq n-2$ be given. Note that $W_n(k)$ is nilpotent if $a_1 = \cdots = a_n = b_n = 1$ and $b_1 = \cdots = b_k = 2$. Now set $b_1 = \cdots = b_k = 2$, and $b_n = 1$, leaving $a_1, \ldots, a_n$ as variables. Then the terms of the characteristic polynomial of $W_n(k)$ are

$$\begin{aligned}\alpha_1 &= a_1 - 1\\ \alpha_i &= (-1)^{i-1}(a_i + a_{i-1} - 2a_n) && \text{for } i = 2, \ldots, k+1\\ \alpha_i &= (-1)^k(a_i - a_{i-1}) && \text{for } i = k+2, \ldots, n.\end{aligned}$$

The Jacobian $J = \frac{\partial(\alpha_1,\ldots,\alpha_n)}{\partial(a_1,\ldots,a_n)}$ is a matrix with ± 1 entries on the main diagonal and on the subdiagonal, and (i, n) entries equal to $(-1)^i 2$ for $i = 2, \ldots, k+1$. Thus, $\det J$ is a (positive or negative) odd integer, and the result follows from Lemma 2.1. □

COROLLARY 3.2. *[8, Theorem 2.6] For $n \geq 2$, every $n \times n$ p-striped sign pattern is spectrally arbitrary.*

Proof. The case $n = 2$ is proved in Example 2.2. Suppose that $n \geq 3$, and consider the $n \times n$ p-striped sign pattern with precisely $p = k + 1 \leq n - 1$ positive columns for some $k \geq 0$. By permutation similarity, it may be assumed that the first $k + 1$ columns are positive. This p-striped sign pattern is a superpattern of $\mathcal{W}_n(k)$, and the result follows by Proposition 3.1. □

If $k = 0$, then $\mathcal{W}_n(0) = \mathcal{V}_n$ is a minimal spectrally arbitrary pattern. For $k = 1$, $\mathcal{W}_n(1)$ is minimally spectrally arbitrary, since at least one of the coefficients α_i has fixed sign if any of the variables $a_1, \ldots, a_n, b_1, b_n$ are set to zero. This is not necessarily true for values $k \geq 2$. For such k, let $\mathcal{I}_k$ denote the family of subsets $I \subseteq \{2, \ldots, k\}$ such that I does not contain two consecutive integers $i, i+1$, and $\{1, \ldots, k+1\}\backslash I$ does not contain three consecutive integers $i, i+1, i+2$. Note that the set of all even integers and the set of all odd integers in $\{2, \ldots, k\}$ both are members of $\mathcal{I}_k$. For $I \in \mathcal{I}_k$, set $a_i = 0$ for each $i \in I$, and let the resulting sign pattern and matrix be denoted by $\mathcal{W}_n^I(k)$ and $W_n^I(k)$, respectively.

THEOREM 3.3. *For each pair $n \geq 4$ and $2 \leq k \leq n-2$, the family of minimal spectrally arbitrary subpatterns of $\mathcal{W}_n(k)$ consists of the patterns $\mathcal{W}_n^I(k)$, where $I \in \mathcal{I}_k$. Furthermore, any superpattern of these patterns is spectrally arbitrary.*

Proof. Let $I \in \mathcal{I}_k$, and set $a_n = b_n = 1$ in $W_n^I(k)$. Then $W_n^I(k)$ is nilpotent if and only if the following coefficients all equal 0:

$$\begin{aligned}\alpha_1 &= a_1 - 1\\ \alpha_i &= (-1)^{i-1}(a_{i-1} - b_{i-1}) && \text{for } i \in I\\ \alpha_i &= (-1)^{i-1}(a_i - b_{i-1}) && \text{for } i-1 \in I,\ i \in \{2, \ldots, k+1\}\backslash I\\ \alpha_i &= (-1)^{i-1}(a_i + a_{i-1} - b_{i-1}) && \text{for } i-1, i \in \{1, \ldots, k+1\}\backslash I\\ \alpha_i &= (-1)^k(a_i - a_{i-1}) && \text{for } i \in \{k+2, \ldots, n\}.\end{aligned}$$

Note that $W_n^I(k)$ is nilpotent if $a_i = 1$ for all variables a_i appearing in the equations above, and for each $i \in \{2, \dots, k+1\}$, the variables $b_{i-1} = 2$ if both i and $i+1$ are contained in $\{1, \dots, k+1\} \backslash I$, and $b_{i-1} = 1$ otherwise. The Jacobian $J = \frac{\partial(\alpha_1, \dots, \alpha_n)}{\partial(a_1, b_1, \dots, b_k, a_{k+1}, \dots, a_{n-1})}$ is the direct sum of a lower-triangular $k \times k$ matrix and an upper-triangular $(n-k) \times (n-k)$ matrix, with ± 1 entries on the main diagonal. The determinant of J has magnitude 1, so J is nonsingular. By Lemma 2.1, $\mathcal{W}_n^I(k)$ is spectrally arbitrary, and each superpattern of $\mathcal{W}_n^I(k)$ is also spectrally arbitrary. By the definition of $\mathcal{I}_k$, if any variable a_i, where $i \in \{2, \dots, k\} \backslash I$, is set to 0, then either a_{i-1} or a_{i+1} also equals 0, and the sign of α_i or α_{i+1} is fixed. Thus, $\mathcal{W}_n^I(k)$ is a minimal spectrally arbitrary sign pattern.

Suppose that $\mathcal{W}$ is a minimal spectrally arbitrary subpattern of $\mathcal{W}_n(k)$ with realization W obtained by setting some of the variables in $W_n(k)$ to 0. Since $\mathcal{W}$ is spectrally arbitrary, no coefficient α_i has fixed sign. Thus, none of the variables $a_1, a_{k+1}, \dots, a_n, b_1, \dots, b_k, b_n$ equals 0. Furthermore, no two consecutive variables a_{i-1} and a_i can both equal zero. Suppose that $i, i+1, i+2$ are three consecutive integers contained in $\{1, \dots, k+1\}$ such that $a_i, a_{i+1}, a_{i+2} \neq 0$. If the entry a_{i+1} is replaced by a zero, then the resulting sign pattern is also spectrally arbitrary, contradicting the minimality of $\mathcal{W}$. It follows that $\mathcal{W} = \mathcal{W}_n^I(k)$, where $I = \{i : 2 \leq i \leq k, a_i = 0\}$. □

4. Sign patterns $\mathcal{V}_n(I)$. For $n \geq 3$, consider the matrix

$$A = \begin{bmatrix} a_0 & -1 & & & & & \\ a_1 & & -1 & & & \text{\Large 0} & \\ \vdots & & & \ddots & & & \\ \vdots & & \text{\Large 0} & & \ddots & & \\ a_{n-3} & & & & & -1 & \\ a_{n-2} & & & & & & -1 \\ a_{n-1} & b_{n-2} & b_{n-3} & \cdots & \cdots & b_1 & b_0 \end{bmatrix}, \tag{4.1}$$

where the entries a_0, b_0, and a_{n-1} are nonzero, and precisely one of a_i and b_i for each $i = 1, \dots, n-2$ is nonzero. The zero-nonzero pattern determined by A is denoted by $\mathcal{V}_n^*(I)$, where $I = \{i : a_i = 0\}$. The matrix A has characteristic polynomial

$$p_A(x) = x^n - \alpha_0 x^{n-1} + \alpha_1 x^{n-2} - \cdots + (-1)^{n-1}\alpha_{n-2}x + (-1)^n \alpha_{n-1},$$

where (on computing the sum of the principal minors of each order)

$$\alpha_0 = a_0 + b_0,$$

$$\alpha_i = a_i + b_i + \sum_{j=0}^{i-1} a_j b_{i-1-j} \quad \text{for} \quad i = 1, \dots, n-2,$$

$$\text{and} \quad \alpha_{n-1} = a_{n-1} + \sum_{j=0}^{n-2} a_j b_{n-2-j}.$$

Define $s_i = a_i + b_i$ for $i = 0, \cdots, n-2$ and $s_{n-1} = a_{n-1}$. Since $\frac{\partial \alpha_i}{\partial s_j}$ is zero whenever $j > i$, the Jacobian $J = \frac{\partial(\alpha_0, \dots, \alpha_{n-1})}{\partial(s_0, \dots, s_{n-1})}$ is lower triangular. The diagonal entries $\frac{\partial \alpha_i}{\partial s_i}$ each equal 1, so the Jacobian has determinant 1 and is therefore nonsingular.

For nilpotency to hold, each coefficient α_i for $i = 0, \ldots, n-1$ must vanish, i.e.,

$$\begin{aligned}
0 &= a_0 + b_0 \\
0 &= a_1 + b_1 + a_0 b_0 \\
0 &= a_2 + b_2 + a_0 b_1 + a_1 b_0 \\
&\vdots \\
0 &= a_{n-2} + b_{n-2} + a_0 b_{n-3} + a_1 b_{n-4} + \cdots + a_{n-3} b_0 \\
0 &= a_{n-1} + a_0 b_{n-2} + a_1 b_{n-3} + \cdots + a_{n-2} b_0
\end{aligned} \tag{4.2}$$

For any fixed I, an induction argument shows that there exist constants $c_0, \ldots, c_{n-1}$, $d_0, \ldots, d_{n-2}, t$ such that the parameters in (4.1) for any nilpotent $A \in V_n^*(I)$ satisfy $a_i = c_i t^{i+1}$ for $i = 0, \ldots, n-1$ and $b_i = d_i t^{i+1}$ for $i = 0, \ldots, n-2$. If c_0 and t are positive, then the sign of a_i for $i = 0, \ldots, n-1$ and the sign of b_i for $i = 0, \ldots, n-2$ are uniquely determined. Thus, if $\mathcal{V}_n^*(I)$ allows nilpotency, then this determines uniquely a sign pattern with a positive $(1,1)$ entry and negative superdiagonal, denoted $\mathcal{V}_n(I)$, which allows nilpotency. By Lemma 2.1, $\mathcal{V}_n(I)$ is a spectrally arbitrary pattern, and each superpattern of $\mathcal{V}_n(I)$ is spectrally arbitrary. To show minimality, first note that if $a_0 = 0$ or $b_0 = 0$, then $a_i = b_i = 0$ for all $i = 0, \ldots, n-2$ and $a_{n-1} = 0$, thus A is clearly not spectrally arbitrary. Thus $a_0 b_0 \neq 0$. If $a_1 = b_1 = 0$, then $a_0 b_0 = 0$ from the second equation in (4.2), contradicting the above. Proceeding similarly, it is not difficult to show that $\mathcal{V}_n(I)$ is an irreducible minimal spectrally arbitrary pattern. The preceding discussion gives the following result.

LEMMA 4.1. *If $\mathcal{V}_n^*(I)$ allows nilpotency, then $\mathcal{V}_n(I)$ exists and is minimally spectrally arbitrary, and each superpattern of $\mathcal{V}_n(I)$ is spectrally arbitrary.*

Note that $\mathcal{V}_n^*(\phi)$ allows nilpotency (let $a_0 = \cdots = a_{n-1} = 1$ and $b_0 = -1$), and that $\mathcal{V}_n(\phi) = \mathcal{V}_n$.

LEMMA 4.2. *For $I \subseteq \{1, \ldots, n-2\}$, let $I^C = \{1, \ldots, n-2\} \backslash I$. Then $\mathcal{V}_n^*(I)$ allows nilpotency if and only if $\mathcal{V}_n^*(I^C)$ allows nilpotency. Also, if $\mathcal{V}_n^*(I)$ allows nilpotency, then $\mathcal{V}_{n'}^*(I')$ allows nilpotency for all $3 \leq n' \leq n$, where $I' = \{i \in I : i \leq n' - 2\}$.*

Proof. Note that $\mathcal{V}_n^*(I)$ and $\mathcal{V}_n^*(I^C)$ are equivalent by transposition and permutation similarity. This proves the first statement of the lemma. If $\mathcal{V}_n^*(I)$ allows nilpotency, then equations (4.2) are satisfied by some $A \in \mathcal{V}_n^*(I)$. In particular, the first n' equations are satisfied, so $\mathcal{V}_{n'}^*(I')$ also allows nilpotency. ∎

There are a large number of spectrally arbitrary patterns arising from patterns $\mathcal{V}_n^*(I)$ but they do not generally seem to fall into easily described categories. Numerical evidence suggests that for $n \geq 4$, precisely $2^{n-3} + 2$ of the 2^{n-2} patterns $\mathcal{V}_n^*(I)$ allow nilpotency. The following theorems with $I = \{k\}$ and $I = \{i : 1 \leq i \leq n-2 \text{ is odd}\}$, respectively, describe two classes, $\mathcal{V}_n(k) = \mathcal{V}_n(\{k\})$ and $\mathcal{V}_n^{\text{alt}}$, of minimal spectrally arbitrary sign patterns arising from $\mathcal{V}_n^*(I)$.

Let $n \geq 3$ and $1 \leq k \leq n-2$ be given and define $\mathcal{V}_{n,k}$ to be the $n \times n$ sign pattern with negative signs in the entries

$$\{(j, j+1) : j = 1, \ldots, n-1\}, \; \{(j, 1) : j = k+2, \ldots, n\}, \text{ and } (n, n);$$

positive signs in the entries

$$\{(j, 1) : j = 1, \ldots, k\} \quad \text{and} \quad (n, n-k);$$

and zeros elsewhere. Note that $\mathcal{V}_{n,k}$ has the same zero-nonzero pattern as $\mathcal{V}_n^*(k)$. To illustrate,

$$\mathcal{V}_{5,1} = \begin{bmatrix} + & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & - \\ - & 0 & 0 & + & - \end{bmatrix} \quad \text{and} \quad \mathcal{V}_{5,2} = \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & - \\ - & 0 & + & 0 & - \end{bmatrix}.$$

THEOREM 4.3. *Let $k \geq 1$ and $k+2 \leq n < 2k + \frac{1}{2}(\sqrt{1+8k}+3)$ be given. Then $\mathcal{V}_n(k)$ exists and is identical to $\mathcal{V}_{n,k}$. Furthermore, it is a minimal spectrally arbitrary pattern, and any superpattern of $\mathcal{V}_n(k)$ is spectrally arbitrary.*

Proof. Let A be as in (4.1), and set

$$a_i = \begin{cases} 1 & \text{for } i = 0, \dots, k-1 \\ k - i & \text{for } i = k, \dots, 2k \\ \frac{1}{2}(i^2 - i) + 2(k^2 - ik) & \text{for } i = 2k+1, \dots, n-1 \end{cases}$$

and

$$b_i = \begin{cases} -1 & \text{for } i = 0 \\ 1 & \text{for } i = k \\ 0 & \text{for otherwise}. \end{cases}$$

The polynomial $\frac{1}{2}(x^2 - x) + 2(k^2 - xk)$ has roots $2k + \frac{1}{2} \pm \frac{1}{2}\sqrt{1+8k}$. Thus, the inequality

$$n - 1 < 2k + \frac{1}{2} + \frac{1}{2}\sqrt{1+8k}$$

implies that $\frac{1}{2}(i^2 - i) + 2(k^2 - ik) < 0$ for all $2k+1 \leq i \leq n-1$. Hence, $A \in \mathcal{V}_n^*(k)$ and $A \in \mathcal{V}_{n,k}$. To prove Theorem 4.3, it suffices, by Lemma 4.1, to show that A is nilpotent, i.e., that the entries of A satisfy equations (4.2). Certainly, $a_0 + b_0 = 1 - 1 = 0$ and

$$a_i + b_i + a_0 b_{i-1} + \cdots + a_{i-1} b_0 = a_i + a_{i-1} b_0 = 1 - 1 = 0$$

for all $i = 1, \dots, k-1$. Also,

$$a_k + b_k + a_0 b_{k-1} + \cdots + a_{k-1} b_0 = b_k + a_{k-1} b_0 = 1 - 1 = 0.$$

Since $b_0 = -1$ and $b_j = 0$ for $j = k+1, \dots, n-2$, on letting $b_{n-1} = 0$, the remaining equations have the form

$$0 = a_i + b_i + a_0 b_{i-1} + \cdots + a_{i-1} b_0 = a_i + a_{i-1-k} - a_{i-1},$$

where $k+1 \leq i \leq n-1$. For $k+1 \leq i \leq \min\{2k, n-1\}$,

$$a_i + a_{i-(k+1)} - a_{i-1} = k - i + 1 - (k - i + 1) = 0.$$

If $n - 1 \leq 2k$, then the proof is concluded. Suppose that $n \geq 2k+2$. The inequality

$$n < 2k + 2 + \frac{1}{2}(\sqrt{1+8k} - 1) \leq 3k + 2,$$

implies that $n-1 \leq 3k$. Thus

$$a_i + a_{i-1-k} - a_{i-1} = \frac{1}{2}(i^2 - i) + 2(k^2 - ik) + k - (i-1-k)$$
$$-(\frac{1}{2}((i-1)^2 - (i-1)) + 2(k^2 - (i-1)k)) = 0$$

for all $i = 2k+1, \ldots, n-1$. This concludes the proof. □

To illustrate Theorem 4.3, consider the case $k = 1$. Since

$$2k + \frac{1}{2}(\sqrt{1+8k} + 3) = 5,$$

it follows from Theorem 4.3 that $\mathcal{V}_3(1)$ and $\mathcal{V}_4(1)$ exist and are minimal spectrally arbitrary patterns such that all of their superpatterns are spectrally arbitrary patterns. Since $\mathcal{V}_5^*(1)$ does not allow nilpotency, $\mathcal{V}_5(1)$ does not exist. On the other hand, $5 < 4 + \frac{1}{2}(\sqrt{17} + 3)$, so $\mathcal{V}_5(2)$ exists and is equal to $\mathcal{V}_{5,2}$. Thus for $n = 5$, $\mathcal{V}_5(k)$ exists if and only if the inequality in Theorem 4.3 holds. In general, this is not true. For instance, the pattern $\mathcal{V}_8^*(2)$ allows nilpotency, as demonstrated by

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \in \mathcal{V}_8^*(2).$$

However, $n \not< 2k + \frac{1}{2}(\sqrt{1+8k} + 3)$ for $n = 8$ and $k = 2$. Note also that the above sign pattern is not equal to $\mathcal{V}_{8,2}$.

A second class of sign patterns arising from patterns $\mathcal{V}_n^*(I)$ is as follows. Let $n \geq 4$, and let $\mathcal{V}_n^{\text{alt}}$ be the $n \times n$ sign pattern with positive signs in the positions

$$\{(4j+1, 1): \ 0 \leq j \leq \left\lfloor \frac{n-1}{4} \right\rfloor\} \ \text{ and } \{(n, n-(4j+1)): \ 0 \leq j \leq \left\lfloor \frac{n-2}{4} \right\rfloor\};$$

negative signs in the positions

$$\{(j, j+1): \ 1 \leq j \leq n-1\},$$
$$\{(4j+3, 1): \ 0 \leq j \leq \left\lfloor \frac{n-3}{4} \right\rfloor\},$$
$$\{(n, n-(4j+3)): \ 0 \leq j \leq \left\lfloor \frac{n-4}{4} \right\rfloor\}, \ \text{ and } (n, n);$$

and zeros elsewhere. To illustrate,

$$\mathcal{V}_7^{\text{alt}} = \begin{bmatrix} + & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 \\ + & 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & - & 0 & + & - \end{bmatrix}, \quad \mathcal{V}_8^{\text{alt}} = \begin{bmatrix} + & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & - \\ - & 0 & + & 0 & - & 0 & + & - \end{bmatrix}.$$

THEOREM 4.4. *For $n \geq 4$, let I consist of all odd integers $i \leq n-2$. Then $\mathcal{V}_n(I)$ exists and is identical to $\mathcal{V}_n^{alt}$. Furthermore, it is a minimal spectrally arbitrary pattern, and any superpattern of $\mathcal{V}_n(I)$ is spectrally arbitrary.*

Proof. Let A be as in (4.1), and assume that n is odd. Let $a_i = b_j = 0$ for all odd $i \leq n-2$ and all even j such that $2 \leq j \leq n-2$, let $b_0 = -1$, and define $b_{n-1} = 0$. For all $0 \leq i \leq \frac{n-1}{2}$, let $a_{2i} = (-1)^i C_i$, where $C_i = \frac{1}{i+1}\binom{2i}{i}$ is the ith Catalan number (see, for example, [10, 11], and note that $C_0 = 1$). Also, let $b_{2i+1} = a_{2i} = (-1)^i C_i$ for all $0 \leq i \leq \frac{n-3}{2}$. Then $A \in \mathcal{V}_n^*(I)$ and $A \in \mathcal{V}_n^{\text{alt}}$. To conclude the proof, it is sufficient, by Lemma 4.1, to show that A is nilpotent. Certainly, $a_0 + b_0 = 0$. For each $i \geq 0$, the Catalan number C_{i+1} satisfies the recursive identity

$$C_{i+1} = \sum_{j=0}^{i} C_j C_{i-j}.$$

(see [11, p. 117]). Thus, for $0 \leq i \leq \frac{n-3}{2}$,

$$a_{2i+1} + b_{2i+1} + a_{2i}b_0 + \cdots + a_0 b_{2i} = b_{2i+1} - a_{2i} = 0$$

and

$$a_{2i+2} + b_{2i+2} + a_{2i+1}b_0 + \cdots + a_0 b_{2i+1} = a_{2i+2} + \sum_{j=0}^{i} a_{2j}a_{2(i-j)} =$$

$$(-1)^{i+1}C_{i+1} + \sum_{j=0}^{i}(-1)^j C_j (-1)^{i-j} C_{i-j} = (-1)^i(-C_{i+1} + \sum_{j=0}^{i} C_j C_{i-j}) = 0.$$

The equations (4.2) are all satisfied, so A is nilpotent.

Assume that n is even. Let $a_i = b_j = 0$ for all odd $i \leq n-2$ and all even j such that $2 \leq j \leq n-2$, and let $b_0 = -1$. For all $0 \leq i \leq \frac{n-2}{2}$, let $a_{2i} = (-1)^i C_i$. Let $a_{n-1} = a_{n-2}$, and let $b_{2i+1} = a_{2i} = (-1)^i C_i$ for all $0 \leq i \leq \frac{n-4}{2}$. Then $A \in \mathcal{V}_n^*(I)$ and $A \in \mathcal{V}_n^{\text{alt}}$. To conclude the proof, it is sufficient, by Lemma 4.1, to show that A is nilpotent. Certainly, $a_0 + b_0 = 0$. For $0 \leq i \leq \frac{n-4}{2}$,

$$a_{2i+1} + b_{2i+1} + a_{2i}b_0 + \cdots + a_0 b_{2i} = b_{2i+1} - a_{2i} = 0$$

and

$$a_{2i+2} + b_{2i+2} + a_{2i+1}b_0 + \cdots + a_0 b_{2i+1} = a_{2i+2} + \sum_{j=0}^{i} a_{2j}a_{2(i-j)} =$$

$$(-1)^{i+1}C_{i+1} + \sum_{j=0}^{i}(-1)^j C_j (-1)^{i-j} C_{i-j} = (-1)^i(-C_{i+1} + \sum_{j=0}^{i} C_j C_{i-j}) = 0.$$

Furthermore,

$$a_{n-1} + a_{n-2}b_0 + \cdots + a_0 b_{n-2} = a_{n-1} - a_{n-2} = 0.$$

The equations (4.2) are all satisfied, so A is nilpotent. $\square$

5. All minimal 3×3 spectrally arbitrary patterns. In the proof of Theorem 5.2, it will be shown that a 3×3 irreducible sign pattern is spectrally arbitrary if and only it allows nilpotency. Our approach to deciding whether or not a 3×3 sign pattern allows nilpotency is more explicit than that in [3, Theorem 4.1]. First, the following lemma is given, which precludes certain 3×3 patterns from allowing nilpotency.

LEMMA 5.1. *Let $\mathcal{A}$ be the sign pattern determined by any $n \times n$ matrix A with nonzero entries a_{ii} for $i = 1, \ldots, n$; $a_{i,i+1}$ for $i = 1, \ldots, n-1$; and a_{n1} (i.e., $D(\mathcal{A})$ is a directed n-cycle with a loop at each vertex). Then $\mathcal{A}$ allows nilpotency if and only if $n = 2$.*

Proof. The characteristic equation of A is

$$0 = \lambda^n - \sum_{i=1}^{n} a_{ii}\lambda^{n-1} + \sum_{1 \le i < j \le n} a_{ii}a_{jj}\lambda^{n-2} - \cdots + (-1)^n \prod_{i=1}^{n} a_{ii} - a_{n1} \prod_{i=1}^{n-1} a_{i,i+1}.$$

If A is to be nilpotent, then

$$\begin{aligned} 0 &= \sum_{i=1}^{n} a_{ii} \\ 0 &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} \\ &\vdots \\ 0 &= \sum_{1 \le i_1 < i_2 < \cdots < i_{n-1} \le n} a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_{n-1} i_{n-1}} \end{aligned}$$

The a_{ii} are roots of the equation $(x - a_{11})(x - a_{22}) \cdots (x - a_{nn}) = 0$, which is

$$x^n + (-1)^n \prod_{i=1}^{n} a_{ii} = 0$$

by the above equations.

If $n = 2$, then this can be satisfied with the two real numbers $\pm\sqrt{|a_{11}a_{22}|}$, i.e., $a_{11} = a$ and $a_{22} = -a$. But for $n \ge 3$, the equation cannot be satisfied for n real values, thus $\mathcal{A}$ does not allow nilpotency. ∎

THEOREM 5.2. *The family of 3×3 minimal spectrally arbitrary sign patterns consists of the sign patterns that are equivalent to one of the patterns $\mathcal{T}_3$, $\mathcal{U}_3$, $\mathcal{V}_3$, and $\mathcal{W}_3$ in Figure 5.1. Furthermore, every 3×3 spectrally arbitrary sign pattern is equivalent to a superpattern of one of these four patterns.*

$$\underset{\mathcal{T}_3}{\begin{bmatrix} + & - & 0 \\ + & 0 & - \\ 0 & + & - \end{bmatrix}} \qquad \underset{\mathcal{U}_3}{\begin{bmatrix} + & - & + \\ + & - & 0 \\ + & 0 & - \end{bmatrix}} \qquad \underset{\mathcal{V}_3}{\begin{bmatrix} + & - & 0 \\ + & 0 & - \\ + & 0 & - \end{bmatrix}} \qquad \underset{\mathcal{W}_3 = \mathcal{W}_3(1)}{\begin{bmatrix} + & + & - \\ + & 0 & - \\ + & 0 & - \end{bmatrix}}$$

FIG. 5.1. *The minimal 3×3 spectrally arbitrary patterns*

Proof. In [1], it is shown that $\mathcal{T}_3$ and $\mathcal{U}_3$ are minimal spectrally arbitrary patterns, and that each superpattern of these two patterns is also spectrally arbitrary. By Theorem 2.5, Proposition 3.1, and the comments following Corollary 3.2, the patterns $\mathcal{V}_3$ and $\mathcal{W}_3$ are both minimal spectrally arbitrary patterns, and each superpattern of these two patterns is also spectrally arbitrary. Since it is easily shown that there are no reducible 3×3 spectrally arbitrary patterns, to conclude the proof, it is necessary to demonstrate that the patterns that are equivalent to these four patterns and their superpatterns are the only (irreducible) 3×3 spectrally arbitrary patterns. This is done by proving that each 3×3 sign pattern not equivalent to any superpattern of $\mathcal{T}_3$, $\mathcal{U}_3$, $\mathcal{V}_3$, or $\mathcal{W}_3$ does not allow nilpotency and, thus, is not a spectrally arbitrary pattern. There are many such sign patterns and a detailed account for each pattern would be quite tedious. Fortunately, this number can be reduced as follows. Up to equivalence, each irreducible 3×3 spectrally arbitrary pattern has one of the following forms

$$\begin{bmatrix} + & \# & \# \\ + & \# & \# \\ 0 & + & - \end{bmatrix} \qquad \begin{bmatrix} + & \# & \# \\ + & \# & \# \\ + & \# & - \end{bmatrix},$$

where each # denotes either a plus, minus, or zero entry. Of $3^4 + 3^5 = 324$ possible sign patterns, 78 are reducible and 115 are equivalent to superpatterns of one or more of the patterns $\mathcal{T}_3$, $\mathcal{U}_3$, $\mathcal{V}_3$, and $\mathcal{W}_3$. Of the remaining 131 patterns, there are 71 patterns $\mathcal{A}$ such that for any matrix $A \in \mathcal{A}$, the characteristic polynomial $p_A(x) = x^3 - \alpha_1 x^2 + \alpha_2 x - \alpha_3$ contains a coefficient α_1, α_2, or α_3 that has a fixed sign, regardless of the specific matrix A. Such patterns cannot allow nilpotency.

The remaining 60 patterns fall into four general classes described below. By Lemma 2.3, it may be assumed that any two of the nonzero strictly upper triangular entries of any given irreducible 3×3 matrix both have magnitude 1.

The first of the four classes consists of the four patterns

$$\begin{bmatrix} + & 0 & - \\ + & - & 0 \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ + & - & 0 \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ + & + & 0 \\ 0 & + & - \end{bmatrix}, \text{ and } \begin{bmatrix} + & 0 & + \\ + & + & 0 \\ 0 & + & - \end{bmatrix}.$$

By Lemma 5.1, such sign patterns do not allow nilpotency.

For the second class, consider

$$\mathcal{A} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & - \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & -j \end{bmatrix} \in \mathcal{A}.$$

The matrix A has the characteristic polynomial

$$\begin{aligned} p_A(x) = {} & x^3 + (j - a - e)x^2 + (ae - aj - bd - cg - ej - fh)x \\ & + aej + ahf - bdj - bgf - cdh + cge. \end{aligned}$$

Assuming that $\mathcal{A}$ allows nilpotency, then values of $a, b, \ldots, j$ exist such that A is nilpotent, i.e., $p_A(x) = x^3$. In this case, $j = a + e$, which implies that

$$\begin{aligned} 0 &= ae - aj - bd - cg - ej - fh \\ &= -a^2 - ae - bd - cg - e^2 - fh < 0, \end{aligned}$$

a contradiction. Thus, $\mathcal{A}$ does not allow nilpotency. The same conclusion is valid if one or more of the entries b, c, d, f, g, or h is equal to 0, and/or both b and d, both c and g, and/or both f and h are nonpositive. These sign patterns and their equivalent patterns account for 35 of the remaining 60 patterns.

For the third sign pattern class (see, e.g., [3]), consider

$$\mathcal{A} = \begin{bmatrix} + & - & - \\ + & - & + \\ + & + & - \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} a & -d & -1 \\ b & -e & h \\ c & f & -j \end{bmatrix} \in \mathcal{A}.$$

The matrix A has the characteristic polynomial

$$p_A(x) = x^3 + (e+j-a)x^2 + (bd+cd+ej-ae-aj-fh)x + afh + bf + bdj + ce + ch - aej.$$

To show that $\mathcal{A}$ does not allow nilpotency, assume that $p_A(x) = x^3$ for appropriate values of $a, b, \ldots, j$. It must hold that $a = e + j$, so

$$c = ae + aj + fh - bd - ej = e^2 + ej + fh + j^2 - bd.$$

Thus the constant term gives

$$0 = -bd^2h - bde + e^3 + bdj + bf + de^2h + dehj + dfh^2 + dhj^2 + 2efh + fhj,$$

so

$$b = \frac{de^2h + dehj + dfh^2 + dhj^2 + e^3 + 2efh + fhj}{d^2h + de - dj - f}.$$

Since $a, b, \ldots, j > 0$, it follows that $d^2h + de - dj - f > 0$. However,

$$\begin{aligned} c &= e^2 + ej + fh + j^2 - bd \\ &= \frac{-defh - 2dfhj - dj^3 - e^2f - efj - f^2h - fj^2}{d^2h + de - dj - f} < 0, \end{aligned}$$

a contradiction, so $\mathcal{A}$ does not allow nilpotency. The same arguments are valid if any of d, f, and h equal 0 such that $d + f > 0$. These sign patterns and their equivalent patterns account for 8 of the 60 patterns.

For the fourth class, let

$$\mathcal{A} = \begin{bmatrix} + & - & 0 \\ + & - & + \\ + & + & - \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} a & -1 & 0 \\ b & -e & 1 \\ c & f & -j \end{bmatrix} \in \mathcal{A}.$$

Assuming that $\mathcal{A}$ allows nilpotency, it is possible to assign values to $a, b, \ldots, j$ such that the characteristic polynomial

$$p_A(x) = x^3 + (e+j-a)x^2 + (b+ej-ae-aj-f)x + af + bj + c - aej$$

equals x^3. If this is true, then $a = j + e$, so

$$b = ae + aj - ej + f = e^2 + j^2 + ej + f$$

and

$$0 = af + bj + c - aej = c + ef + 2fj + j^3 > 0,$$

a contradiction. Thus, $\mathcal{A}$ does not allow nilpotency. The same arguments and conclusion are true if c or f equals 0. The cases

$$\begin{bmatrix} + & - & - \\ + & - & 0 \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & - & - \\ + & - & + \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & + & - \\ 0 & + & - \end{bmatrix}, \text{ and } \begin{bmatrix} + & + & + \\ + & + & - \\ 0 & + & - \end{bmatrix}$$

are proven to not allow nilpotency in the same way. The sign patterns above and their equivalent patterns account for 13 of the 60 patterns.

It may be verified by inspection that every one of the 60 sign pattern belongs to one of the four classes above, no members of which allow nilpotency. This concludes the proof. ∎

6. Concluding remarks. Since our interest is on minimal spectrally arbitrary patterns, we address the question of the least number of nonzero entries required by such a pattern.

CONJECTURE 6.1. *For $n \geq 2$, an $n \times n$ sign pattern that is spectrally arbitrary has at least $2n$ nonzero entries.*

Conjecture 6.1 is verified for $n = 2$ by Example 2.2, and Theorem 5.2 verifies the conjecture for $n = 3$ (since there are no 3×3 reducible spectrally arbitrary sign patterns). For all $n \geq 3$, this bound is realized by $\mathcal{V}_n$ (Theorem 2.4). It is also realized by the antipodal tridiagonal sign pattern T_n in [1, 2] for all values of n for which T_n is known to be spectrally arbitrary (i.e., $2 \leq n \leq 16$).

Let $\mathbb{Q}[X]$ be the set of polynomials with rational coefficients. A set $S \subseteq \mathbb{R}$ is algebraically independent if, for all $s_1, \ldots, s_n \in S$ and each nonzero polynomial $p(x_1, \ldots, x_n) \in \mathbb{Q}[X]$, $p(s_1, \ldots, s_n) \neq 0$ (see [6, p. 316] for further details). Let $\mathbb{Q}(S)$ denote the field of rational expressions

$$\left\{ \frac{p(s_1, \ldots, s_m)}{q(t_1, \ldots, t_n)} \ : \ p(x_1, \ldots, x_m), q(x_1, \ldots, x_n) \in \mathbb{Q}[X], s_1, \ldots, s_m, t_1, \ldots, t_n \in S \right\},$$

and let the *transcendental degree* of S be

$$tr.d.S = \sup\{|T| \ : \ T \subseteq S, \ T \text{ is algebraically independent}\}.$$

The following theorem very nearly verifies Conjecture 6.1.

THEOREM 6.2. *For $n \geq 2$, an irreducible $n \times n$ sign pattern that is spectrally arbitrary has at least $2n - 1$ nonzero entries.*

Proof. Let $\mathcal{A}$ be an irreducible $n \times n$ spectrally arbitrary sign pattern with $n_{\mathcal{A}}$ nonzero entries. Choose a set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{R}$ that is algebraically independent. By Lemma 2.3, $\mathcal{A}$ has a realization $A = [a_{ij}]$ with characteristic polynomial

$$p_A(x) = x^n - \alpha_1 x^{n-1} + \cdots + (-1)^n \alpha_n$$

and $n - 1$ (off-diagonal) entries with magnitude 1. Since for each $1 \leq i \leq n$, α_i is a polynomial in the entries $\{a_{ij} : 1 \leq i, j \leq n\}$ with rational coefficients, it follows that $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Q}(a_{ij} : 1 \leq i, j \leq n)$, so

$$n = tr.d.\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \leq tr.d.\mathbb{Q}(a_{ij} : 1 \leq i, j \leq n) \leq n_{\mathcal{A}} - (n - 1).$$

Thus, $n_{\mathcal{A}} \geq 2n-1$. □

It is clear from the proof of Theorem 5.2 that a 3×3 irreducible sign pattern allows nilpotency if and only if it is a spectrally arbitrary pattern. This is not generally true, as the following 4×4 sign pattern demonstrates. Let

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & - & 0 & + \\ - & 0 & 0 & - \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -c & 0 & 1 \\ -b & 0 & 0 & -d \end{bmatrix} \in \mathcal{A}.$$

By Lemma 2.3, it may be assumed without loss of generality that each realization of $\mathcal{A}$ has the form of A above. The characteristic polynomial of A is

$$p_A(x) = x^4 - (a-d)x^3 - (ad-c)x^2 - (a-d)cx - acd + b.$$

If $(a-d)c = 0$, then $a-d=0$, so $p_A(x)$ cannot equal $x^4 - \alpha x^3$ for any nonzero α. Thus, $\mathcal{A}$ does not allow the spectrum $\{0,0,0,\alpha\}$ for any nonzero α, and thus $\mathcal{A}$ is not spectrally arbitrary. However, $\mathcal{A}$ does allow nilpotency, since A is nilpotent for $a=b=c=d=1$.

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