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Geometric inequalities via a general comparison principle for interacting gases

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Abstract

The article builds on several recent advances in the Monge-Kantorovich theory of mass transport which have – among other things – led to new and quite natural proofs for a wide range of geometric inequalities such as the ones formulated by Brunn-Minkowski, Sobolev, Gagliardo-Nirenberg, Beckner, Gross, Talagrand, Otto-Villani and their extensions by many others. While this paper continues in this spirit, we however propose here a basic framework to which all of these inequalities belong, and a general unifying principle from which many of them follow. This basic inequality relates the relative total energy – internal, potential and interactive – of two arbitrary probability densities, their Wasserstein distance, their barycenters and their entropy production functional. The framework is remarkably encompassing as it implies many old geometric – Gaussian and Euclidean – inequalities as well as new ones, while allowing a direct and unified way for computing best constants and extremals. As expected, such inequalities also lead to exponential rates of convergence to equilibria for solutions of Fokker-Planck and McKean-Vlasov type equations. The principle also leads to a remarkable correspondence between ground state solutions of certain quasilinear – or semilinear – equations and stationary solutions of – nonlinear – Fokker-Planck type equations.

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Contents

1 Introduction

The recent advances in the Monge-Kantorovich theory of mass transport have – among other things – led to new and quite natural proofs for a wide range of geometric inequalities. Most notable are McCann's generalization of the Brunn-Minkowski's inequality [24], Otto-Villani's [27] and Cordero-Gangbo-Houdre [14] extensions of the Log Sobolev inequality of Gross [20] and Bakry-Emery [4], as well as Cordero-Nazaret-Villani's proof [12] of the Sobolev and the Gagliardo-Nirenberg inequalities. We refer to the superb recent monograph of Villani [30] for more details on these remarkable developments.

This paper continues in this spirit, but our emphasis here is on developing a framework for a unified and compact approach to a substantial number of these inequalities which originate in disparate areas of analysis and geometry. The main idea is to try to describe the evolution of the total – internal, potential and interactive – energy of a system along an optimal transport that takes one configuration to another, taking into account the entropy production functional, the transport cost (Wasserstein distance), as well as the displacement of their centres of mass. Once this general comparison principle is established, then several – new and old – inequalities follow directly by simply considering different examples of – admissible – internal energies, and various confinement and interactive potentials. Others (e.g., Concentration of measure phenomenon and Poincaré's inequality) will in turn follow from the well known hierarchy between these inequalities.

Besides the obvious pedagogical relevance of a streamlining approach, we find it interesting and intriguing that most of these inequalities appear as different manifestations of one basic principle in the theory of interacting gases that compares the energies of two states of a system after one is transported "at minimal cost" into another. Here is our framework which is already present in McCann's thesis [23]. Let Ω be an open and convex subset of $\mathbb{R}^n$. The set of probability densities over Ω is denoted by $\mathcal{P}_a(\Omega) = \{\rho : \Omega \to \mathbb{R};\ \rho \geq 0 \text{ and } \int_\Omega \rho(x)dx = 1\}$ and $\operatorname{supp} \rho$ will stand for the support of $\rho \in \mathcal{P}_a(\Omega)$, that is the closure of $\{x \in \Omega : \rho \neq 0\}$, while $|\Omega|$ will denote the Lebesgue measure of $\Omega \subset \mathbb{R}^n$. Let $F : [0, \infty) \to \mathbb{R}$ be a differentiable function on $(0, \infty)$, and let V and W be C^2-real valued functions on $\mathbb{R}^n$. The associated *Free Energy Functional* is then defined on $\mathcal{P}_a(\Omega)$ as:

$$\mathrm{H}_V^{F,W}(\rho) := \int_\Omega \left[F(\rho) + \rho V + \frac{1}{2}(W \star \rho)\rho \right] \mathrm{d}x,$$

which is the sum of the internal energy $\mathrm{H}^{\mathrm{F}}(\rho) := \int_\Omega F(\rho)dx$, the potential energy $\mathrm{H}_V(\rho) := \int_\Omega \rho V dx$ and the interaction energy $\mathrm{H}^W(\rho) := \frac{1}{2}\int_\Omega \rho(W \star \rho)\,\mathrm{d}x$. Of importance is also the concept of *relative energy of ρ_0 with respect to ρ_1* simply defined as: $\mathrm{H}_V^{F,W}(\rho_0|\rho_1) := \mathrm{H}_V^{F,W}(\rho_0) - \mathrm{H}_V^{F,W}(\rho_1)$, where ρ_0 and ρ_1 are two probability densities. The *relative entropy production of ρ with respect to ρ_V* is normally defined as

$$I_2(\rho|\rho_V) = \int_\Omega \rho \Big| \nabla \left(F'(\rho) + V + W \star \rho\right) \Big|^2 \mathrm{d}x$$

in such a way that if $\rho_{_V}$ is a probability density that satisfies

$$\nabla \left(F'(\rho_{_V}) + V + W \star \rho_{_V}\right) = 0 \quad \text{a.e.}$$

then

$$I_2(\rho|\rho_{_V}) = \int_\Omega \rho |\nabla \left(F'(\rho) - F'(\rho_{_V}) + W \star (\rho - \rho_{_V}\right)|^2 \,\mathrm{d}x.$$

Our notation for the density $\rho_{_V}$ reflects this paper's emphasis on its dependence on the confinement potential, though it obviously also depends on F and W.

We need the notion of Wasserstein distance W_2 between two probability measures ρ_0 and ρ_1 on $\mathbb{R}^n$, defined as:

$$W_2^2(\rho_0, \rho_1) := \inf_{\gamma \in \Gamma(\rho_0,\rho_1)} \int_{R^n \times R^n} |x - y|^2 d\gamma(x, y),$$

where $\Gamma(\rho_0, \rho_1)$ is the set of Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals ρ_0 and ρ_1, respectively. The *barycentre* (or centre of mass) of a probability density ρ, denoted $\mathrm{b}(\rho) := \int_{R^n} x\rho(x)dx$ will play a role in the presence of an interactive potential.

In this paper, we shall also deal with non-quadratic versions of the entropy. For that we call *Young function*, any strictly convex C^1-function $c : \mathbb{R}^n \to \mathbb{R}$ such that $c(0) = 0$ and $\lim_{|x| \to \infty} \frac{c(x)}{|x|} = \infty$. We denote by c^* its Legendre conjugate defined by $c^*(y) = \sup_{z \in R^n}\{y \cdot z - c(z)\}$. For any probability density ρ on Ω, we define the *generalized relative entropy production-type function of ρ with respect to ρ_V measured against c^** by

$$\mathcal{I}_{c^*}(\rho|\rho_{_V}) := \int_\Omega \rho c^\star \left(-\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x,$$

which is closely related to the *generalized relative entropy production function of ρ with respect to ρ_V measured against c^** defined as:

$$I_{c^*}(\rho|\rho_V) := \int_\Omega \rho \nabla \left(F'(\rho) + V + W \star \rho\right) \cdot \nabla c^\star \left(\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x.$$

Indeed, the convexity inequality $c^*(z) \le z \cdot \nabla c^*(z)$ satisfied by any Young function c, readily implies that $\mathcal{I}_{c^*}(\rho|\rho_V) \le I_{c^*}(\rho|\rho_V)$. Note that when $c(x) = \frac{|x|^2}{2}$, we have

$$I_{c^*}(\rho|\rho_V) =: I_2(\rho|\rho_V) = \int_\Omega \rho \Big| \nabla \left(F'(\rho) + V + W \star \rho\right) \Big|^2 \mathrm{d}x = 2\mathcal{I}_{c^*}(\rho|\rho_V),$$

and we denote $\mathcal{I}_{c^*}(\rho|\rho_V)$ by $\mathcal{I}_2(\rho|\rho_V)$.

Throughout this paper, the internal energy will be given by a differentiable function $F : [0,\infty) \to \mathbb{R}$ on $(0,\infty)$ with $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ convex and non-increasing. We denote by $P_F(x) := xF'(x) - F(x)$ its associated pressure function. The confinement potential will be given by a C^2-function $V : \mathbb{R}^n \to \mathbb{R}$ with $D^2 V \ge \lambda I$, while the interaction potential W will be an even C^2-function with $D^2 W \ge \nu I$ where $\lambda, \nu \in \mathbb{R}$, and where I stands for the identity map.

In section 2, we start by establishing the following inequality relating the free energies of two arbitrary probability densities, their Wasserstein distance, their barycenters and their relative entropy production functional. The fact that it yields many of the admittedly powerful geometric inequalities is remarkable.

Basic comparison principle for interactive gases: If Ω is any open, bounded and convex subset of $\mathbb{R}^n$, then for any $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$ satisfying $\operatorname{supp} \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, and any Young function $c : \mathbb{R}^n \to \mathbb{R}$, we have:

$$\mathrm{H}^{F,W}_{V+c}(\rho_0|\rho_1) + \frac{\lambda+\nu}{2} W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \le \mathrm{H}^{-nP_F, 2x\cdot\nabla W}_{c+\nabla V\cdot x}(\rho_0) + \mathcal{I}_{c^*}(\rho_0|\rho_V). \quad (1)$$

Furthermore, equality holds in (1) whenever $\rho_0 = \rho_1 = \rho_{V+c}$, where the latter satisfies

$$\nabla \left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad \text{a.e.} \quad (2)$$

To give an idea about the strength of the above inequality, assume $V = W = 0$ and apply it with ρ_0 being any probability density ρ satisfying $\operatorname{supp} \rho \subset \Omega$ and $\rho_1 = \rho_c$ the reference density. We obtain:

The General Euclidean Sobolev Inequality:

$$\mathrm{H}^{F+nP_F}(\rho) \le \int_\Omega \rho c^\star \left(-\nabla(F' \circ \rho)\right) \mathrm{d}x + K_c, \quad (3)$$

where K_c is the unique constant determined by the equation

$$F'(\rho_c) + c = K_c \text{ and } \int_\Omega \rho_c = 1. \quad (4)$$

Applied to various – displacement convex – functionals F, we shall see in section 3 that (3) already implies the Sobolev, the Gagliardo-Nirenberg and the Euclidean p-Log Sobolev inequalities, allowing in the process a direct and unified way for computing best constants and extremals. This formulation also points to an interesting fact: that the various Sobolev inequalities are nothing but another manifestation of how free energy is controlled by entropy production in appropriate systems.

In section 4, we notice that inequality (1) simplifies considerably in the case where c is a quadratic Young function of the form $c(x) := c_\sigma(x) = \frac{1}{2\sigma}|\,x\,|^2$ for $\sigma > 0$, and we obtain:

The General Logarithmic Sobolev Inequality: For all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp}\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have for any $\sigma > 0$,

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) + \frac{1}{2}(\lambda + \nu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \le \frac{\sigma}{2} I_2(\rho_0|\rho_V). \tag{5}$$

Minimizing the above inequality over $\sigma > 0$ then yields:

The HBWI inequality for interactive gases:

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) \le W_2(\rho_0, \rho_1)\sqrt{I_2(\rho_0|\rho_V)} - \frac{\lambda + \nu}{2} W_2^2(\rho_0, \rho_1) + \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2. \tag{6}$$

This extends the HWI inequality established in [27] and [10], with the additional "B" referring to the new barycentric terms, and constitutes yet another extension of various powerful inequalities by Gross [20], Bakry-Emery[4], Talagrand [29], Otto-Villani [27], Cordero [13] and others.

In section 5, we describe how these inequalities combined with the following energy dissipation equation

$$\frac{d}{dt}\,\mathrm{H}_V^{F,W}\left(\rho(t)|\rho_V\right) = -I_2\left(\rho(t)|\rho_V\right), \tag{7}$$

provide rates of convergence to equilibria for solutions to McKean-Vlasov type equations

$$\begin{cases} \frac{\partial \rho}{\partial t} = \operatorname{div}\left\{\rho\nabla\left(F'(\rho) + V + W \star \rho\right)\right\} & \text{in} \quad (0, \infty) \times I\!R^n \\ \\ \rho(t = 0) = \rho_0 & \text{in} \quad \{0\} \times I\!R^n. \end{cases} \tag{8}$$

One can then recover the recent results of Carillo, McCann and Villani in [10], which estimate the rate of convergence of various quantities to the equilibrium state.

In section 6, we apply inequality (1) to the most basic system – where no potential nor interaction energies are involved– to obtain:

The Energy-Entropy Duality Formula: For any probability density $\rho_0 \in W^{1,\infty}(\Omega)$ with support in Ω, and any $\rho_1 \in \mathcal{P}_a(\Omega)$, we have

$$-\mathrm{H}_c^F(\rho_1) \le -\mathrm{H}^{F + nP_F}(\rho_0) + \int_\Omega \rho_0 c^\star\left(-\nabla(F' \circ \rho_0)\right)\, \mathrm{d}x. \tag{9}$$

Moreover, equality holds whenever $\rho_0 = \rho_1 = \rho_c$ where ρ_c is a probability density on Ω such that $\nabla(F'(\rho_c) + c) = 0$ a.e.

Motivated by the recent work of Cordero-Nazaret-Villani [12], we show that (9) yields a statement of the following type:

$$\sup\{J(\rho);\ \int_\Omega \rho(x)dx = 1\} \le \inf\{I(f);\ \int_\Omega \psi(f(x))dx = 1\}, \tag{10}$$

where

$$I(f)=\int_\Omega \left[c^*(-\nabla f(x))-G\left(\psi\circ f(x)\right)\right]dx \tag{11}$$

and

$$J(\rho)=-\int_\Omega [F(\rho(y))+c(y)\rho(y)]dy \tag{12}$$

with $G(x)=(1-n)F(x)+nxF'(x)$ and where ψ is computable from F and c. Moreover, we have equality in (10) whenever there exists $\bar f$ (and $\bar\rho=\psi(\bar f)$) that satisfies the first order equation:

$$-(F'\circ\psi)'(\bar f)\nabla\bar f(x)=\nabla c(x) \text{ a.e.} \tag{13}$$

In this case, the extrema are achieved at $\bar f$ (resp. $\bar\rho=\psi(\bar f)$). The latter is therefore a solution for the quasilinear (or semi-linear) equation

$$\operatorname{div}\{\nabla c^*(-\nabla f)\}-(G\circ\psi)'(f)=\psi'(f) \tag{14}$$

since it is the L^2-Euler-Lagrange equation of I on $\{f\in C_0^\infty(\Omega);\ \int_\Omega\psi(f(x))dx=1\}$. Equally interesting is the fact that $\psi(\bar f)$ is also a stationary solution of the (non-linear) Fokker-Planck equation:

$$\frac{\partial u}{\partial t}=\operatorname{div}\{u\nabla(F'(u)+c)\} \tag{15}$$

since J is nothing but the Free Energy functional on $\mathcal{P}_a(\Omega)$, whose gradient flow with respect to the Wasserstein distance is precisely the evolution equation (15). In other words, this is pointing to a remarkable correspondence between Fokker-Planck evolution equations and certain quasilinear or semi-linear equations which appear as Euler-Lagrange equations of the entropy production functionals.

In conclusion to this introduction, we mention that this paper is an expanded version of the unpublished but distributed manuscript [2]. This unifying and compact approach to so many important inequalities eventually led us to make the paper as self-contained as possible so that it can serve as a quick introduction to these basic tools of modern analysis. We should however warn the reader that we have barely scratched the surface of the huge litterature that exists on these basic inequalities, their various generalizations and on the hierarchy and relationships between them. Therefore, our references are in no way complete nor exhaustive. Fortunately many books and surveys have already appeared on these topics and we refer the reader to the monograph of Villani mentioned above, as well as to the book of Ledoux [22] and the recent survey of Gardner [18].

2 Basic inequality between two configurations of interacting gases

Here is our starting point.

Theorem 2.1 *Let $F:[0,\infty)\to I\!R$ be differentiable function on $(0,\infty)$ with $F(0)=0$ and $x\mapsto x^nF(x^{-n})$ convex and non-increasing, and let $P_F(x):=xF'(x)-F(x)$ be* *its associated pressure function. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^2-confinement potential with $D^2V \geq \lambda I$, and let W be an even C^2-interaction potential with $D^2W \geq \nu I$ where $\lambda, \nu \in \mathbb{R}$, and I denotes the identity map. If Ω is any open, bounded and convex subset of $\mathbb{R}^n$, then for any $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$, satisfying $\operatorname{supp}\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, and any Young function $c : \mathbb{R}^n \to \mathbb{R}$, we have:*

$$\mathrm{H}^{F,W}_{V+c}(\rho_0|\rho_1) + \frac{\lambda+\nu}{2} W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \mathrm{H}^{-nP_F, 2x\cdot\nabla W}_{c+\nabla V\cdot x}(\rho_0) + +\mathcal{I}_{c^*}(\rho_0|\rho_V). \quad (16)$$

Furthermore, equality holds in (16) whenever $\rho_0 = \rho_1 = \rho_{V+c}$, where the latter satisfies

$$\nabla\left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad a.e. \quad (17)$$

In particular, we have for any $\rho \in \mathcal{P}_c(\Omega)$ with $\operatorname{supp}\rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}^{F+nP_F,\, W-2x\cdot\nabla W}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda+\nu}{2} W_2^2(\rho,\rho_{V+c}) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_{V+c})|^2 \leq \mathcal{I}_{c^*}(\rho|\rho_V) - \mathrm{H}^{P_F,W}(\rho_{V+c}) + K_{V+c}, \quad (18)$$

where K_{V+c} is a constant such that

$$F'(\rho_{V+c}) + V + c + W \star \rho_{V+c} = K_{V+c} \text{ while } \int_\Omega \rho_{V+c} = 1. \quad (19)$$

The proof is based on the recent advances in the theory of mass transport as developed by Brenier [8], Gangbo-McCann [16], [17], Caffarelli [9] and many others. For a survey, see Villani [30]. Here is a brief summary of the needed results.
Fix a non-negative C^1, strictly convex function $d : \mathbb{R}^n \to \mathbb{R}$ such that $d(0) = 0$. Given two probability measures μ and ν on $\mathbb{R}^n$, the minimum cost for transporting μ onto ν is given by

$$W_d(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{R^n\times R^n} d(x-y)\,d\gamma(x,y), \quad (20)$$

where $\Gamma(\mu,\nu)$ is the set of Borel probability measures with marginals μ and ν, respectively. When $d(x) = |\,x\,|^2$, we have that $W_d = W_2^2$, where W_2 is the Wasserstein distance. We say that a Borel map $T : \mathbb{R}^n \to \mathbb{R}^n$ pushes μ forward to ν, if $\mu(T^{-1}(B)) = \nu(B)$ for any Borel set $B \subset \mathbb{R}^n$. The map T is then said to be d-optimal if

$$W_d(\mu,\nu) = \int_{R^n} d(x - Tx)\,d\mu(x) = \inf_S \int_{R^n} d(x - Sx)\,d\mu(x), \quad (21)$$

where the infimum is taken over all Borel maps $S : \mathbb{R}^n \to \mathbb{R}^n$ that push μ forward to ν. For quadratic cost functions $d(z) = \frac{1}{2}|z|^2$, Brenier [8] characterized the optimal transport map T as the gradient of a convex function. An analogous result holds for general cost functions d, provided convexity is replaced by an appropriate notion of d-concavity. See [16], [9] for details.

Here is the lemma which leads to our main inequality (16). It is essentially a compendium of various observations by several authors. It describes the evolution of a generalized energy functional along optimal transport. The key idea behind it, is the concept of *displacement convexity* introduced by McCann [24]. For generalized cost functions, and when $V = 0$, it was first obtained by Otto [26] for the Tsallis entropy functionals and by Agueh [1] in general. The case of a nonzero confinement potential V and an interaction potential W was included in [14], [10]. Here, we state the results when the cost function is quadratic, $d(x) = |\,x\,|^2$.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, and let ρ_0 and ρ_1 be probability densities on Ω, with $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. Let T be the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\,x\,|^2$. Then*

1) Assume $F : [0, \infty) \to \mathbb{R}$ is differentiable on $(0, \infty)$, $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, then the internal energy satisfies:

$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla \left(F'(\rho_0) \right) \, dx. \tag{22}$$

2) Assume $V : \mathbb{R}^n \to \mathbb{R}$ is such that $D^2 V \geq \lambda I$ for some $\lambda \in \mathbb{R}$, then the potential energy satisfies

$$\mathrm{H}_V(\rho_1) - \mathrm{H}_V(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla V \, dx + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1). \tag{23}$$

3) Assume $W : \mathbb{R}^n \to \mathbb{R}$ is even, and $D^2 W \geq \nu I$ for some $\nu \in \mathbb{R}$, then the interaction energy satisfies

$$\mathrm{H}^W(\rho_1) - \mathrm{H}^W(\rho_0) \geq \int_\Omega \rho_0 (T - I) \cdot \nabla (W \star \rho_0) \, dx + \frac{\nu}{2} \left(W_2^2(\rho_0, \rho_1) - |\mathrm{b}(\rho_0) - b(\rho_1)|^2 \right). \tag{24}$$

Proof: If T ($T = \nabla \psi$, where ψ is convex) is the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\,x\,|^2$, one can then define a path of probability densities joining them, by letting ρ_t be the push-forward measure of ρ_0 by the map $T_t = (1 - t)I + tT$. The key idea behind the estimate for the internal energy is the fact first noticed by McCann [24]), that under the above assumptions on F, the function $t \mapsto \mathrm{H}^F(\rho_t)$ is convex on $[0, 1]$, which – at least for smooth ρ_t – essentially leads to (22) via the following inequality for the internal energy:

$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq [\frac{d}{dt} \mathrm{H}^F(\rho_t)]_{t=0} = - \int_\Omega F'(\rho_0) \, \mathrm{div} \left(\rho_0 (T - I) \right) \, \mathrm{d}x. \tag{25}$$

We shall use here another approach due to Agueh [1] as it is more elementary and is applicable to other cost functions.

First note that $T = \nabla \psi$ is diagonalizable with positive eigenvalues for ρ_0 a.e., and satisfies the Monge-Ampère equation

$$0 \neq \rho_0(x) = \rho_1 \left(T(x) \right) \det \nabla T(x) \qquad \rho_0 \ \text{ a.e.} \tag{26}$$

So, $\rho_1 \left(T(x) \right) \neq 0$ for ρ_0 a.e. Here, $\nabla T(x) = \nabla^2 \psi(x)$ denotes the derivative in the sense of Aleksandrov of ψ (see McCann [24]). Set $A(x) = x^n F(x^{-n})$, which is non-increasing by assumption, hence the pressure P_F is non-negative and $x \mapsto \frac{F(x)}{x}$ is also non-increasing. Use that $F(0)=0$, $T_{\#}\rho_0=\rho_1$ and (26), to obtain that

$$
\begin{aligned}
\mathrm{H}^F(\rho_1)=\int_{[\rho_1\neq 0]}\frac{F(\rho_1(y))}{\rho_1(y)}\rho_1(y)\,\mathrm{d}y &= \int_\Omega \frac{F(\rho_1(Tx))}{\rho_1(Tx)}\rho_0(x)\,\mathrm{d}x \\
&= \int_\Omega F\left(\frac{\rho_0(x)}{\det\nabla T(x)}\right)\det\nabla T(x)\,\mathrm{d}x. \qquad (27)
\end{aligned}
$$

Comparing the geometric mean $(\det\nabla T(x))^{1/n}$ with the arithmetic mean $\frac{\operatorname{tr}\nabla T(x)}{n}$, we get $\frac{1}{\det\nabla T(x)}\geq\left(\frac{n}{\operatorname{tr}\nabla T(x)}\right)^n$, and since $x\mapsto\frac{F(x)}{x}$ is non-decreasing, we obtain

$$
F\left(\frac{\rho_0(x)}{det\,\nabla T(x)}\right)\det\nabla T(x)\geq\Lambda^n F\left(\frac{\rho_0(x)}{\Lambda^n}\right)=\rho_0(x)A\left(\frac{\Lambda}{\rho_0(x)^{1/n}}\right), \qquad (28)
$$

where $\Lambda:=\frac{\operatorname{tr}\nabla T(x)}{n}$. Next, we use that $A'(x)=-nx^{n-1}P_F(x^{-n})$ and that A is convex, to obtain that

$$
\begin{aligned}
\rho_0(x)A\left(\frac{\Lambda}{\rho_0(x)^{1/n}}\right) &\geq \rho_0(x)\left[A\left(\frac{1}{\rho_0(x)^{1/n}}\right)+A'\left(\frac{1}{\rho_0(x)^{1/n}}\right)\left(\frac{\Lambda-1}{\rho_0(x)^{1/n}}\right)\right] \\
&= \rho_0(x)\left[\frac{F(\rho_0(x))}{\rho_0(x)}-n(\Lambda-1)\frac{P_F(\rho_0(x))}{\rho_0(x)}\right] \\
&= F(\rho_0(x))-P_F(\rho_0(x))\operatorname{tr}(\nabla T(x)-I). \qquad (29)
\end{aligned}
$$

We combine (27) - (29), to conclude that

$$
\begin{aligned}
\mathrm{H}^F(\rho_1)-\mathrm{H}^F(\rho_0) &\geq -\int_\Omega P_F(\rho_0(x))\ tr\,(\nabla T(x)-I)\,\mathrm{d}x \\
&= -\int_\Omega P_F(\rho_0(x))\ \operatorname{div}(T(x)-I)\,\mathrm{d}x \\
&\geq \int_\Omega \rho_0\,(T-I)\cdot\nabla\left(F'(\rho_0)\right)\,\mathrm{d}x. \qquad (30)
\end{aligned}
$$

(2) As noted in [14], the fact that $D^2V\geq\lambda I$, which means that

$$
V(b)-V(a)\geq\nabla V(a)\cdot(b-a)+\frac{\lambda}{2}|\,a-b\,|^2
$$

for all $a,b\in I\!R^n$, easily implies (23) via the following inequality for the corresponding potential energy:

$$
\begin{aligned}
\mathrm{H}_V(\rho_1)-\mathrm{H}_V(\rho_0) &\geq [\frac{d}{dt}\mathrm{H}_V(\rho_t)]_{t=0}+\frac{\lambda}{2}\int_\Omega|(T-I)(x)|^2\rho_0(x)dx \\
&= -\int_\Omega V\operatorname{div}(\rho_0(T-I))\,\mathrm{d}x+\frac{\lambda}{2}W_2^2(\rho_0,\rho_1). \qquad (31)
\end{aligned}
$$

(3) The proof of (24) is due to Cordero-Gangbo-Houdré [14], and is also included here for completeness. Rewrite the interaction energy as follows:

$$
\begin{aligned}
\mathrm{H}^W(\rho_1) &= \frac{1}{2}\int_{\Omega\times\Omega} W(x-y)\rho_1(x)\rho_1(y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_{\Omega\times\Omega} W(T(x)-T(y))\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_{\Omega\times\Omega} W\left(x-y+(T-I)(x)-(T-I)(y)\right)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\geq \frac{1}{2}\int_{\Omega\times\Omega} \left[W(x-y)+\nabla W(x-y)\cdot\left((T-I)(x)-(T-I)(y)\right)\rho_0(x)\rho_0(y)\right]\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{\nu}{4}\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&= \mathrm{H}^W(\rho_0)+\frac{1}{2}\int_{\Omega\times\Omega} \nabla W(x-y)\cdot\left((T-I)(x)-(T-I)(y)\right)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{\nu}{4}\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y, \qquad (32)
\end{aligned}
$$

where we used above that $D^2W \geq \nu I$. The last term of the subsequent inequality can be written as:

$$
\begin{aligned}
&\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_{\Omega} |(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2\Big|\int_{R^n}(T-I)(x)\rho_0(x)\,\mathrm{d}x\Big|^2 \\
&\quad = 2\int_{\Omega} |(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2|\mathrm{b}(\rho_1)-\mathrm{b}(\rho_0)|^2. \qquad (33)
\end{aligned}
$$

And since ∇W is odd (because W is even), we get for the second term of (32)

$$
\begin{aligned}
&\int_{\Omega\times\Omega} \left[\nabla W(x-y)\cdot\left((T-I)(x)-(T-I)(y)\right)\right]\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_{\Omega\times\Omega} \nabla W(x-y)\cdot(T-I)(x)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_{\Omega\times\Omega} \rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x. \qquad (34)
\end{aligned}
$$

Combining (32) - (34), we obtain that

$$
\begin{aligned}
&\mathrm{H}^W(\rho_1)-\mathrm{H}^W(\rho_0) \\
&\quad \geq \int_{\Omega\times\Omega} \rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x + \frac{\nu}{2}\left(\int_{\Omega}|(T-I)(x)|^2\rho_0 dx - |\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\right).
\end{aligned}
$$

This complete the proof of (24).

Proof of Theorem 2.1: Adding (22), (23) and (24), one gets

$$
\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \qquad (35) \\
&\quad \leq \int_{\Omega}(x-Tx)\cdot\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\,\mathrm{d}x.
\end{aligned}
$$

Since $\rho_0\nabla(F'(\rho_0)) = \nabla\left(P_F(\rho_0)\right)$, we integrate by part $\int_\Omega \rho_0\nabla\left(F'(\rho_0)\right)\cdot x\,\mathrm{d}x$, and obtain that

$$\int_\Omega x\cdot\nabla(F'(\rho_0)+V+W\star\rho_0)\rho_0 = \mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0).$$

This leads to

$$\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \\
\leq\quad &\mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0)-\int_\Omega\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\cdot T(x)\,\mathrm{d}x.
\end{aligned}\tag{36}$$

Now, use Young's inequality to get

$$\begin{aligned}
&-\nabla\left(F'\left(\rho_0(x)\right)+V(x)+(W\star\rho_0)(x)\right)\cdot T(x) \\
&\quad\leq c\left(T(x)\right)+c^\star\left(-\nabla\left(F'(\rho_0(x))+V(x)+(W\star\rho_0)(x)\right)\right),
\end{aligned}\tag{37}$$

and deduce that

$$\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\mu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \\
&\leq \mathrm{H}_{x\cdot\nabla V}^{-nP_F,2x\cdot\nabla W}(\rho_0)+\int_\Omega\rho_0c^\star\left(-\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\right)+\int_\Omega c(Tx)\rho_0\,\mathrm{d}x.
\end{aligned}\tag{38}$$

Finally, use again that T pushes ρ_0 forward to ρ_1, to rewrite the last integral on the right hand side of (38) as $\int_\Omega c(y)\rho_1(y)dy$ to obtain (16).
Now, set $\rho_0=\rho_1:=\rho_{V+c}$ in (36). We have that $T=I$, and equality then holds in (36). Therefore, equality holds in (16) whenever equality holds in (37), where $T(x)=x$. This occurs when (17) is satisfied.
(18) is straightforward when choosing $\rho_0:=\rho$ and $\rho_1:=\rho_{V+c}$ in (16).

3 The General Euclidean Sobolev Inequality

We start with the following general inequality, which can be seen as an extension of the various Euclidean Sobolev inequalities, since once applied to appropriate functionals F and c, one gets the Sobolev, the Gagliardo-Nirenberg and the Euclidean p-Log Sobolev inequalities.

Theorem 3.1 (The General Sobolev Inequality) *Under the hypothesis of Theorem 2.1, assume that V and W are also convex. Then, for any* Young function $c:\mathbb{R}^n\to\mathbb{R}$, *and any $\rho\in\mathcal{P}_c(\Omega)$ with $\mathrm{supp}\,\rho\subset\Omega$ and $P_F(\rho)\in W^{1,\infty}(\Omega)$, we have*

$$\mathrm{H}_{-V^*(\nabla V)}^{F+nP_F,\,W-2x\cdot\nabla W}(\rho)\leq\int_\Omega\rho c^\star\left(-\nabla\left(F'(\rho)+V+W\star\rho\right)\right)\,dx-\mathrm{H}^{P_F,W}(\rho_{V+c})+K_{V+c},\tag{39}$$

where ρ_{V+c} is the probability density and K_{V+c} is the constant satisfying

$$F'(\rho_{V+c})+V+c+W\star\rho_{V+c}=K_{V+c}.\tag{40}$$

In particular, if $V = W = 0$, we have

$$\mathrm{H}^{F+nP_F}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho)\right)\, dx + K_c, \tag{41}$$

where K_c is the unique constant determined by the equation

$$F'(\rho_c) + c = K_c \text{ and } \int_\Omega \rho_c = 1. \tag{42}$$

Proof: This follows immediately from inequality (18) in Theorem 2.1. Indeed, if $\lambda+\nu \geq 0$, then the term involving the Wasserstein distance can be omitted from the equation, while if W is convex, then the barycentric term can also be omitted. If V is strictly convex, then $V - x \cdot \nabla V = -V^*(\nabla V)$.
Now if $V = W = 0$, we obtain the remarkably simple inequality:

$$\mathrm{H}^{F+nP_F}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho)\right) \mathrm{d}x - \mathrm{H}^{P_F}(\rho_c) + K_c, \tag{43}$$

where K_c is the unique constant determined by (42). Finally, we obtain (41) by noting that $\mathrm{H}^{P_F}(\rho_c)$ is always positive.

3.1 Euclidean Log-Sobolev inequalities

The following optimal Euclidean p-Log Sobolev inequality was first established by Beckner in [5] for $p = 1$, and by Del-Pino and Dolbeault [15] for $1 < p < n$. The case where $p > n$ was established recently and independently by I. Gentil [19] who used the Prékopa-Leindler inequality and the Hopf-Lax semi-group associated to the Hamilton-Jacobi equation.

Corollary 3.2 (General Euclidean Log-Sobolev inequality) *Let $\Omega \subset \mathbb{R}^n$ be open bounded and convex, and let $c : \mathbb{R}^n \to \mathbb{R}$ be a Young functional such that its conjugate $c^\star$ is p-homogeneous for some $p > 1$. Then,*

$$\int_{R^n} \rho \ln \rho \, dx \leq \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) dx\right), \tag{44}$$

for all probability densities ρ on $\mathbb{R}^n$, such that $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\mathbb{R}^n)$. Here, $\sigma_c := \int_{R^n} e^{-c}\, dx$. Moreover, equality holds in (44) if $\rho(x) = K_\lambda e^{-\lambda^q c(x)}$ for some $\lambda > 0$, where $K_\lambda = \left(\int_{R^n} e^{-\lambda^q c(x)}\, dx\right)^{-1}$ and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

Proof: Use $F(x) = x\ln(x)$ and $V = W = 0$ in (18). Note that $P_F(x) = x$, and then, $H^{P_F}(\rho) = 1$ for any $\rho \in \mathcal{P}_a(\mathbb{R}^n)$. So, $\rho_c(x) = \frac{e^{-c(x)}}{\sigma_c}$. We then have for $\rho \in \mathcal{P}_a(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ such that $\operatorname{supp}\rho \subset \Omega$,

$$\int_\Omega \rho \ln \rho \, \mathrm{d}x \leq \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x - n - \ln\left(\int_{R^n} e^{-c(x)}\, \mathrm{d}x\right), \tag{45}$$

with equality when $\rho = \rho_c$.

Now assume that $c^\star$ is p-homogeneous and set $\Gamma_\rho^c = \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x$. Using $c_\lambda(x) := c(\lambda x)$ in (45), we get for $\lambda > 0$ that

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \leq \int_{R^n} \rho c^\star\left(-\frac{\nabla \rho}{\lambda \rho}\right) \mathrm{d}x + n \ln \lambda - n - \ln \sigma_c, \tag{46}$$

for all $\rho \in \mathcal{P}_a(I\!\!R^n)$ satisfying $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\Omega)$. Equality holds in (46) if $\rho_\lambda(x) = \left(\int_{R^n} e^{-\lambda^q c(x)}\,\mathrm{d}x\right)^{-1} e^{-\lambda^q c(x)}$. Hence

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \leq -n - \ln \sigma_c + \inf_{\lambda>0}\left(G_\rho(\lambda)\right),$$

where

$$G_\rho(\lambda) = n\ln(\lambda) + \frac{1}{\lambda^p}\int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) = n\ln(\lambda) + \frac{\Gamma_\rho^c}{\lambda^p}.$$

The infimum of $G_\rho(\lambda)$ over $\lambda > 0$ is attained at $\bar\lambda_\rho = \left(\frac{p}{n}\Gamma_\rho^c\right)^{1/p}$. Hence

$$\begin{aligned}
\int_{R^n} \rho \ln \rho \,\mathrm{d}x &\leq G_\rho(\bar\lambda_\rho) - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{n}\Gamma_\rho^c\right) + \frac{n}{p} - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\Gamma_\rho^c\right),
\end{aligned}$$

for all probability densities ρ on $I\!\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$, and $\rho \in W^{1,\infty}(I\!\!R^n)$.

Corollary 3.3 *(*Optimal Euclidean p-Log Sobolev inequality*)*

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\, dx \leq \frac{n}{p}\ln\left(C_p \int_{R^n} |\,\nabla f\,|^p\, dx\right), \tag{47}$$

holds for all $p \geq 1$, and for all $f \in W^{1,p}(I\!\!R^n)$ such that $\|\, f\,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right)\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if} \quad p > 1, \\ \\ \frac{1}{n\sqrt{\pi}}\left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if} \quad p = 1, \end{cases} \tag{48}$$

and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).
For $p > 1$, equality holds in (47) for $f(x) = Ke^{-\lambda^q \frac{|\,x-\bar{x}\,|^q}{q}}$ for some $\lambda > 0$ and $\bar{x} \in I\!\!R^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\,\lambda x\,|^q}\, dx\right)^{-1/p}$.

Proof: First assume that $p > 1$, and set $c(x) = (p-1)|\,x\,|^q$ and $\rho = |\,f\,|^p$ in (44), where $f \in C_c^\infty(I\!R^n)$ and $\|\,f\,\|_p = 1$. We have that $c^\star(x) = \frac{|\,x\,|^p}{p^p}$, and then, $\int_{R^n} \rho c^* \left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x = \int_{R^n} |\,\nabla f\,|^p \,\mathrm{d}x$. Therefore, (44) reads as

$$\int_{R^n} |\,f\,|^p \ln(|\,f\,|^p)\,\mathrm{d}x \le \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} |\,\nabla f\,|^p \,\mathrm{d}x\right). \tag{49}$$

Now, it suffices to note that

$$\sigma_c := \int_{R^n} e^{-(p-1)|\,x\,|^q}\,\mathrm{d}x = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{q}+1\right)}{(p-1)^{\frac{n}{q}}\Gamma\left(\frac{n}{2}+1\right)}. \tag{50}$$

To prove the case where $p = 1$, it is sufficient to apply the above to $p_\epsilon = 1+\epsilon$ for some arbitrary $\epsilon > 0$. Note that

$$C_{p\epsilon} = \left(\frac{1+\epsilon}{n}\right)\left(\frac{\epsilon}{e}\right)^\epsilon \pi^{-\frac{1+\epsilon}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n\epsilon}{1+\epsilon}+1)}\right]^{\frac{1+\epsilon}{n}},$$

so that when ϵ go to 0, we have

$$\lim_{\epsilon\to 0} C_{p_\epsilon} = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{1}{n}} = C_1.$$

3.2 Sobolev and Gagliardo-Nirenberg inequalities

Corollary 3.4 (Gagliardo-Nirenberg inequalities) *Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right)$ such that $r \ne p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(I\!R^n)$ we have*

$$\|f\|_r \le C(p,r)\|\nabla f\|_p^\theta \|f\|_{r\gamma}^{1-\theta}, \tag{51}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{52}$$

$p^* = \frac{np}{n-p}$ and where the best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: Let $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \ne \gamma > 1 - \frac{1}{n}$, which follows from the fact that $p \ne r \in \left(0, \frac{np}{n-p}\right)$. For this value of γ, the function F satisfies the conditions of Theorem 3.1. Let $c(x) = \frac{r\gamma}{q}|\,x\,|^q$ so that $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}|\,x\,|^p$, and set $V = W = 0$. Inequality (18) then gives for all $f \in C_c^\infty(I\!R^n)$ such that $\|\,f\,\|_r = 1$,

$$\left(\frac{1}{\gamma-1}+n\right)\int_{R^n} |\,f\,|^{r\gamma} \le \frac{r\gamma}{p}\int_{R^n} |\,\nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty. \tag{53}$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \;\text{ a.e.}, \tag{54}$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side of (53) are not easy to calculate, so one can obtain θ and the best constant by a standard scaling procedure. Namely, write (53) as

$$\frac{r\gamma}{p}\frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma-1}+n\right)\frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r^{r\gamma}} \geq H^{P_F}(\rho_\infty) - C_\infty =: C, \tag{55}$$

for some constant C. Then apply (55) to $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

The limiting case where r is the critical Sobolev exponent $r = p^* = \frac{np}{n-p}$ (and then $\gamma = 1 - \frac{1}{n}$) leads to the Sobolev inequalities:

Corollary 3.5 *(*Sobolev inequalities*) If $1 < p < n$, then for any $f \in W^{1,p}(\mathbb{R}^n)$,*

$$\| f \|_{p^*} \leq C(p,n) \| \nabla f \|_p \tag{56}$$

for some constant $C(p,n) > 0$.

Proof: It follows directly from (53), by using $\gamma = 1 - \frac{1}{n}$ and $r = p^*$. Note that the scaling argument cannot be used here to compute the best constant $C(p,n)$ in (56), since $\| \nabla f_\lambda \|_p^p = \lambda^{p-n} \| \nabla f \|_p^p$ and $\| f_\lambda \|_r^p = \lambda^{p-n} \| f \|_r^p$ scale the same way in (55). Instead, one can proceed directly from (53) to have that

$$\| f \|_{p^*} = 1 \leq \left(\frac{r\gamma}{p\left[H^{P_F}(\rho_\infty) - C_\infty\right]}\right)^{1/p} \| \nabla f \|_p = \left(\frac{p^*(n-1)}{np\left[H^{P_F}(\rho_\infty) - C_\infty\right]}\right)^{1/p} \| \nabla f \|_p,$$

which shows that

$$C(p,n) = \left(\frac{p^*(n-1)}{np\left[H^{P_F}(\rho_\infty) - C_\infty\right]}\right)^{1/p}, \tag{57}$$

where $\rho_\infty = h_\infty^{p^*} = \left(\frac{p^*}{nq}| x |^q - \frac{C_\infty}{n-1}\right)^{-n}$ is obtained from (54), and C_∞ can be found using that ρ_∞ is a probability density,

$$C_\infty = (1-n)\left[\int_{R^n}\left(\frac{p^*}{nq}| x |^q + 1\right)^{-n} \mathrm{d}x\right]^{p/n}. \tag{58}$$

4 The General Logarithmic Sobolev Inequality

In this section, we consider the case where c is a quadratic Young function of the form $c(x) := c_\sigma(x) = \frac{1}{2\sigma}| x |^2$ for $\sigma > 0$. In this case, our basic inequality (1) simplifies considerably to yield Theorem 4.1 below, which relates the total energy of two arbitrary probability densities, their Wasserstein distance, their barycenters and their entropy production functional. This gives yet another remarkable extension of various powerful inequalities by Gross [20], Bakry-Emery[4], Talagrand [29], Otto-Villani [27], Cordero[13] and others.

Theorem 4.1 (General Logarithmic Sobolev Inequality) *Under the hypothesis of Theorem 2.1, we have for all $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$, satisfying $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, and any $\sigma > 0$,*

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) + \frac{1}{2}(\mu + \nu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \frac{\sigma}{2} I_2(\rho_0|\rho_U). \tag{59}$$

Proof: Apply inequality (16) with a quadratic Young functional $c(x) = \frac{1}{2\sigma}|\,x\,|^2$, $V = U - c$ and $\lambda = \mu - \frac{1}{\sigma}$ to obtain

$$\begin{aligned} \mathrm{H}_U^{F,W}(\rho_0|\rho_1) \quad &+ \quad \frac{1}{2}(\mu + \nu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \\ &\leq \quad \mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla\left(F'(\rho_0) + U - c + W \star \rho_0\right)\right) \mathrm{d}x. \end{aligned} \tag{60}$$

Now we show the identity:

$$\mathcal{I}_{c_\sigma^*}(\rho_0|\rho_V) + H_{c_\sigma + x\cdot\nabla V}^{-nP_F, 2x\cdot\nabla W}(\rho_0) = \mathcal{I}_{c_\sigma^*}(\rho_0|\rho_{V+c_\sigma}) = \frac{\sigma}{2} I_2(\rho_0|\rho_{V+c_\sigma}).$$

Indeed, by elementary computations, we have

$$\begin{aligned} &\int_\Omega \rho_0 c^* \left(-\nabla\left(F' \circ \rho_0 + U - c + W \star \rho_0\right)\right) \mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 \Big|\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, \Big|^2 \mathrm{d}x + \frac{1}{2\sigma} \int_\Omega \rho_0 |\,x\,|^2 \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla\left(F'(\rho_0)\right) \mathrm{d}x \\ &\qquad - \int_\Omega \rho_0 x \cdot \nabla U \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla (W \star \rho_0) \,\mathrm{d}x, \end{aligned}$$

and

$$\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) = -\mathrm{H}^{nP_F}(\rho_0) + \int_\Omega \rho_0 x \cdot \nabla(W \star \rho_0)\,\mathrm{d}x + \int_\Omega \rho_0 x \cdot \nabla U \,\mathrm{d}x - \frac{1}{2\sigma} \int_\Omega |\,x\,|^2 \rho_0 \,\mathrm{d}x.$$

By combining the last 2 identities, we can rewrite the right hand side of (60) as

$$\begin{aligned} &\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla(F' \circ \rho_0 + U - c + W \star \rho_0)\right) \mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 |\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, |^2 \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla\left(F' \circ \rho_0\right) \mathrm{d}x - \int_\Omega n P_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 |\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, |^2 \mathrm{d}x + \int_\Omega \operatorname{div}(\rho_0 x) F'(\rho_0)\,\mathrm{d}x - \int_\Omega n P_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 \Big|\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, \Big|^2 \mathrm{d}x + n \int_\Omega \rho_0 F'(\rho_0)\,\mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho_0)\,\mathrm{d}x \\ &\qquad - \int_\Omega n P_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 \Big|\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, \Big|^2 \mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho_0)\,\mathrm{d}x + n \int_\Omega F \circ \rho_0 \,\mathrm{d}x \\ &\quad = \frac{\sigma}{2} \int_\Omega \rho_0 \Big|\, \nabla\left(F'(\rho_0) + U + W \star \rho_0\right)\, \Big|^2 \mathrm{d}x. \end{aligned} \tag{61}$$

Inserting (61) into (60), we conclude (59).

4.1 HWBI inequalities

We now establish the HWBI inequality which extends the HWI inequality established in [27] and [10], with the additional "B" referring here to the new barycentric term.

Theorem 4.2 (HWBI inequality) *Under the hypothesis of Theorem 2.1, we have for all $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$, satisfying $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \le W_2(\rho_0,\rho_1)\sqrt{I_2(\rho_0|\rho_U)} - \frac{\mu+\nu}{2}W_2^2(\rho_0,\rho_1) + \frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2. \tag{62}$$

Proof: Rewrite (59) as

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) + \frac{\mu+\nu}{2}W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \le \frac{1}{2\sigma}W_2^2(\rho_0,\rho_1) + \frac{\sigma}{2}I_2(\rho_0|\rho_U). \tag{63}$$

Now minimize the right hand side of (63) over $\sigma > 0$. The minimum is obviously achieved at $\bar\sigma = \frac{W_2(\rho_0,\rho_1)}{\sqrt{I_2(\rho_0|\rho_U)}}$. This yields (62).

Setting $W = 0$ (and then $\nu = 0$) in Theorem 4.2, we obtain in particular, the following HWI inequality first established by Otto-Villani [27] in the case of the classical entropy $F(x) = x\ln x$, and extended later on, for generalized entropy functions F by Carillo, McCann and Villani in [10].

Corollary 4.3 (HWI inequalities [10]) *Under the hypothesis on Ω and F in Theorem 2.1, let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function with $D^2U \ge \mu I$, where $\mu \in \mathbb{R}$. Then we have for all $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$ satisfying $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$\mathrm{H}_U^{F}(\rho_0|\rho_1) \le W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_U)} - \frac{\mu}{2}W_2^2(\rho_0,\rho_1). \tag{64}$$

If $U + W$ is uniformly convex (i.e., $\mu + \nu > 0$) inequality (59) yields the following extensions of the Log-Sobolev inequality:

Corollary 4.4 *(*Log-Sobolev inequalities with interaction potentials)
In addition to the hypothesis on Ω, F, U and W in Theorem 2.1, assume $\mu + \nu > 0$. Then for all $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$ satisfying $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \le \frac{1}{2(\mu+\nu)}I_2(\rho_0|\rho_U). \tag{65}$$

In particular, if $b(\rho_0) = b(\rho_1)$, we have that

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \le \frac{1}{2(\mu+\nu)}I_2(\rho_0|\rho_U). \tag{66}$$

Furthermore, if W is convex, then we have the following inequality, established in [10]

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \le \frac{1}{2\mu}I_2(\rho_0|\rho_U). \tag{67}$$

Proof: (65) follows easily from (59) by choosing $\sigma = \frac{1}{\mu+\nu}$, and (67) follows from (65), using $\nu = 0$ because W is convex.
In particular, setting $W = 0$ in Corollary 4.4, one obtains the following generalized Log-Sobolev inequality obtained in [11], and in [14] for generalized cost functions.

Corollary 4.5 (Generalized Log-Sobolev inequalities [11], [14])
Assume that Ω and F satisfy the assumptions in Theorem 2.1, and that $U : I\!R^n \to I\!R$ is a C^2- uniformly convex function with $D^2U \geq \mu I$, where $\mu > 0$. Then for all $\rho_0, \rho_1 \in \mathcal{P}_c(\Omega)$ satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$\mathrm{H}_U^F(\rho_0|\rho_1) \leq \frac{1}{2\mu} I_2(\rho_0|\rho_U). \tag{68}$$

One can also deduce the following generalization of Talagrand's inequality. We note in particular that when $W = 0$, the result below is obtained previously by Blower [6], Otto-Villani [27] and Bobkov-Ledoux [7] for the Tsallis entropy $F(x) = x \ln x$, and by Carillo-McCann-Villani [10] for generalized entropy functions F.

Corollary 4.6 *(*Generalized Talagrand Inequality with interaction potentials)
In addition to the hypothesis on Ω, F, U and W in Theorem 2.1, assume $\mu + \nu > 0$. Then for all probability densities ρ on Ω, we have

$$\frac{\nu+\mu}{2} W_2^2(\rho, \rho_U) - \frac{\nu}{2} |\mathrm{b}(\rho) - \mathrm{b}(\rho_U)|^2 \leq \mathrm{H}_U^{F,W}(\rho|\rho_U). \tag{69}$$

In particular, if $b(\rho) = b(\rho_U)$, we have that

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu+\nu}}. \tag{70}$$

Furthermore, if W is convex, then the following inequality established in [10] holds:

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu}}. \tag{71}$$

Proof: (69) follows from (59) if we use $\rho_0 := \rho_U$, $\rho_1 := \rho$, notice that $I_2(\rho_U|\rho_U) = 0$, and then let σ go to ∞. (71) follows from (69), where we use $\nu = 0$ because W is convex.

4.2 Inequalities with Boltzmann reference measures

To each confinement potential $U : I\!R^n \to I\!R$ with $D^2U \geq \mu I$ where $\mu \in I\!R$, one associates a Boltzmann reference measure denoted by ρ_U which is the normalized $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,\mathrm{d}x$ is assumed to be finite. To deduce inequalities involving such reference measures, we can apply Proposition 4.1 with $F(x) = x \ln x$ and $W = 0$ to get Gross' Log-Sobolev inequality (when $U(x) = \frac{1}{2}|x|^2$) and its extension by Bakry and Emery in [4] (when U uniformly convex). We first state the following HWI-type inequality from which we deduce Otto-Villani's HWI inequality [27], and the Log-Sobolev inequality of Gross [20] and Bakry-Emery [4].

Corollary 4.7 *Let $U : I\!\!R^n \to I\!\!R$ be a C^2-function with $D^2U \geq \mu I$ where $\mu \in I\!\!R$. Then for any $\sigma > 0$, the following holds for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} f\rho_U\, dx = 1$:*

$$\int_{R^n} f\ln(f)\,\rho_U\,dx + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(f\rho_U, \rho_U) \leq \frac{\sigma}{2}\int_{R^n} \frac{|\nabla f|^2}{f}\,\rho_U\,dx. \tag{72}$$

Proof: First assume that f has compact support, and set $F(x) = x\ln x$, $\rho_0 = f\rho_U$, $\rho_1 = \rho_U$ and $W = 0$ in (59). We have that

$$H_U^F(f\rho_U|\rho_U) + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(f\rho_U, \rho_U) \leq \frac{\sigma}{2}\int_{R^n} \Big|\frac{\nabla(f\rho_U)}{f\rho_U} + U\Big|^2 f\rho_U\,\mathrm{d}x. \tag{73}$$

By direct computations,

$$\frac{\nabla(f\rho_U)}{f\rho_U} = \frac{\nabla f}{f} - \nabla U, \tag{74}$$

and

$$\begin{aligned} \mathrm{H}_U^{F,W}(f\rho_U|\rho_U) &\leq \int_{R^n} [f\rho_U\ln(f\rho_U) + Uf\rho_U - \rho_U\ln\rho_U - U\rho_U]\,\mathrm{d}x \\ &= \int_{R^n} (f\rho_U\ln f)\,\mathrm{d}x + \ln\sigma_U\int_{R^n}(\rho_U - f\rho_U)\,\mathrm{d}x \\ &= \int_{R^n} f\ln(f)\rho_U\,\mathrm{d}x. \end{aligned} \tag{75}$$

Combining (73) - (75), we get (72). We finish the proof using a standard approximation argument.

Corollary 4.8 *(*Otto-Villani's HWI inequality [27]) *Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$, where $\mu > 0$. Then, for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} f\rho_U\,dx = 1$,*

$$\int_{R^n} f\ln(f)\rho_U\,dx \leq W_2(\rho_U, f\rho_U)\sqrt{I(f\rho_U|\rho_U)} - \frac{\mu}{2}W_2^2(f\rho_U, \rho_U), \tag{76}$$

where

$$I(f\rho_U|\rho_U) = \int_{R^n} \frac{|\nabla f|^2}{f}\rho_U\,dx.$$

Proof: It is similar to the proof of Theorem 4.2. Rewrite (72) as

$$\int_{R^n} f\ln(f)\rho_U\,\mathrm{d}x + \frac{\mu}{2}W_2^2(f\rho_U, \rho_U) \leq \frac{\mu}{2\sigma}W_2^2(f\rho_U, \rho_U) + \frac{\sigma}{2}I(f\rho_U|\rho_U),$$

and show that the minimum over $\sigma > 0$ of the right hand side is attained at $\bar{\sigma} = \frac{W_2(f\rho_U, \rho_U)}{\sqrt{I(f\rho_U|\rho_U)}}$.

Setting $f := g^2$ and $\sigma := \frac{1}{\mu}$ in (76), one obtains the following extension of Gross' [20] Log-Sobolev inequality first established by Bakry and Emery in [4].

Corollary 4.9 (Original Log Sobolev inequality [4], [20]) *Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$. Then, for any function g such that $g^2\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} g^2\rho_U\, dx = 1$, we have*

$$\int_{R^n} g^2 \ln(g^2)\, \rho_U\, dx \leq \frac{2}{\mu} \int_{R^n} |\, \nabla g\, |^2\, \rho_U\, dx. \tag{77}$$

As pointed out by Rothaus in [28], the above Log-Sobolev inequality implies the Poincaré's inequality.

Corollary 4.10 (Poincaré's inequality) *Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$. Then, for any function f such that $f\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} f\rho_U\, dx = 0$, we have*

$$\int_{R^n} f^2\rho_U\, dx \leq \frac{1}{\mu} \int_{R^n} |\, \nabla f\, |^2 \rho_U\, dx. \tag{78}$$

Proof: From (77), we have that

$$\int_{R^n} f_\epsilon \ln(f_\epsilon)\, \rho_U\, \mathrm{d}x \leq \frac{1}{2\mu} \int_{R^n} \frac{|\, \nabla f_\epsilon\, |^2}{f_\epsilon} \rho_U\, \mathrm{d}x, \tag{79}$$

where $f_\epsilon = 1 + \epsilon f$ for some $\epsilon > 0$. Using that $\int_{R^n} f\rho_U\, \mathrm{d}x = 0$, we have for small ϵ,

$$\int_{R^n} f_\epsilon \ln(f_\epsilon)\rho_U\, \mathrm{d}x = \frac{\epsilon^2}{2} \int_{R^n} f^2\rho_U\, \mathrm{d}x + o(\epsilon^3), \tag{80}$$

and

$$\int_{R^n} \frac{|\, \nabla f_\epsilon\, |^2}{f_\epsilon}\, \rho_U\, \mathrm{d}x = \epsilon^2 \int_{R^n} |\, \nabla f\, |^2 \rho_U\, \mathrm{d}x + o(\epsilon^3). \tag{81}$$

We combine (79) - (81) to have that

$$\int_{R^n} f^2\rho_U\, \mathrm{d}x \leq \frac{1}{\mu} \int_{R^n} |\, \nabla f\, |^2 \rho_U\, \mathrm{d}x + o(\epsilon). \tag{82}$$

We let ϵ go to 0 in (82) to conclude (78).

If we apply Corollary 4.6 to $F(x) = x \ln x$ when $W = 0$, we obtain the following extension of Talagrand's inequality established by Otto and Villani in [27].

Corollary 4.11 (Original Talagrand's inequality [29], [27]) *Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$. Then, for any nonnegative function f such that $\int_{R^n} f\rho_U\, dx = 1$, we have*

$$W_2(f\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu} \int_{R^n} f \ln(f)\rho_U\, dx}. \tag{83}$$

In particular, if $f = \frac{I_B}{\rho_U(B)}$ for some measurable subset B of $I\!R^n$, where $d\gamma(x) = \rho_U(x)\mathrm{d}x$ and $I\!I_B$ is the characteristic function of B, one obtains the following inequality in the concentration of measures in Gauss space, first proved by Talagrand building on an argument by Marton (see details in Villani [30]).

Corollary 4.12 (Concentration of measure inequality) *Let $U : I\!R^n \to I\!R$ be a C^2-uniformly convex function with $D^2 U \geq \mu I$ where $\mu > 0$. Then, for any ϵ-neighborhood B_ϵ of a measurable set B in $I\!R^n$, we have*

$$\gamma(B_\epsilon) \geq 1 - e^{-\frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2}, \tag{84}$$

where $\epsilon \geq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}$.

Proof: Using $f = f_B = \frac{I_B}{\gamma(B)}$ in (83), we have that

$$W_2(f_B\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)},$$

and then, we obtain from the triangle inequality that

$$W_2(f_B\rho_U, f_{R^n\setminus B_\epsilon}\rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)} + \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{1-\gamma(B_\epsilon)}\right)}. \tag{85}$$

But since $|\,x - y\,| \geq \epsilon$ for all $(x,y) \in B \times (I\!R^n \setminus B_\epsilon)$, we have that

$$W_2(f_B\rho_U, \rho_U) \geq \epsilon. \tag{86}$$

We combine (85) and (86) to deduce that

$$\ln\left(\frac{1}{1-\gamma(I\!R^n\setminus B_\epsilon)}\right) \geq \frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2,$$

which leads to (84).

5 Trends to equilibrium

We use Corollary 4.5 and Corollary 4.6 to recover rates of convergence for solutions to equation

$$\begin{cases} \frac{\partial\rho}{\partial t} = \mathrm{div}\left\{\rho\nabla\left(F'(\rho) + V + W \star \rho\right)\right\} & \text{in} \quad (0,\infty) \times I\!R^n \\ \\ \rho(t=0) = \rho_0 & \text{in} \quad \{0\} \times I\!R^n, \end{cases} \tag{87}$$

recently shown by Carillo, McCann and Villani in [10]. Here we consider the case where $V+W$ is uniformly convex and W convex, and the case when only $V+W$ is uniformly convex but the barycenter $b(\rho(t))$ of any solution $\rho(t,x)$ of (87) is invariant in t. For a background and other cases of convergence to equilibrium for this equation, we refer to [10] and the references therein.

Corollary 5.1 (Trend to equilibrium) *Let $F : [0,\infty) \to \mathbb{R}$ be strictly convex, differentiable on $(0,\infty)$ and satisfies $F(0)=0$, $\lim_{x\to\infty} \frac{F(x)}{x} = \infty$, and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing. Let V, $W : \mathbb{R}^n \to [0,\infty)$ be respectively C^2-confinement and interaction potentials with $D^2V \geq \lambda I$ and $D^2W \geq \nu I$, where $\lambda, \nu \in \mathbb{R}$. Assume that the initial probability density ρ_0 has finite total energy. Then*

1. *If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and W is convex (i.e. $\nu \geq 0$), then, for any solution ρ of (87), such that $\mathrm{H}_V^{F,W}(\rho(t)) < \infty$, we have:*
$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq e^{-2\lambda t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{88}$$
and
$$W_2(\rho(t),\rho_V) \leq e^{-\lambda t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda}}. \tag{89}$$

2. *If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and if we assume that the barycenter $b(\rho(t))$ of any solution $\rho(t,x)$ of (87) is invariant in t, then, for any solution ρ of (87) such that $\mathrm{H}_V^{F,W}(\rho(t)) < \infty$, we have:*
$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq e^{-2(\lambda+\nu)t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{90}$$
and
$$W_2(\rho(t),\rho_V) \leq e^{-2(\lambda+\nu)t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda+\nu}}. \tag{91}$$

Proof: Under the assumptions on F, V and W in Corollary 5.1, it is known (see [10], and references therein) that the total energy $\mathrm{H}_V^{F,W}$ – which is a Lyapunov functional associated with (87) – has a unique minimizer ρ_V defined by
$$\rho_V \nabla\left(F'(\rho_V) + V + W \star \rho_V\right) = 0 \quad \text{a.e.}$$
Now, let ρ be a – smooth – solution of (87). We have the following energy dissipation equation
$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) = -I_2(\rho(t)|\rho_V). \tag{92}$$
Combining (92) with (67), we have that
$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq -2\lambda\mathrm{H}_V^{F,W}(\rho(t)|\rho_V). \tag{93}$$

We integrate (93) over $[0,t]$ to conclude (88). (89) follows directly from (71) and (88). To prove (90), we use (92) and (66) to have that

$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq -2(\lambda+\nu)\mathrm{H}_V^{F,W}(\rho(t)|\rho_V). \tag{94}$$

We integrate (94) over $[0,t]$ to conclude (90). As before, (91) is a consequence of (90) and (70).

We now apply Corollary 5.1 to obtain rates of convergence to equilibrium for some equations of the form (87) studied in the literature by many authors.

- If $W=0$ and $F(x)=x\ln x$ in which case (87) is the linear Fokker-Planck equation $\frac{\partial\rho}{\partial t}=\Delta\rho+\mathrm{div}(\rho\nabla V)$, Corollary 5.1 gives an exponential decay in relative entropy of solutions of this equation to the Gaussian density $\rho_V=\frac{e^{-V}}{\sigma_V}$, $\sigma_V=\int_{R^n}e^{-V}\,dx$, at the rate 2λ when $D^2V\geq\lambda I$ for some $\lambda>0$, and an exponential decay in the Wasserstein distance, at the rate λ.
- If $W=0$, $F(x)=\frac{x^m}{m-1}$ where $1\neq m\geq 1-\frac{1}{n}$, and $V(x)=\lambda\frac{|x|^2}{2}$ for some $\lambda>0$, in which case (87) is the rescaled porous medium equation ($m>1$), or fast diffusion equation ($1-\frac{1}{n}\leq m<1$), that is $\frac{\partial\rho}{\partial t}=\Delta\rho^m+\mathrm{div}(\lambda x\rho)$, Corollary 5.1 gives an exponential decay in relative entropy of solutions of this equation to the Barenblatt-Prattle profile $\rho_V(x)=\left[\left(C+\frac{\lambda(1-m)}{2m}|\,x\,|^2\right)^{\frac{1}{m-1}}\right]^+$ (where $C>0$ is such that $\int_{R^n}\rho(x)\,dx=1$) at the rate 2λ, and an exponential decay in the Wasserstein distance at the rate λ.

6 The Energy-Entropy Duality Formula

In this section, we apply Theorem 2.1 with $V=W=0$, to obtain the following intriguing duality formula.

Proposition 6.1 (The Energy-Entropy Duality Formula) *Under the hypothesis of Theorem 2.1, we have for any $\rho_0,\rho_1\in\mathcal{P}_c(\Omega)$ satisfying supp $\rho_0\subset\Omega$ and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$, and any Young function $c:I\!R^n\to I\!R$:*

$$-\mathrm{H}_c^F(\rho_1)\leq -\mathrm{H}^{F+nP_F}(\rho_0)+\int_\Omega\rho_0c^\star\left(-\nabla(F'\circ\rho_0)\right)\,dx. \tag{95}$$

Moreover, equality holds whenever $\rho_0=\rho_1=\rho_c$ where ρ_c is a probability density on Ω such that $\nabla(F'(\rho_c)+c)=0$ a.e.

Motivated by the recent work of Cordero-Nazaret-Villani [12], we show that this inequality points to a remarkable correspondence between ground state solutions of some quasilinear PDEs or semi-linear equations which appear as Euler-Lagrange equations of the entropy production functionals and stationary solutions of Fokker-Planck type equations.

Corollary 6.1 *Under the hypothesis of Theorem 2.1, let $\psi : I\!\!R \to [0,\infty)$ differentiable be chosen in such a way that $\psi(0) = 0$ and $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ where $p > 1$, and K is chosen to be 1 for simplicity. Then, for any Young function c with p-homogeneous Legendre transform c^*, we have the following inequality:*

$$\sup\{-\int_\Omega F(\rho)+c\rho; \rho \in \mathcal{P}_a(\Omega)\} \leq \inf\{\int_\Omega c^*(-\nabla f)-G_F\circ\psi(f); f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} \tag{96}$$

where $G_F(x) := (1-n)F(x) + nxF'(x)$.
Furthermore, equality holds in (96) if there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies

$$-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x) \text{ a.e.} \tag{97}$$

Moreover, $\bar{f}$ solves

$$\begin{array}{ll} \mathrm{div}\{\nabla c^*(-\nabla f)\} - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f)\cdot \nu = 0 & \text{on } \partial\Omega, \end{array} \tag{98}$$

for some $\lambda \in I\!\!R$, while $\bar{\rho}$ is a stationary solution of

$$\begin{array}{ll} \frac{\partial \rho}{\partial t} = \mathrm{div}\{\rho \nabla \left(F'(\rho) + V\right)\} & \text{in } (0,\infty)\times\Omega \\ \rho\nabla\left(F'(\rho)+V\right)\cdot\nu = 0 & \text{on } (0,\infty)\times\partial\Omega. \end{array} \tag{99}$$

Proof: Assume that c^* is p-homogeneous, and let $Q''(x) = x^{\frac{1}{q}}F''(x)$ where q is the conjugate of p. Let

$$J(\rho) := -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy$$

and

$$\tilde{J}(\rho) := -\int_\Omega (F + nP_F)(\rho(x))dx + \int_\Omega c^*(-\nabla(Q'(\rho(x)))dx.$$

Equation (16) (where we use $V = W = 0$, and then $\lambda = \nu = 0$) then becomes

$$J(\rho_1) \leq \tilde{J}(\rho_0) \tag{100}$$

for all probability densities ρ_0, ρ_1 on Ω such that $\mathrm{supp}\,\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. If $\bar{\rho}$ satisfies

$$-\nabla(F'(\bar{\rho}(x))) = \nabla c(x) \text{ a.e.},$$

then equality holds in (100), and $\bar{\rho}$ is an extremal of the variational problems

$$\sup\{J(\rho);\ \rho \in \mathcal{P}_a(\Omega)\} = \inf\{\tilde{J}(\rho); \rho \in \mathcal{P}_a(\Omega), \mathrm{supp}\,\rho \subset \Omega, P_F(\rho) \in W^{1,\infty}(\Omega)\}.$$

In particular, $\bar{\rho}$ is a solution of

$$\begin{array}{ll} \mathrm{div}\{\rho\nabla(F'(\rho)+c)\} = 0 & \text{in } \Omega \\ \rho\nabla(F'(\rho)+c)\cdot\nu = 0 & \text{on } \partial\Omega. \end{array} \tag{101}$$

Suppose now $\psi : I\!R \to [0,\infty)$ differentiable, $\psi(0) = 0$ and that $\bar{f} \in C_0^\infty(\Omega)$ satisfies $-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x)$ a.e. Then equality holds in (100), and $\bar{f}$ and $\bar{\rho} = \psi(\bar{f})$ are extremals of the following variational problems

$$\inf\{I(f);\ f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} = \sup\{J(\rho); \rho \in \mathcal{P}_a(\Omega)\}$$

where

$$I(f) = \tilde{J}(\psi(f)) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla(Q' \circ \psi(f))).$$

If now ψ is such that $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = 1$, then $|\,(Q' \circ \psi)'\,| = 1$ and

$$I(f) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla f)),$$

because c^* is p-homogeneous. This proves (96). The Euler-Lagrange equation of the variational problem

$$\inf\Big\{\int_\Omega c^*(-\nabla(f)) - [F \circ \psi + nP_F \circ \psi](f);\ \int_\Omega \psi(f) = 1\Big\}$$

reads as

$$\begin{array}{ll} \operatorname{div}\{\nabla c^*(-\nabla f)\} - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f) \cdot \nu = 0 & \text{on } \partial\Omega \end{array} \tag{102}$$

where $\lambda \in I\!R$ is a Lagrange multiplier, and $G(x) = (1-n)F(x) + nxF'(x)$. This proves (98). To prove that the maximizer $\bar{\rho}$ of

$$\sup\{-\int_\Omega (F(\rho) + c\rho)\ \mathrm{d}x;\ \rho \in \mathcal{P}_a(\Omega)\}$$

is a stationary solution of (99), we refer to [21] and [25].
Now, we apply Corollary 6.1 to the functions $F(x) = x\ln x, \psi(x) = |\,x\,|^p$ and $c(x) = (p-1)|\,\mu x\,|^q$, with $\mu > 0$ and $c^*(x) = \frac{1}{p}\Big|\,\frac{x}{\mu}\,\Big|^p$ and $\frac{1}{p} + \frac{1}{q} = 1$, to derive a duality between stationary solutions of Fokker-Planck equations, and ground state solutions of some semi-linear equations. We note here that the condition $|\,\psi^{\frac{1}{p}}(F' \circ \psi)\,| = K$ holds for $K = p$. We obtain the following:

Corollary 6.2 *Let $p > 1$ and let q be its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$). For all $f \in W^{1,p}(I\!R^n)$, such that $\|\,f\,\|_p = 1$, any probability density ρ such that $\int_{R^n} \rho(x)|x|^q dx < \infty$, and any $\mu > 0$, we have*

$$J_\mu(\rho) \le I_\mu(f), \tag{103}$$

where

$$J_\mu(\rho) := -\int_{R^n} \rho \ln(\rho)\ dy - (p-1)\int_{R^n} |\,\mu y\,|^q \rho(y)\ dy,$$

and

$$I_\mu(f) := -\int_{R^n} |\,f\,|^p \ln(|\,f\,|^p) + \int_{R^n} \Big|\,\frac{\nabla f}{\mu}\,\Big|^p - n.$$

Furthermore, if $h \in W^{1,p}(I\!R^n)$ *is such that* $h \geq 0$, $\| h \|_p = 1$, *and*

$$\nabla h(x) = -\mu^q x |\, x \,|^{q-2} h(x) \quad a.e.,$$

then

$$J_\mu(h^p) = I_\mu(h).$$

Therefore, h *(resp.,* $\rho = h^p$*) is an extremum of the variational problem:*

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(I\!R^n),\ \| \rho \|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(I\!R^n), \| f \|_p = 1\}.$$

It follows that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, i.e., h is a solution of

$$\mu^{-p}\Delta_p f + pf |\, f \,|^{p-2} \ln(|\, f \,|) = \lambda f |\, f \,|^{p-2}, \tag{104}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^p$ is a stationary solution of the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \Delta u + \mathrm{div}(p\mu^q |x|^{q-2} x u). \tag{105}$$

We can also apply Corollary 6.1 to recover the duality associated to the Gagliardo-Nirenberg inequalities obtained recently in [12].

Corollary 6.3 *Let* $1 < p < n$, *and* $r \in \left(0, \frac{np}{n-p}\right]$ *such that* $r \neq p$. *Set* $\gamma := \frac{1}{r} + \frac{1}{q}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, for* $f \in W^{1,p}(I\!R^n)$ *such that* $\| f \|_r = 1$, *for any probability density* ρ *and for all* $\mu > 0$, *we have*

$$J_\mu(\rho) \leq I_\mu(f) \tag{106}$$

where

$$J_\mu(\rho) := -\frac{1}{\gamma - 1}\int_{R^n} \rho^\gamma - \frac{r\gamma\mu^q}{q}\int_{R^n} |\, y \,|^q \rho(y)(y)\, dy,$$

and

$$I_\mu(f) := -\left(\frac{1}{\gamma - 1} + n\right)\int_{R^n} |\, f \,|^{r\gamma} + \frac{r\gamma}{p\mu^p}\int_{R^n} |\, \nabla f \,|^p.$$

Furthermore, if $h \in W^{1,p}(I\!R^n)$ *is such that* $h \geq 0$, $\| h \|_r = 1$, *and*

$$\nabla h(x) = -\mu^q x |\, x \,|^{q-2} h^{\frac{r}{p}}(x) \quad a.e.,$$

then

$$J_\mu(h^r) = I_\mu(h).$$

Therefore, h *(resp.,* $\rho = h^r$*) is an extremum of the variational problems*

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(I\!R^n),\ \| \rho \|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(I\!R^n), \| f \|_r = 1\}.$$

Proof: Again, the proof follows from Corollary 6.1, by using now $\psi(x) = |\, x\,|^r$ and $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma \geq 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right]$. Indeed, for this value of γ, the function F satisfies the conditions of Corollary 6.1. The Young function is now $c(x) = \frac{r\gamma}{q}|\, \mu x\,|^q$, that is, $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\left|\, \frac{x}{\mu}\,\right|^p$, and the condition $|\, \psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ holds with $K = r\gamma$.
Moreover, if $h \geq 0$ satisfies (97), which is here,

$$-\nabla h(x) = \mu^q x|\, x\,|^{q-2} h^{\frac{r}{p}}(x) \text{ a.e.},$$

then h is extremal in the minimization problem defined in Corollary 6.3.
As above, we also note that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, that is, h is a solution of

$$\mu^{-p}\Delta_p f + \left(\frac{1}{\gamma-1} + n\right) f|\, f\,|^{r\gamma-2} = \lambda f|\, f\,|^{r-2}, \tag{107}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^r$ is a stationary solution of the evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u^\gamma + \mathrm{div}(r\gamma\mu^q|x|^{q-2}xu). \tag{108}$$

Example: In particular, when $\mu = 1, p = 2, \gamma = 1 - \frac{1}{n}$ and then $r = 2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, then Corollary 6.3 yields a duality between solutions of (107), which here the Yamabe equation:

$$-\Delta f = \lambda f|\, f\,|^{2^*-2},$$

(where λ is the Lagrange multiplier due to the constraint $\|\, f\,\|_{2^*} = 1$), and stationary solutions of (108), which is here the rescaled fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \mathrm{div}\left(\frac{2n-2}{n-2}xu\right).$$

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