Geometric inequalities via a general comparison principle for interacting gases

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Abstract

Using the Monge-Kantorovich theory of mass transport, we establish an inequality for the relative total energy – internal, potential and interactive – of two arbitrary probability densities, their Wasserstein distance, their barycenters and their entropy production functional. This inequality is remarkably encompassing as it implies most known geometrical – Gaussian and Euclidean – inequalities as well as new ones, while allowing a direct and unified way for computing best constants and extremals. As expected, such inequalities also lead to exponential rates of convergence to equilibria for solutions of Fokker-Planck and McKean-Vlasov type equations. The proposed inequality also leads to a remarkable correspondence between ground state solutions of certain quasilinear (or semi-linear) equations and stationary solutions of (non-linear) Fokker-Planck type equations

The article is written in a self-contained fashion as it offers a streamlined, unified and compact approach to a substantial number of inequalities originating in disparate areas of analysis and geometry. Some of the ideas presented here are known to the experts and may already be in the literature. They are included for the same pedagogical reasons that motivated the survey style of the paper.

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Contents

1 Introduction

The recent advances in the Monge-Kantorovich theory of mass transport have – among other things – led to new and quite natural proofs for a wide range of geometric inequalities. Most notable are McCann's generalization of the Brunn-Minkowski's inequality [21], Otto-Villani's [24] extension of the Log Sobolev inequality of Gross [18] and Bakry-Emery [2], as well as Cordero-Nazaret-Villani's proof of the Gagliardo-Nirenberg inequalities [11].

While this paper continues in this spirit, we however propose here a basic framework – already present in McCann's thesis [20] – to which most geometric inequalities belong, and a general inequality from which most of them follow. Besides the obvious pedagogical relevance of a streamlining approach, we find it interesting and intriguing that most of these inequalities appear as different manifestations of one basic principle in the theory of interacting gases that compares the different types of – internal, potential and interactive – energies of two states of a system after one is transported "at minimal cost" into another.

The main idea is to try to describe the evolution of a generalized energy functional along an optimal transport that takes one configuration to another, taking into account the – relative – entropy production functional, the transport cost (Wasserstein distance), as well as their centres of mass. Once this general comparison principle is established, then various – new and old – inequalities follow by simply considering different examples of – admissible – internal energies, of confinement and interactive potentials. Here is our framework:

Let $F : [0, \infty) \to I\!R$ be a differentiable function on $(0, \infty)$, V and W be C^2-real valued functions on $I\!R^n$ and let $\Omega \subset I\!R^n$ be open and convex. The set of probability densities over Ω is denoted by $\mathcal{P}_a(\Omega) = \{\rho : \Omega \to I\!\!R;\ \rho \geq 0 \,\text{and}\, \int_\Omega \rho(x)dx = 1\}$. The associated *Free Energy Functional* is then defined on $\mathcal{P}_a(\Omega)$ as:

$$\mathrm{H}_V^{F,W}(\rho) := \int_\Omega \left[F(\rho) + \rho V + \frac{1}{2}(W \star \rho)\rho \right] \mathrm{d}x,$$

which is the sum of the internal energy

$$\mathrm{H}^{\mathrm{F}}(\rho) := \int_\Omega F(\rho)dx,$$

the potential energy

$$\mathrm{H}_V(\rho) := \int_\Omega \rho V dx$$

and the interaction energy

$$\mathrm{H}^W(\rho) := \frac{1}{2}\int_\Omega \rho(W \star \rho)\,\mathrm{d}x.$$

Of importance is also the concept of *relative energy of ρ_0 with respect to ρ_1* simply defined as:

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) := \mathrm{H}_V^{F,W}(\rho_0) - \mathrm{H}_V^{F,W}(\rho_1).$$

where ρ_0 and ρ_1 are two probability densities. The *relative entropy production of ρ with respect to ρ_V* is normally defined as

$$I_2(\rho|\rho_V) = \int_\Omega \rho \Big| \nabla \left(F'(\rho) + V + W \star \rho\right) \Big|^2 \mathrm{d}x$$

in such a way that if $\rho_{_V}$ is a probability density that satisfies

$$\nabla \left(F'(\rho_{_V}) + V + W \star \rho_{_V}\right) = 0 \quad \text{a.e.}$$

then

$$I_2(\rho|\rho_{_V}) = \int_\Omega \rho |\nabla \left(F'(\rho) - F'(\rho_{_V}) + W \star (\rho - \rho_{_V}\right)|^2 \,\mathrm{d}x.$$

Our notation for the density $\rho_{_V}$ reflects this paper's emphasis on its dependence on the confinement potential, though it obviously also depends on F and W.
We need the notion of Wasserstein distance W_2 between two probability measures ρ_0 and ρ_1 on $I\!\!R^n$, defined as:

$$W_2^2(\rho_0, \rho_1) := \inf_{\gamma \in \Gamma(\rho_0, \rho_1)} \int_{R^n \times R^n} |x-y|^2 d\gamma(x,y),$$

where $\Gamma(\rho_0, \rho_1)$ is the set of Borel probability measures on $I\!\!R^n \times I\!\!R^n$ with marginals ρ_0 and ρ_1, respectively.
The *barycentre* (or centre of mass) of a probability density ρ, denoted

$$\mathrm{b}(\rho) := \int_{R^n} x\rho(x)dx$$

will play a role in the presence of an interactive potential.

In this paper, we shall also deal with non-quadratic versions of the entropy. For that we call *Young function*, any strictly convex C^1-function $c : I\!R^n \to I\!R$ such that $c(0) = 0$ and $\lim_{|x| \to \infty} \frac{c(x)}{|x|} = \infty$. We denote by c^* its Legendre conjugate defined by

$$c^*(y) = \sup_{z \in R^n} \{y \cdot z - c(z)\}.$$

For any probability density ρ on Ω, we define the *generalized relative entropy production-type function of ρ with respect to ρ_V measured against c^** by

$$\mathcal{I}_{c^*}(\rho|\rho_V) := \int_\Omega \rho c^\star \left(-\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x,$$

which is closely related to the *generalized relative entropy production function of ρ with respect to ρ_V measured against c^** defined as:

$$I_{c^*}(\rho|\rho_V) := \int_\Omega \rho \nabla \left(F'(\rho) + V + W \star \rho\right) \cdot \nabla c^\star \left(\nabla \left(F'(\rho) + V + W \star \rho\right)\right) \mathrm{d}x.$$

Indeed, the convexity inequality $c^*(z) \leq z \cdot \nabla c^*(z)$ satisfied by any Young function c, readily implies that $\mathcal{I}_{c^*}(\rho|\rho_V) \leq I_{c^*}(\rho|\rho_V)$. Note that when $c(x) = \frac{|x|^2}{2}$, we have

$$I_{c^*}(\rho|\rho_V) =: I_2(\rho|\rho_V) = \int_\Omega \rho \Big| \nabla \left(F'(\rho) + V + W \star \rho\right) \Big|^2 \mathrm{d}x = 2\mathcal{I}_{c^*}(\rho|\rho_V),$$

and we denote $\mathcal{I}_{c^*}(\rho|\rho_V)$ by $\mathcal{I}_2(\rho|\rho_V)$.

The following general inequality –established in section 2– is the main result of this paper. It relates the free energies of two –almost– arbitrary probability densities, their Wasserstein distance, their barycenters and their relative entropy production functional. The fact that it yields many admittedly powerful geometric inequalities is remarkable.

Basic comparison principle for interactive gases:
Let Ω be open, bounded and convex subset of $I\!R^n$, let $F : [0, \infty) \to I\!R$ be differentiable function on $(0, \infty)$ with $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Let $V : I\!R^n \to I\!R$ be a C^2-confinement potential with $D^2V \geq \lambda I$, and let W be an even C^2-interaction potential with $D^2W \geq \nu I$ where $\lambda, \nu \in I\!R$, and I denotes the identity map. Then, for any Young function $c : I\!R^n \to I\!R$, we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp} \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}^{F,W}_{V+c}(\rho_0|\rho_1) + \frac{\lambda + \nu}{2} W_2^2(\rho_0, \rho_1) - \frac{\nu}{2} |\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \mathrm{H}^{-nP_F, 2x\cdot\nabla W}_{c+\nabla V \cdot x}(\rho_0) + \mathcal{I}_{c^*}(\rho_0|\rho_V). \quad (1)$$

Furthermore, equality holds in (1) whenever $\rho_0 = \rho_1 = \rho_{V+c}$, where the latter satisfies

$$\nabla \left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad \text{a.e.} \qquad (2)$$

Quadratic case of the comparison principle for interactive gases:
The above equation simplifies considerably when c is a quadratic Young functional of the form $c(x) := c_\sigma(x) = \frac{1}{2\sigma}|\,x\,|^2$ for $\sigma > 0$, since then we have the identity:

$$\mathcal{I}_{c_\sigma^*}(\rho_0|\rho_V) + H^{-nP_F,2x\cdot\nabla W}_{c_\sigma + x\cdot\nabla V}(\rho_0) = \mathcal{I}_{c_\sigma^*}(\rho_0|\rho_{V+c_\sigma}) = \frac{\sigma}{2} I_2(\rho_0|\rho_{V+c_\sigma}).$$

Inequality (1) then yields: For all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp}\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have for any $\sigma > 0$,

$$\mathrm{H}^{F,W}_V(\rho_0|\rho_1) + \frac{1}{2}(\lambda + \nu - \frac{1}{\sigma})W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \le \frac{\sigma}{2} I_2(\rho_0|\rho_V). \tag{3}$$

Minimizing the above inequality over $\sigma > 0$ then yields the *HBWI inequality for interactive gases:*

$$\mathrm{H}^{F,W}_V(\rho_0|\rho_1) \le W_2(\rho_0,\rho_1)\sqrt{I_2(\rho_0|\rho_V)} - \frac{\lambda+\nu}{2} W_2^2(\rho_0,\rho_1) + \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2. \tag{4}$$

This extends the HWI inequality established in [24] and [9], with the additional "B" referring to the new barycentric terms.

In the remainder of this introduction, we describe various particular cases of inequalities (1) and (3) and show how they easily yield various – new and old – geometric inequalities.

Systems with no potential nor interaction energy – Euclidean geometric inequalities:

We start with the most basic system – where no potential nor interaction energies are involved– since it already contains many important features of the approach and their applications. Assume that $V = W = 0$ and that F is differentiable on $(0,\infty)$, $F(0) = 0$ and that $x \mapsto x^n F(x^{-n})$ is convex and non-increasing. Let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function and let $c : I\!R^n \to I\!R$ be any Young function. Inequality (1) gives that for any probability density $\rho_0 \in W^{1,\infty}(\Omega)$ with support in Ω, and any $\rho_1 \in \mathcal{P}_a(\Omega)$,

$$-\mathrm{H}^F_c(\rho_1) \le -\mathrm{H}^{F+nP_F}(\rho_0) + \int_\Omega \rho_0 c^\star\left(-\nabla(F'\circ\rho_0)\right) \mathrm{d}x, \tag{5}$$

and subsequently,

$$-\mathrm{H}^F_c(\rho_1) \le -\mathrm{H}^{F+nP_F}(\rho_0) + \mathcal{I}_{c^*}(\rho_0|\rho_\infty) \tag{6}$$

where ρ_∞ is a probability density such that $\nabla(F'(\rho_\infty)) = 0$ a.e.
Moreover, equality holds whenever $\rho_0 = \rho_1 = \rho_c$ where ρ_c is a probability density on Ω such that $\nabla(F'(\rho_c) + c) = 0$ a.e.

Applying the above inequality with any $\rho_0 = \rho$ and $\rho_1 = \rho_c$, we obtain the remarkably simple inequality:

$$H^{F+nP_F}(\rho) \le \int_\Omega \rho c^\star\left(-\nabla(F'\circ\rho)\right) \mathrm{d}x - H^{P_F}(\rho_c) + K_c, \tag{7}$$

where K_c is the unique constant determined by the equation

$$F'(\rho_c) + c = K_c \text{ and } \int_\Omega \rho_c = 1. \tag{8}$$

Applied to various – displacement convex – functionals F, one recovers several known inequalities.

For example, by taking $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma > 1 - \frac{1}{n}$, which satisfies the above assumptions, and by letting $c(x) = \frac{r\gamma}{2}|\, x\,|^2$ where $r \in \left(0, \frac{2n}{n-2}\right)$ we get that

$$\left(\frac{1}{\gamma-1} + n\right) \int_{R^n} |\, f\,|^{r\gamma} \leq \frac{r\gamma}{2} \int_{R^n} |\, \nabla f\,|^2 - \mathrm{H}^{P_F}(\rho_c) + K_c \tag{9}$$

for all $f \in C_c^\infty(I\!R^n)$ such that $\|\, f\,\|_r = 1$. A standard scaling argument on f now yields the Gagliardo-Nirenberg inequalities (See Corollary 3.3).

By taking $F(x) = x\ln(x)$ then $P_F(x) = x$, and if $c : I\!R^n \to I\!R$ is any Young function such that $c^\star$ is p-homogeneous for some $p > 1$, then $\rho_c = \frac{e^{-c(x)}}{\sigma_c}$ where $\sigma_c = \int_{R^n} e^{-c(x)}\,\mathrm{d}x$. Inequality (7) then yields for any $\rho \in \mathcal{P}_a(I\!R^n)$

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \leq \int_{R^n} \rho c^\star \left(-\frac{\nabla \rho}{\rho}\right) \mathrm{d}x - n - \ln\left(\int_{R^n} e^{-c(x)}\,\mathrm{d}x\right), \tag{10}$$

with equality when $\rho = \rho_c$. This time around, a scaling argument on the Young function c (Corollary 3.1) yields the Euclidean p-Log Sobolev inequality for any $p > 1$

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \leq \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} \rho c^\star \left(-\frac{\nabla \rho}{\rho}\right) \mathrm{d}x\right). \tag{11}$$

Such inequalities were first established by Beckner in [3] for $p = 1$, and by Del-Pino and Dolbeault [14] for $1 < p < n$. The case where $p > n$ was established recently and independently by I. Gentil [17] who used the Prékopa-Leindler inequality and the Hopf-Lax semi-group associated to the Hamilton-Jacobi equation.

Motivated by the recent work of [11], one can see that (5) yields a stronger statement of the following type

$$\sup\{J(\rho);\ \int_\Omega \rho(x)dx = 1\} \leq \inf\{I(f);\ \int_\Omega \psi(f(x))dx = 1\}, \tag{12}$$

where

$$I(f) = \int_\Omega \left[c^*(-\nabla f(x)) - G\left(\psi \circ f(x)\right)\right] dx \tag{13}$$

and

$$J(\rho) = -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy \tag{14}$$

with $G(x) = (1-n)F(x) + nxF'(x)$ and where ψ satisfies $|\, \psi^{\frac{1}{p}}(F' \circ \psi)'\,| = 1$. Here we have assumed that c^* is p-homogeneous. Moreover, we have equality in (12) whenever there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies the first order equation:

$$-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x) \text{ a.e.} \tag{15}$$

In this case, the extrema are achieved at $\bar{f}$ (resp. $\bar{\rho} = \psi(\bar{f})$). The latter is therefore a solution for the quasilinear (or semi-linear) equation

$$\operatorname{div}\{\nabla c^*(-\nabla f)\} - (G \circ \psi)'(f) = \psi'(f) \tag{16}$$

since it is essentially the L^2-Euler-Lagrange equation of I on the manifold

$$\{f \in C_0^\infty(\Omega);\ \int_\Omega \psi(f(x))dx = 1\}.$$

Equally interesting is the fact that $\psi(\bar{f})$ is also a stationary solution of the (non-linear) Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \operatorname{div}\{u\nabla(F'(u) + c)\} \tag{17}$$

since J is nothing but the Free Energy functional on $\mathcal{P}_a(\Omega)$, whose gradient flow with respect to the Wasserstein distance is precisely the evolution equation (17).

In other words, this points to a remarkable correspondence between Fokker-Planck evolution equations and certain quasilinear or semi-linear equations which appear as Euler-Lagrange equations of the entropy production functionals. Behind this correspondence lies a non-trivial "change of variable" that is given by the solution of the Monge transport problem. It essentially maps the solutions of the evolution equation associated to (13) to those of the Fokker-Planck equations (17). A typical example is the correspondence between the "Yamabe" equation

$$-\Delta f = |f|^{2^*-2} f \text{ on } I\!R^n, \tag{18}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, and the non-linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \operatorname{div}(x.u), \tag{19}$$

which –after appropriate scaling– reduces to the fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}}. \tag{20}$$

The correspondence was motivated by the work of [11] where mass transport is used to establish Sobolev-type inequalities. Solutions of (18) can be obtained by minimizing the energy functional on the unit sphere of L^{2^*}, that is:

$$\inf\Big\{ \Big(\frac{n-1}{n-2}\Big)^2 \int_{R^n} |\nabla f|^2 dx;\ f \in C_0^\infty(I\!R^n),\ \int_{R^n} |f|^{2^*} dx = 1\Big\}. \tag{21}$$

Using mass transport, they show that the above infimum is equal to the supremum of the functional

$$J(\rho) = n\int_{R^n} \rho(x)^{\frac{n-1}{n}} dx - \frac{1}{2}\int_{R^n} |x|^2 \rho(x) dx \tag{22}$$

over the space of probability densities.

Cordero et al. also deal with the Gagliardo-Nirenberg inequalities and obtain best constant results that Del Pino-Dolbeault had obtained earlier by carefully analyzing porous media evolution equations [14]. The link between the two methods becomes much clearer via the above correspondence. More details in section (6).

Systems with non-trivial potential but no interaction energy – Gaussian-type inequalities:

Assume now that F is as above but that $W = 0$, while $V : \mathbb{R}^n \to \mathbb{R}$ is a C^2-confinement potential with $D^2V \geq \lambda I$, where $\lambda \in \mathbb{R}$, and that $c : \mathbb{R}^n \to \mathbb{R}$ is again a Young function. Our basic inequality then yields: for all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp} \rho_0 \subset \Omega$, $\rho_0 > 0$ a.e. on Ω and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$-\mathrm{H}^{F}_{V+c}(\rho_1) + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1) \leq -\mathrm{H}^{F+nP_F}_{V-x\cdot\nabla V}(\rho_0) + \mathcal{I}_{c^*}(\rho_0|\rho_V) \tag{23}$$

where ρ_V is defined by $\nabla(F'(\rho_V) + V) = 0$ a.e. Furthermore, equality holds in (23) whenever $\rho_0 = \rho_1 = \rho_{V+c}$ where the latter satisfies

$$\nabla \left(F'(\rho_{V+c}) + V + c \right) = 0 \quad \text{a.e.} \tag{24}$$

In particular, we have for any probability density ρ such that $\operatorname{supp} \rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}^{F+nP_F}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda}{2} W_2^2(\rho, \rho_{V+c}) \leq \mathcal{I}_{c^*}(\rho|\rho_V) - \mathrm{H}^{P_F}(\rho_{V+c}) + K_{V+c} \tag{25}$$

where K_{V+c} is the unique constant such that

$$F'(\rho_{V+c}) + V + c = K_{V+c} \text{ while } \int_\Omega \rho_{V+c} = 1. \tag{26}$$

If V is a convex potential (i.e., $\lambda \geq 0$), then the term involving the Wasserstein distance can be omitted, and if V is strictly convex, then we have the identity $V(x) - x \cdot \nabla V(x) = -V^*(\nabla V(x)$ in such a way that a correcting "moment" appears in the inequality:

$$\mathrm{H}^{F+nP_F}_{-V^*(\nabla V)}(\rho) \leq \mathcal{I}_{c^*}(\rho|\rho_V) - \mathrm{H}^{P_F}(\rho_{V+c}) + K_{V+c}. \tag{27}$$

Again, the pressure P_F is always positive and we obtain the inequality:

$$\mathrm{H}^{F+nP_F}_{-V^*(\nabla V)}(\rho) \leq \mathcal{I}_{c^*}(\rho|\rho_V) + K_{V+c}. \tag{28}$$

If we now consider the quadratic case (i.e., inequality (3)), we then get for any $\sigma > 0$,

$$\mathrm{H}^F_V(\rho_0|\rho_1) + \frac{1}{2}(\lambda - \frac{1}{\sigma}) W_2^2(\rho_0, \rho_1) \leq \frac{\sigma}{2} I_2(\rho_0|\rho_V). \tag{29}$$

By letting $\rho_0 = \rho_V$, this already gives a *generalized Talagrand's inequality* : If V is uniformly convex (i.e., $\lambda > 0$), then for any probability density ρ on Ω,

$$W_2(\rho, \rho_V) \leq \sqrt{\frac{2\mathrm{H}^F_V(\rho|\rho_V)}{\lambda}}, \tag{30}$$

which in the case where $F(x) = x \ln x$ gives

$$W_2(f\rho_V, \rho_V) \leq \sqrt{\frac{2}{\lambda} \int_{R^n} f \ln(f) \rho_V \, dx}. \tag{31}$$

where here ρ_V is the normalized Gaussian $\frac{e^{-V}}{\sigma_V}$ and $\sigma_V = \int_{R^n} e^{-V} \, dx$.

Back to (29) and after minimization over $\sigma > 0$, one gets the *HWI inequality*:

$$\mathrm{H}_V^F(\rho_0|\rho_1) \leq W_2(\rho_0, \rho_1)\sqrt{I_2(\rho_0|\rho_V)} - \frac{\lambda}{2} W_2(\rho_0, \rho_1)^2. \tag{32}$$

This inequality, first established by Otto-Villani [24] contains many Gaussian inequalities. For example, it yields:
The generalized Log-Sobolev inequality : If V is uniformly convex (i.e., $\lambda > 0$), then for all probability densities ρ_0 and ρ_1 on Ω with $\operatorname{supp} \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}_V^F(\rho_0|\rho_1) \leq \frac{1}{2\lambda} I_2(\rho_0|\rho_V)^2. \tag{33}$$

which in the case where $F(x) = x \ln x$ yields the Log-Sobolev inequality of Gross [18] and Bakry-Emery [2]: for any function g such that $g^2 \rho_V \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} g^2 \rho_V \, dx = 1$, we have

$$\int_{R^n} g^2 \ln(g^2)\, \rho_V dx \leq \frac{2}{\lambda} \int_{R^n} |\nabla g|^2 \rho_V dx. \tag{34}$$

The general case of non-trivial confinement and interaction potentials:

Let F be as above, let again $V : I\!\!R^n \to I\!\!R$ be a C^2-confinement potential with $D^2 V \geq \lambda I$, but let now W be an even C^2-interaction potential with $D^2 W \geq \nu I$ where $\lambda, \nu \in I\!\!R$ (not necessarily positive). In this case, the general inequality (1) applied with $\rho_1 = \rho_{V+c}$ yields for any probability density ρ such that $\operatorname{supp} \rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$\begin{aligned} \mathrm{H}_{V - x\cdot\nabla V}^{F+nP_F,\, W-2x\cdot\nabla W}(\rho) + \frac{\lambda+\nu}{2} W_2^2(\rho, \rho_{V+c}) - \frac{\nu}{2} |\mathrm{b}(\rho) - \mathrm{b}(\rho_{V+c})|^2 \\ \leq \mathcal{I}_{c^*}(\rho|\rho_V) - \mathrm{H}^{P_F, W}(\rho_{V+c}) + K_{V+c} \end{aligned} \tag{35}$$

where K_{V+c} is such that

$$F'(\rho_{V+c}) + V + c + W \star \rho_{V+c} = K_{V+c} \text{ and } \int_\Omega \rho_{V+c} = 1. \tag{36}$$

If $\lambda + \nu \geq 0$, then the term involving the Wasserstein distance can be omitted from the equation. If W is convex, then the barycentric term can also be omitted, and if V is strictly convex, then we have

$$\mathrm{H}_{-V^*(\nabla V)}^{F+nP_F,\, W-2x\cdot\nabla W}(\rho) \leq I_{c^*}(\rho|\rho_V) + K_{V+c}. \tag{37}$$

On the other hand, the HWBI inequalities (4) obtained in the quadratic case have many interesting consequences. For example, *The generalized Log-Sobolev inequality with interaction potentials*: If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$), then for all probability density functions ρ_0 and ρ_1 on Ω with $\operatorname{supp}\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \leq \frac{1}{2(\lambda+\nu)} I_2(\rho_0|\rho_V), \tag{38}$$

The generalized Talagrand's inequality with interaction potentials: If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$), then for any probability density function ρ on Ω,

$$\frac{\nu+\lambda}{2} W_2^2(\rho,\rho_V) - \frac{\nu}{2}|\mathrm{b}(\rho)-\mathrm{b}(\rho_V)|^2 \leq \mathrm{H}_V^{F,W}(\rho|\rho_V). \tag{39}$$

In addition, if W is convex (i.e., $\nu \geq 0$), we obtain in particular:

$$\mathrm{H}_V^{F,W}(\rho_0|\rho_1) \leq \frac{1}{2\lambda} I_2(\rho_0|\rho_V). \tag{40}$$

and

$$W_2(\rho,\rho_V) \leq \sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho|\rho_V)}{\lambda}}. \tag{41}$$

Finally, these inequalities combined with the following energy dissipation equation

$$\frac{d}{dt}\,\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) = -I_2(\rho(t)|\rho_V), \tag{42}$$

provide rates of convergence to equilibria for solutions to the McKean-Vlasov type equation

$$\begin{cases} \frac{\partial\rho}{\partial t} = \operatorname{div}\{\rho\nabla(F'(\rho)+V+W\star\rho)\} & \text{in} \quad (0,\infty)\times I\!R^n \\ \rho(t=0)=\rho_0 & \text{in} \quad \{0\}\times I\!R^n. \end{cases} \tag{43}$$

One can then recover the recent results of Carillo, McCann and Villani in [9], which state that if $V+W$ is uniformly convex and if W is also convex then

$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq e^{-2\lambda t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{44}$$

and

$$W_2(\rho(t),\rho_V) \leq e^{-\lambda t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda}}. \tag{45}$$

If on the other hand, $V+W$ is uniformly convex, while the barycenter $b(\rho(t))$ of any solution $\rho(t,x)$ of (43) is invariant in t, then

$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V) \leq e^{-2(\lambda+\nu)t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{46}$$

and

$$W_2(\rho(t),\rho_V) \leq e^{-2(\lambda+\nu)t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda+\nu}}. \tag{47}$$

Throughout this paper, $\operatorname{supp}\rho$ denotes the support of $\rho \in \mathcal{P}_a(\Omega)$, that is, the closure of $\{x\in\Omega :\ \rho\neq 0\}$, $|\Omega|$ is the Lebesgue measure of $\Omega \subset I\!R^n$, and $q>1$ stands for the conjugate index of some $p<1$, $\frac{1}{p}+\frac{1}{q}=1$.

2 Main inequality between two configurations of interacting gases

Theorem 2.1 *Let Ω be open, bounded and convex subset of $I\!R^n$, let $F : [0, \infty) \to I\!R$ be differentiable function on $(0, \infty)$ with $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Let $V : I\!R^n \to I\!R$ be a C^2-confinement potential with $D^2V \geq \lambda I$, and let W be an even C^2-interaction potential with $D^2W \geq \nu I$ where $\lambda, \nu \in I\!R$, and I denotes the identity map. Then, for any Young function $c : I\!R^n \to I\!R$, we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying $\mathrm{supp}\, \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$\begin{aligned} &\mathrm{H}^{F,W}_{V+c}(\rho_0|\rho_1) + \frac{\lambda+\nu}{2} W_2^2(\rho_0, \rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \\ &\qquad \leq \mathrm{H}^{-nP_F, 2x\cdot\nabla W}_{c+\nabla V\cdot x}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla \left(F'(\rho_0) + V + W \star \rho_0\right)\right)\, dx. \end{aligned} \tag{48}$$

Furthermore, equality holds in (48) whenever $\rho_0 = \rho_1 = \rho_{V+c}$, where the latter satisfies

$$\nabla \left(F'(\rho_{V+c}) + V + c + W \star \rho_{V+c}\right) = 0 \quad a.e. \tag{49}$$

In particular, we have for any probability density ρ on Ω with $\mathrm{supp}\, \rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$\begin{aligned} &\mathrm{H}^{F+nP_F,\, W-2x\cdot\nabla W}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda+\nu}{2} W_2^2(\rho, \rho_{V+c}) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_{V+c})|^2 \\ &\qquad \leq \int_\Omega \rho c^\star \left(-\nabla \left(F'(\rho) + V + W \star \rho\right)\right)\, dx - \mathrm{H}^{P_F,W}(\rho_{V+c}) + K_{V+c}, \end{aligned} \tag{50}$$

where K_{V+c} is such that

$$F'(\rho_{V+c}) + V + c + W \star \rho_{V+c} = K_{V+c} \text{ while } \int_\Omega \rho_{V+c} = 1. \tag{51}$$

The proof is based on the recent advances in the theory of mass transport as developed by Brenier [7], Gangbo-McCann [15], [16], Caffarelli [8] and many others. For a survey, see Villani [27]. Here is a brief summary of the needed results.

Fix a non-negative C^1, strictly convex function $d : I\!R^n \to I\!R$ such that $d(0) = 0$. Given two probability measures μ and ν on $I\!R^n$, the minimum cost for transporting μ onto ν is given by

$$W_d(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{R^n \times R^n} d(x-y)\, d\gamma(x, y), \tag{52}$$

where $\Gamma(\mu, \nu)$ is the set of Borel probability measures with marginals μ and ν, respectively. When $d(x) = |\, x\,|^2$, we have that $W_d = W_2^2$, where W_2 is the Wasserstein distance.

We say that a Borel map $T : I\!R^n \to I\!R^n$ pushes μ forward to ν, if $\mu(T^{-1}(B)) = \nu(B)$ for any Borel set $B \subset I\!R^n$. The map T is then said to be d-optimal if

$$W_d(\mu,\nu) = \int_{R^n} d(x - Tx)d\mu(x) = \inf_S \int_{R^n} d(x - Sx)d\mu(x), \tag{53}$$

where the infimum is taken over all Borel maps $S : I\!R^n \to I\!R^n$ that push μ forward to ν.

For quadratic cost functions $d(z) = \frac{1}{2}|z|^2$, Brenier [7] characterized the optimal transport map T as the gradient of a convex function. An analogous result holds for general cost functions d, provided convexity is replaced by an appropriate notion of d-concavity. See [15], [8] for details.

Here is the lemma which leads to our main inequality (48). It is essentially a compendium of various observations by several authors. It describes the evolution of a generalized energy functional along optimal transport. The key idea lying behind it, is the concept of *displacement convexity* introduced by McCann [21]. For generalized cost functions, and when $V = 0$, it was first obtained by Otto [23] for the Tsallis entropy functionals and by Agueh [1] in general. The case of a nonzero confinement potential V and an interaction potential W was included in [13], [9]. Here, we state the results when the cost function is quadratic, $d(x) = |\,x\,|^2$.

Lemma 2.2 *Let $\Omega \subset I\!R^n$ be open, bounded and convex, and let ρ_0 and ρ_1 be probability densities on Ω, with supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. Let T be the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\,x\,|^2$. Then*

- *Assume $F : [0,\infty) \to I\!R$ is differentiable on $(0,\infty)$, $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, then the following inequality holds for the internal energy:*
$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq \int_\Omega \rho_0 (T - I)\cdot \nabla \left(F'(\rho_0)\right)\, dx. \tag{54}$$
- *Assume $V : I\!R^n \to I\!R$ is such that $D^2 V \geq \lambda I$ for some $\lambda \in I\!R$, then the potential energy satisfies*
$$\mathrm{H}_V(\rho_1) - \mathrm{H}_V(\rho_0) \geq \int_\Omega \rho_0 (T - I)\cdot \nabla V\, dx + \frac{\lambda}{2} W_2^2(\rho_0,\rho_1). \tag{55}$$
- *Assume $W : I\!R^n \to I\!R$ is even, and $D^2 W \geq \nu I$ for some $\nu \in I\!R$, then the interaction energy satisfies*
$$\begin{aligned}\mathrm{H}^W(\rho_1) - \mathrm{H}^W(\rho_0) \;\; &\geq \;\; \int_\Omega \rho_0 (T - I)\cdot \nabla (W \star \rho_0)\, dx \\ &\qquad + \frac{\nu}{2}\left(W_2^2(\rho_0,\rho_1) - |\mathrm{b}(\rho_0) - b(\rho_1)|^2\right).\end{aligned} \tag{56}$$

Proof: If T is the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ for the quadratic cost $d(x) = |\,x\,|^2$, define a path of probability densities joining them, by letting ρ_t be the push-forward measure of ρ_0 by the map $T_t = (1-t)I + tT$. It is known from the correspondence between Lagrangian and Eulerian coordinates that – at least for smooth ρ_t – the trajectory T_t satisfies

$$\begin{cases} \frac{\partial T_t}{\partial t} &= U_{\rho_t}(t, T_t) \\ T_0 &= I_\Omega, \end{cases}$$

where the velocity U_{ρ_t} is such that

$$\begin{cases} \frac{\partial \rho_t}{\partial t} + \mathrm{div}\,(\rho_t U_{\rho_t}) = 0 \\ \rho_{t=0} = \rho_0. \end{cases}$$

(1) Under the above assumptions on F, it turns out (see McCann [21]) that the function $t \mapsto \mathrm{H}^F(\rho_t)$ is convex on $[0,1]$, which essentially leads to (54) via the following inequality for the internal energy:

$$\mathrm{H}^F(\rho_1) - \mathrm{H}^F(\rho_0) \geq [\frac{d}{dt}\mathrm{H}^F(\rho_t)]_{t=0} = -\int_\Omega F'(\rho_0)\,\mathrm{div}\,(\rho_0(T-I))\,\mathrm{d}x. \tag{57}$$

(2) As noted in [13], the fact that $D^2V \geq \lambda I$, which means that

$$V(b) - V(a) \geq \nabla V(a)\cdot(b-a) + \frac{\lambda}{2}|\,a-b\,|^2$$

for all $a, b \in I\!R^n$, easily implies (55) via the following inequality for the corresponding potential energy:

$$\begin{aligned} \mathrm{H}_V(\rho_1) - \mathrm{H}_V(\rho_0) &\geq [\frac{d}{dt}\mathrm{H}_V(\rho_t)]_{t=0} + \frac{\lambda}{2}\int_\Omega |(T-I)(x)|^2\rho_0(x)dx \\ &= -\int_\Omega V\mathrm{div}\,(\rho_0(T-I))\,\mathrm{d}x + \frac{\lambda}{2}W_2^2(\rho_0,\rho_1). \end{aligned} \tag{58}$$

(3) The proof of (56) can be found in Cordero-Gangbo-Houdré [13]. But for completeness, we repeat the argument of these authors here. Indeed, following [13], we write the interaction energy as follows:

$$\begin{aligned} \mathrm{H}^W(\rho_1) &= \frac{1}{2}\int_{\Omega\times\Omega} W(x-y)\rho_1(x)\rho_1(y)\,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\int_{\Omega\times\Omega} W(T(x)-T(y))\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\int_{\Omega\times\Omega} W\left(x-y+(T-I)(x)-(T-I)(y)\right)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\ &\geq \frac{1}{2}\int_{\Omega\times\Omega} [W(x-y)+\nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))\,\rho_0(x)\rho_0(y)]\,\mathrm{d}x\mathrm{d}y \\ &\quad +\frac{\nu}{4}\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\ &= \mathrm{H}^W(\rho_0) + \frac{1}{2}\int_{\Omega\times\Omega} \nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))\,\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\ &\quad +\frac{\nu}{4}\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{59}$$

where we used above that $D^2W \geq \nu I$. The last term of the subsequent inequality can be written as:

$$
\begin{aligned}
&\int_{\Omega\times\Omega} |(T-I)(x)-(T-I)(y)|^2\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_\Omega |(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2\Big|\int_{R^n}(T-I)(x)\rho_0(x)\,\mathrm{d}x\Big|^2 \\
&\quad = 2\int_\Omega |(T-I)(x)|^2\rho_0(x)\,\mathrm{d}x - 2|\mathrm{b}(\rho_1)-\mathrm{b}(\rho_0)|^2. \qquad (60)
\end{aligned}
$$

And since ∇W is odd (because W is even), we get for the second term of (59)

$$
\begin{aligned}
&\int_{\Omega\times\Omega} [\nabla W(x-y)\cdot((T-I)(x)-(T-I)(y))]\,\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_{\Omega\times\Omega} \nabla W(x-y)\cdot(T-I)(x)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = 2\int_{\Omega\times\Omega} \rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x. \qquad (61)
\end{aligned}
$$

Combining (59) - (61), we obtain that

$$
\begin{aligned}
&\mathrm{H}^W(\rho_1)-\mathrm{H}^W(\rho_0) \\
&\quad \geq \int_{\Omega\times\Omega}\rho_0(T-I)\cdot\nabla(W\star\rho_0)\,\mathrm{d}x + \frac{\nu}{2}\left(\int_\Omega |(T-I)(x)|^2\rho_0 dx - |\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2\right).
\end{aligned}
$$

This proves (56).

Proof of Theorem 2.1: Adding (54), (55) and (56), one gets

$$
\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \qquad (62)\\
&\quad \leq \int_\Omega (x-Tx)\cdot\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\,\mathrm{d}x.
\end{aligned}
$$

Since $\rho_0\nabla(F'(\rho_0)) = \nabla\left(P_F(\rho_0)\right)$, we integrate by part $\int_\Omega \rho_0\nabla\left(F'(\rho_0)\right)\cdot x\,\mathrm{d}x$, and obtain that

$$
\int_\Omega x\cdot\nabla(F'(\rho_0)+V+W\star\rho_0)\rho_0 = \mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0).
$$

This leads to

$$
\begin{aligned}
&\mathrm{H}_V^{F,W}(\rho_0)-\mathrm{H}_V^{F,W}(\rho_1)+\frac{\lambda+\nu}{2}W_2^2(\rho_0,\rho_1)-\frac{\nu}{2}|\mathrm{b}(\rho_0)-\mathrm{b}(\rho_1)|^2 \qquad (63)\\
&\leq\quad \mathrm{H}_{x\cdot\nabla V}^{-nP_F,\,2x\cdot\nabla W}(\rho_0)-\int_\Omega\rho_0\nabla\left(F'(\rho_0)+V+W\star\rho_0\right)\cdot T(x)\,\mathrm{d}x.
\end{aligned}
$$

Now, use Young's inequality to get

$$
\begin{aligned}
&-\nabla\left(F'\left(\rho_0(x)\right)+V(x)+(W\star\rho_0)(x)\right)\cdot T(x) \qquad (64)\\
&\quad \leq c\left(T(x)\right)+c^\star\left(-\nabla\left(F'(\rho_0(x))+V(x)+(W\star\rho_0)(x)\right)\right),
\end{aligned}
$$

and deduce that

$$\mathrm{H}_V^{F,W}(\rho_0) - \mathrm{H}_V^{F,W}(\rho_1) + \frac{\lambda+\mu}{2} W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \qquad (65)$$
$$\leq \mathrm{H}_{x\cdot\nabla V}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^\star\left(-\nabla\left(F'(\rho_0) + V + W\star\rho_0\right)\right) + \int_\Omega c(Tx)\rho_0 \, \mathrm{d}x.$$

Finally, use again that T pushes ρ_0 forward to ρ_1, to rewrite the last integral on the right hand side of (65) as $\int_\Omega c(y)\rho_1(y)dy$ to obtain (48).
Now, set $\rho_0 = \rho_1 := \rho_{V+c}$ in (63). We have that $T = I$, and equality then holds in (63). Therefore, equality holds in (48) whenever equality holds in (64), where $T(x) = x$. This occurs when (49) is satisfied.
(50) is straightforward when choosing $\rho_0 := \rho$ and $\rho_1 := \rho_{V+c}$ in (48).

3 Optimal Euclidean Sobolev inequalities

3.1 Euclidean Log-Sobolev inequalities

The following optimal Euclidean p-Log Sobolev inequality was established by Beckner [3] in the case where $p = 1$, by Del Pino- Dolbeault [14] for $1 < p < n$, and independently by Gentil for all $p > 1$.

Corollary 3.1 *(*General Euclidean Log-Sobolev inequality*)*
Let $\Omega \subset I\!R^n$ be open bounded and convex, and let $c : I\!R^n \to I\!R$ be a Young functional such that its conjugate $c^\star$ is p-homogeneous for some $p > 1$. Then,

$$\int_{R^n} \rho \ln \rho \, dx \leq \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) dx\right), \qquad (66)$$

for all probability densities ρ on $I\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(I\!R^n)$. Here, $\sigma_c := \int_{R^n} e^{-c} \, dx$. Moreover, equality holds in (66) if $\rho(x) = K_\lambda e^{-\lambda^q c(x)}$ for some $\lambda > 0$, where $K_\lambda = \left(\int_{R^n} e^{-\lambda^q c(x)} \, dx\right)^{-1}$ and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

Proof: Use $F(x) = x\ln(x)$ and $V = W = 0$ in (50). Note that $P_F(x) = x$, and then, $H^{P_F}(\rho) = 1$ for any $\rho \in \mathcal{P}_a(I\!R^n)$. So, $\rho_c(x) = \frac{e^{-c(x)}}{\sigma_c}$. We then have for $\rho \in \mathcal{P}_a(I\!R^n) \cap W^{1,\infty}(I\!R^n)$ such that $\operatorname{supp}\rho \subset \Omega$,

$$\int_\Omega \rho \ln \rho \, \mathrm{d}x \leq \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x - n - \ln\left(\int_{R^n} e^{-c(x)} \, \mathrm{d}x\right), \qquad (67)$$

with equality when $\rho = \rho_c$.

Now assume that $c^\star$ is p-homogeneous and set $\Gamma_\rho^c = \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x$. Using $c_\lambda(x) := c(\lambda x)$ in (67), we get for $\lambda > 0$ that

$$\int_{R^n} \rho \ln \rho \, \mathrm{d}x \leq \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\lambda\rho}\right) \mathrm{d}x + n\ln\lambda - n - \ln\sigma_c, \qquad (68)$$

for all $\rho \in \mathcal{P}_a(I\!\!R^n)$ satisfying $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\Omega)$. Equality holds in (68) if $\rho_\lambda(x) = \left(\int_{R^n} e^{-\lambda^q c(x)}\,\mathrm{d}x\right)^{-1} e^{-\lambda^q c(x)}$. Hence

$$\int_{R^n} \rho \ln \rho \, \mathrm{d}x \leq -n - \ln \sigma_c + \inf_{\lambda>0} \left(G_\rho(\lambda)\right),$$

where

$$G_\rho(\lambda) = n\ln(\lambda) + \frac{1}{\lambda^p}\int_{R^n} \rho c^\star \left(-\frac{\nabla\rho}{\rho}\right) = n\ln(\lambda) + \frac{\Gamma^c_\rho}{\lambda^p}.$$

The infimum of $G_\rho(\lambda)$ over $\lambda > 0$ is attained at $\bar\lambda_\rho = \left(\frac{p}{n}\Gamma^c_\rho\right)^{1/p}$. Hence

$$\begin{aligned}\int_{R^n} \rho \ln \rho \, \mathrm{d}x &\leq G_\rho(\bar\lambda_\rho) - n - \ln(\sigma_c)\\ &= \frac{n}{p}\ln\left(\frac{p}{n}\Gamma^c_\rho\right) + \frac{n}{p} - n - \ln(\sigma_c)\\ &= \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\Gamma^c_\rho\right),\end{aligned}$$

for all probability densities ρ on $I\!\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$, and $\rho \in W^{1,\infty}(I\!\!R^n)$.

Corollary 3.2 *(Optimal Euclidean p-Log Sobolev inequality)*

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\, dx \leq \frac{n}{p}\ln\left(C_p \int_{R^n} |\,\nabla f\,|^p\, dx\right), \tag{69}$$

holds for all $p \geq 1$, and for all $f \in W^{1,p}(I\!\!R^n)$ such that $\|\, f\,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right)\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if } p>1,\\[2ex] \frac{1}{n\sqrt{\pi}}\left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if } p=1,\end{cases} \tag{70}$$

and q is the conjugate of p ($\frac{1}{p}+\frac{1}{q}=1$).
For $p>1$, equality holds in (69) for $f(x) = Ke^{-\lambda^q\frac{|\,x-\bar x\,|^q}{q}}$ for some $\lambda > 0$ and $\bar x \in I\!\!R^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\,\lambda x\,|^q}\,dx\right)^{-1/p}$.

Proof: First assume that $p > 1$, and set $c(x) = (p-1)|\,x\,|^q$ and $\rho = |\,f\,|^p$ in (66), where $f \in C_c^\infty(I\!\!R^n)$ and $\|\,f\,\|_p = 1$. We have that $c^\star(x) = \frac{|\,x\,|^p}{p^p}$, and then, $\int_{R^n} \rho c^* \left(-\frac{\nabla\rho}{\rho}\right)\mathrm{d}x = \int_{R^n} |\,\nabla f\,|^p\,\mathrm{d}x$. Therefore, (66) reads as

$$\int_{R^n} |\,f\,|^p \ln(|\,f\,|^p)\,\mathrm{d}x \leq \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\int_{R^n}|\,\nabla f\,|^p\,\mathrm{d}x\right). \tag{71}$$

Now, it suffices to note that

$$\sigma_c := \int_{R^n} e^{-(p-1)|\,x\,|^q}\,\mathrm{d}x = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{q}+1\right)}{(p-1)^{\frac{n}{q}}\Gamma\left(\frac{n}{2}+1\right)}. \tag{72}$$

To prove the case where $p = 1$, it is sufficient to apply the above to $p_\epsilon = 1 + \epsilon$ for some arbitrary $\epsilon > 0$. Note that

$$C_{p\epsilon} = \left(\frac{1+\epsilon}{n}\right)\left(\frac{\epsilon}{e}\right)^{\epsilon} \pi^{-\frac{1+\epsilon}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n\epsilon}{1+\epsilon}+1)}\right]^{\frac{1+\epsilon}{n}},$$

so that when ϵ go to 0, we have

$$\lim_{\epsilon\to 0} C_{p_\epsilon} = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{1}{n}} = C_1.$$

3.2 Sobolev and Gagliardo-Nirenberg inequalities

Corollary 3.3 *(*Gagliardo-Nirenberg inequalities*)*
Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right)$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(I\!R^n)$ we have

$$\|f\|_r \le C(p,r) \|\nabla f\|_p^\theta \, \|f\|_{r\gamma}^{1-\theta}, \tag{73}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{74}$$

$p^* = \frac{np}{n-p}$ and where the best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: Let $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma > 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right)$. For this value of γ, the function F satisfies the conditions of Theorem 2.1. Let $c(x) = \frac{r\gamma}{q} |\, x\,|^q$ so that $c^*(x) = \frac{1}{p(r\gamma)^{p-1}} |\, x\,|^p$, and set $V = W = 0$. Inequality (50) then gives for all $f \in C_c^\infty(I\!R^n)$ such that $\|\, f\,\|_r = 1$,

$$\left(\frac{1}{\gamma-1} + n\right)\int_{R^n} |\, f\,|^{r\gamma} \le \frac{r\gamma}{p}\int_{R^n} |\,\nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty. \tag{75}$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\, x\,|^{q-2} h^{\frac{r}{p}}(x) \ \text{ a.e.}, \tag{76}$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side of (75) are not easy to calculate, so one can obtain θ and the best constant by a standard scaling procedure. Namely, write (75) as

$$\frac{r\gamma}{p}\frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma-1} + n\right)\frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r^{r\gamma}} \ge H^{P_F}(\rho_\infty) - C_\infty =: C, \tag{77}$$

for some constant C. Then apply (77) to $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

The limiting case where r is the critical Sobolev exponent $r = p^* = \frac{np}{n-p}$ (and then $\gamma = 1 - \frac{1}{n}$) leads to the Sobolev inequalities:

Corollary 3.4 *(Sobolev inequalities)*
If $1 < p < n$, then for any $f \in W^{1,p}(I\!R^n)$,

$$\| f \|_{p^*} \le C(p,n) \| \nabla f \|_p \tag{78}$$

for some constant $C(p,n) > 0$.

Proof: It follows directly from (75), by using $\gamma = 1 - \frac{1}{n}$ and $r = p^*$.

Note that the scaling argument cannot be used here to compute the best constant $C(p,n)$ in (78), since $\| \nabla f_\lambda \|_p^p = \lambda^{p-n} \| \nabla f \|_p^p$ and $\| f_\lambda \|_r^p = \lambda^{p-n} \| f \|_r^p$ scale the same way in (77). Instead, one can proceed directly from (75) to have that

$$\| f \|_{p^*} = 1 \le \left(\frac{r\gamma}{p\,[H^{P_F}(\rho_\infty) - C_\infty]} \right)^{1/p} \| \nabla f \|_p = \left(\frac{p^*(n-1)}{np\,[H^{P_F}(\rho_\infty) - C_\infty]} \right)^{1/p} \| \nabla f \|_p,$$

which shows that

$$C(p,n) = \left(\frac{p^*(n-1)}{np\,[H^{P_F}(\rho_\infty) - C_\infty]} \right)^{1/p}, \tag{79}$$

where $\rho_\infty = h_\infty^{p^*} = \left(\frac{p^*}{nq} | x |^q - \frac{C_\infty}{n-1} \right)^{-n}$ is obtained from (76), and C_∞ can be found using that ρ_∞ is a probability density,

$$C_\infty = (1-n) \left[\int_{R^n} \left(\frac{p^*}{nq} | x |^q + 1 \right)^{-n} \mathrm{d}x \right]^{p/n}. \tag{80}$$

4 Optimal geometric inequalities

4.1 HWBI inequalities

We now establish HWBI inequalities relating the total energy of two arbitrary probability densities, their Wasserstein distance, their barycenters and their entropy production functional, and we deduce extensions of various powerful inequalities by Gross [18], Bakry-Emery[2], Talagrand [26], Otto-Villani [24], Cordero[12] and others.

Theorem 4.1 *(HWBI inequality)*
Let Ω be an open, bounded and convex subset of $I\!R^n$. Let $F : [0,\infty) \to I\!R$ be a differentiable function on $(0,\infty)$ with $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Let $U : I\!R^n \to I\!R$ be a C^2-confinement potential with $D^2U \ge \mu I$, and let W be an even C^2-interaction potential with $D^2W \ge \nu I$ where $\mu, \nu \in I\!R$. Then we have for all probability densities ρ_0 and ρ_1 on Ω satisfying $supp\,\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \le W_2(\rho_0,\rho_1)\sqrt{I_2(\rho_0|\rho_U)} - \frac{\mu+\nu}{2} W_2^2(\rho_0,\rho_1) + \frac{\nu}{2} |\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2. \tag{81}$$

The proof of Theorem 4.1 relies on the following proposition.

Proposition 4.1 *Under the above hypothesis on Ω and F, let $U, W : I\!R^n \to I\!R$ be C^2-functions with $D^2U \geq \mu I$ and $D^2W \geq \nu I$, where $\mu, \nu \in I\!R$, and W is even. Then for any $\sigma > 0$, we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) + \frac{1}{2}(\mu + \nu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \frac{\sigma}{2} I_2(\rho_0|\rho_U), \tag{82}$$

Proof: Use (48) with $c(x) = \frac{1}{2\sigma}|\, x\,|^2$, $V = U - c$ and $\lambda = \mu - \frac{1}{\sigma}$ to obtain

$$\begin{aligned} \mathrm{H}_U^{F,W}(\rho_0|\rho_1) \quad &+ \quad \frac{1}{2}(\mu + \nu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) + \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \\ &\leq \quad \mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla \left(F'(\rho_0) + U - c + W \star \rho_0\right)\right) \mathrm{d}x. \end{aligned} \tag{83}$$

By elementary computations, we have

$$\begin{aligned} &\int_\Omega \rho_0 c^* \left(-\nabla \left(F' \circ \rho_0 + U - c + W \star \rho_0\right)\right) \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big| \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) \Big|^2 \mathrm{d}x + \frac{1}{2\sigma}\int_\Omega \rho_0 |\, x\,|^2 \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla \left(F'(\rho_0)\right) \mathrm{d}x \\ &\qquad - \int_\Omega \rho_0 x \cdot \nabla U \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla(W \star \rho_0)\,\mathrm{d}x, \end{aligned}$$

and

$$\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) = -\mathrm{H}^{nP_F}(\rho_0) + \int_\Omega \rho_0 x \cdot \nabla(W \star \rho_0)\,\mathrm{d}x + \int_\Omega \rho_0 x \cdot \nabla U \,\mathrm{d}x - \frac{1}{2\sigma}\int_\Omega |\, x\,|^2 \rho_0 \,\mathrm{d}x.$$

By combining the last 2 identities, we can rewrite the right hand side of (83) as

$$\begin{aligned} &\mathrm{H}_{c+\nabla(U-c)\cdot x}^{-nP_F, 2x\cdot\nabla W}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla(F' \circ \rho_0 + U - c + W \star \rho_0)\right) \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 |\, \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) |^2 \,\mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla \left(F' \circ \rho_0\right) \mathrm{d}x - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 |\, \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) |^2 \mathrm{d}x + \int_\Omega \mathrm{div}\,(\rho_0 x) F'(\rho_0)\,\mathrm{d}x - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big| \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) \Big|^2 \mathrm{d}x + n\int_\Omega \rho_0 F'(\rho_0)\,\mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho_0)\,\mathrm{d}x \\ &\qquad - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big| \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) \Big|^2 \mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho_0)\,\mathrm{d}x + n\int_\Omega F \circ \rho_0 \,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big| \nabla \left(F'(\rho_0) + U + W \star \rho_0\right) \Big|^2 \mathrm{d}x. \end{aligned} \tag{84}$$

Inserting (84) into (83), we conclude (82).

Proof of Theorem 4.1: To establish the HWBI inequality (81), we rewrite (82) as

$$\begin{aligned}\mathrm{H}_U^{F,W}(\rho_0|\rho_1) + \frac{\mu+\nu}{2} W_2^2(\rho_0,\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \\ \leq \frac{1}{2\sigma} W_2^2(\rho_0,\rho_1) + \frac{\sigma}{2} I_2(\rho_0|\rho_U),\end{aligned} \tag{85}$$

then minimize the right hand side of (85) over $\sigma > 0$. The minimum is obviously achieved at $\bar{\sigma} = \frac{W_2(\rho_0,\rho_1)}{\sqrt{I_2(\rho_0|\rho_U)}}$. This yields (81).

Setting $W = 0$ (and then $\nu = 0$) in Theorem 4.1, we obtain in particular, the following HWI inequality first established by Otto-Villani [24] in the case of the classical entropy $F(x) = x \ln x$, and extended later on, for generalized entropy functions F by Carillo, McCann and Villani in [9].

Corollary 4.2 *(*HWI inequalities [9]*)*
Under the hypothesis on Ω and F in Theorem 4.1, let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function with $D^2U \geq \mu I$, where $\mu \in \mathbb{R}$. Then we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$\mathrm{H}_U^F(\rho_0|\rho_1) \leq W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_U)} - \frac{\mu}{2} W_2^2(\rho_0,\rho_1). \tag{86}$$

If $U + W$ is uniformly convex (i.e., $\mu + \nu > 0$) inequality (82) yields the following extensions of the Log-Sobolev inequality:

Corollary 4.3 *(*Log-Sobolev inequalities with interaction potentials*)*
In addition to the hypothesis on Ω, F, U and W in Theorem 4.1, assume $\mu + \nu > 0$. Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) - \frac{\nu}{2}|\mathrm{b}(\rho_0) - \mathrm{b}(\rho_1)|^2 \leq \frac{1}{2(\mu+\nu)} I_2(\rho_0|\rho_U). \tag{87}$$

In particular, if $b(\rho_0) = b(\rho_1)$, we have that

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \leq \frac{1}{2(\mu+\nu)} I_2(\rho_0|\rho_U). \tag{88}$$

Furthermore, if W is convex, then we have the following inequality, established in [9]

$$\mathrm{H}_U^{F,W}(\rho_0|\rho_1) \leq \frac{1}{2\mu} I_2(\rho_0|\rho_U). \tag{89}$$

Proof: (87) follows easily from (82) by choosing $\sigma = \frac{1}{\mu+\nu}$, and (89) follows from (87), using $\nu = 0$ because W is convex.

In particular, setting $W = 0$ in Corollary 4.3, one obtains the following generalized Log-Sobolev inequality obtained in [10], and in [13] for generalized cost functions.

Corollary 4.4 *(*Generalized Log-Sobolev inequalities *[10], [13])*
Assume that Ω and F satisfy the assumptions in Theorem 4.1, and that $U : I\!R^n \to I\!R$ is a C^2- uniformly convex function with $D^2U \geq \mu I$, where $\mu > 0$. Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying $\mathrm{supp}\,\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$\mathrm{H}_U^F(\rho_0|\rho_1) \leq \frac{1}{2\mu} I_2(\rho_0|\rho_U). \tag{90}$$

One can also deduce the following generalization of Talagrand's inequality. We note in particular that when $W = 0$, the result below is obtained previously by Blower [4], Otto-Villani [24] and Bobkov-Ledoux [5] for the Tsallis entropy $F(x) = x \ln x$, and by Carillo-McCann-Villani [9] for generalized entropy functions F.

Corollary 4.5 *(*Generalized Talagrand Inequality with interaction potentials*)*
In addition to the hypothesis on Ω, F, U and W in Theorem 4.1, assume $\mu + \nu > 0$. Then for all probability densities ρ on Ω, we have

$$\frac{\nu+\mu}{2} W_2^2(\rho, \rho_U) - \frac{\nu}{2} |\mathrm{b}(\rho) - \mathrm{b}(\rho_U)|^2 \leq \mathrm{H}_U^{F,W}(\rho|\rho_U). \tag{91}$$

In particular, if $b(\rho) = b(\rho_U)$, we have that

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu+\nu}}. \tag{92}$$

Furthermore, if W is convex, then the following inequality established in [9] holds:

$$W_2(\rho, \rho_U) \leq \sqrt{\frac{2\mathrm{H}_U^{F,W}(\rho|\rho_U)}{\mu}}. \tag{93}$$

Proof: (91) follows from (82) if we use $\rho_0 := \rho_U$, $\rho_1 := \rho$, notice that $I_2(\rho_U|\rho_U) = 0$, and then let σ go to ∞. (93) follows from (91), where we use $\nu = 0$ because W is convex.

4.2 Gaussian inequalities

Proposition 4.1 applied to $F(x) = x \ln x$ when $W = 0$, yields the following extension of Gross' Log-Sobolev inequality established by Bakry and Emery in [2]. First, we state the following HWI-type inequality from which we deduce Otto-Villani's HWI inequality [24], and the Log-Sobolev inequality of Gross [18] and Bakry-Emery [2].

Corollary 4.6 *Let $U : I\!R^n \to I\!R$ be a C^2-function with $D^2U \geq \mu I$ where $\mu \in I\!R$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,dx$. Then for any $\sigma > 0$, the following holds for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(I\!R^n)$ and $\int_{R^n} f\rho_U\,dx = 1$:*

$$\int_{R^n} f \ln(f)\, \rho_U\, dx + \frac{1}{2}(\mu - \frac{1}{\sigma}) W_2^2(f\rho_U, \rho_U) \leq \frac{\sigma}{2} \int_{R^n} \frac{|\nabla f|^2}{f} \rho_U\, dx. \tag{94}$$

Proof: First assume that f has compact support, and set $F(x) = x\ln x$, $\rho_0 = f\rho_U$, $\rho_1 = \rho_U$ and $W = 0$ in (82). We have that

$$H_U^F(f\rho_U|\rho_U) + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(f\rho_U, \rho_U) \le \frac{\sigma}{2}\int_{R^n} \Big| \frac{\nabla(f\rho_U)}{f\rho_U} + U \Big|^2 f\rho_U \,\mathrm{d}x. \tag{95}$$

By direct computations,

$$\frac{\nabla(f\rho_U)}{f\rho_U} = \frac{\nabla f}{f} - \nabla U, \tag{96}$$

and

$$\begin{aligned} \mathrm{H}_U^{F,W}(f\rho_U|\rho_U) &\le \int_{R^n} [f\rho_U \ln(f\rho_U) + Uf\rho_U - \rho_U \ln \rho_U - U\rho_U]\,\mathrm{d}x \qquad (97)\\ &= \int_{R^n} (f\rho_U \ln f)\,\mathrm{d}x + \ln \sigma_U \int_{R^n} (\rho_U - f\rho_U)\,\mathrm{d}x \\ &= \int_{R^n} f\ln(f)\rho_U\,\mathrm{d}x. \end{aligned}$$

Combining (95) - (97), we get (94). We finish the proof using a standard approximation argument.

Corollary 4.7 *(*Otto-Villani's HWI inequality [24]*)*
Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \ge \mu I$, where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,dx$. Then, for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} f\rho_U\,dx = 1$,

$$\int_{R^n} f\ln(f)\rho_U\,dx \le W_2(\rho_U, f\rho_U)\sqrt{I(f\rho_U|\rho_U)} - \frac{\mu}{2}W_2^2(f\rho_U, \rho_U), \tag{98}$$

where

$$I(f\rho_U|\rho_U) = \int_{R^n} \frac{|\nabla f|^2}{f}\rho_U\,dx.$$

Proof: It is similar to the proof of Theorem 4.1. Rewrite (94) as

$$\int_{R^n} f\ln(f)\rho_U\,\mathrm{d}x + \frac{\mu}{2}W_2^2(f\rho_U, \rho_U) \le \frac{\mu}{2\sigma}W_2^2(f\rho_U, \rho_U) + \frac{\sigma}{2}I(f\rho_U|\rho_U),$$

and show that the minimum over $\sigma > 0$ of the right hand side is attained at $\bar\sigma = \frac{W_2(f\rho_U,\rho_U)}{\sqrt{I(f\rho_U|\rho_U)}}$.

Now, setting $f := g^2$ and $\sigma := \frac{1}{\mu}$ in (98), one obtains the following extension of Gross' [18] Log-Sobolev inequality first established by Bakry and Emery in [2].

Corollary 4.8 *(*Original Log Sobolev inequality [2], [18]*)*
Let $U : I\!\!R^n \to I\!\!R$ be a C^2-uniformly convex function with $D^2U \ge \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\,dx$. Then, for any function g such that $g^2\rho_U \in W^{1,\infty}(I\!\!R^n)$ and $\int_{R^n} g^2\rho_U\,dx = 1$, we have

$$\int_{R^n} g^2\ln(g^2)\,\rho_U\,dx \le \frac{2}{\mu}\int_{R^n} |\nabla g|^2\,\rho_U\,dx. \tag{99}$$

As pointed out by Rothaus in [25], the above Log-Sobolev inequality implies the Poincaré's inequality.

Corollary 4.9 *(*Poincaré's inequality*)*
Let $U : I\!R^n \to I\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\, dx$. Then, for any function f such that $f\rho_U \in W^{1,\infty}(I\!R^n)$ and $\int_{R^n} f\rho_U\, dx = 0$, we have

$$\int_{R^n} f^2 \rho_U \, dx \leq \frac{1}{\mu} \int_{R^n} |\nabla f|^2 \rho_U \, dx. \tag{100}$$

Proof: From (99), we have that

$$\int_{R^n} f_\epsilon \ln(f_\epsilon)\, \rho_U \,\mathrm{d}x \leq \frac{1}{2\mu} \int_{R^n} \frac{|\nabla f_\epsilon|^2}{f_\epsilon} \rho_U \,\mathrm{d}x, \tag{101}$$

where $f_\epsilon = 1 + \epsilon f$ for some $\epsilon > 0$. Using that $\int_{R^n} f\rho_U \,\mathrm{d}x = 0$, we have for small ϵ,

$$\int_{R^n} f_\epsilon \ln(f_\epsilon) \rho_U \,\mathrm{d}x = \frac{\epsilon^2}{2} \int_{R^n} f^2 \rho_U \,\mathrm{d}x + o(\epsilon^3), \tag{102}$$

and

$$\int_{R^n} \frac{|\nabla f_\epsilon|^2}{f_\epsilon} \rho_U \,\mathrm{d}x = \epsilon^2 \int_{R^n} |\nabla f|^2 \rho_U \,\mathrm{d}x + o(\epsilon^3). \tag{103}$$

We combine (101) - (103) to have that

$$\int_{R^n} f^2 \rho_U \,\mathrm{d}x \leq \frac{1}{\mu} \int_{R^n} |\nabla f|^2 \rho_U \,\mathrm{d}x + o(\epsilon). \tag{104}$$

We let ϵ go to 0 in (104) to conclude (100).

If we apply Corollary 4.5 to $F(x) = x \ln x$ when $W = 0$, we obtain the following extension of Talagrand's inequality established by Otto and Villani in [24].

Corollary 4.10 *(*Original Talagrand's inequality [26], [24]*)*
Let $U : I\!R^n \to I\!R$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$, and denote by ρ_U the normalized Gaussian $\frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{R^n} e^{-U}\, dx$. Then, for any nonnegative function f such that $\int_{R^n} f\rho_U\, dx = 1$, we have

$$W_2(f\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu} \int_{R^n} f \ln(f) \rho_U \, dx}. \tag{105}$$

In particular, if $f = \frac{I_B}{\gamma(B)}$ for some measurable subset B of $I\!R^n$, where $d\gamma(x) = \rho_U(x)\mathrm{d}x$ and $I\!I_B$ is the characteristic function of B, we obtain the following inequality in the concentration of measures in Gauss space, first proved by Bobkov and Götze in [6].

Corollary 4.11 *(*Concentration of measure inequality [6]*)*
Let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-uniformly convex function with $D^2U \geq \mu I$ where $\mu > 0$, and denote by γ the normalized Gaussian measure with density $\rho_U = \frac{e^{-U}}{\sigma_U}$, where $\sigma_U = \int_{\mathbb{R}^n} e^{-U}\,dx$. Then, for any ϵ-neighborhood B_ϵ of a measurable set B in $\mathbb{R}^n$, we have

$$\gamma(B_\epsilon) \geq 1 - e^{-\frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2}, \tag{106}$$

where $\epsilon \geq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}$.

Proof: Using $f = f_B = \frac{I_B}{\gamma(B)}$ in (105), we have that

$$W_2(f_B\rho_U, \rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)},$$

and then, we obtain from the triangle inequality that

$$W_2(f_B\rho_U, f_{R^n\backslash B_\epsilon}\rho_U) \leq \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)} + \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{1-\gamma(B_\epsilon)}\right)}. \tag{107}$$

But since $|\,x - y\,| \geq \epsilon$ for all $(x, y) \in B \times (\mathbb{R}^n\backslash B_\epsilon)$, we have that

$$W_2(f_B\rho_U, \rho_U) \geq \epsilon. \tag{108}$$

We combine (107) and (108) to deduce that

$$\ln\left(\frac{1}{1-\gamma(\mathbb{R}^n\backslash B_\epsilon)}\right) \geq \frac{\mu}{2}\left(\epsilon - \sqrt{\frac{2}{\mu}\ln\left(\frac{1}{\gamma(B)}\right)}\right)^2,$$

which leads to (106).

5 Trends to equilibrium

We use Corollary 4.4 and Corollary 4.5 to recover rates of convergence for solutions to equation

$$\begin{cases} \frac{\partial\rho}{\partial t} = \operatorname{div}\{\rho\nabla\left(F'(\rho) + V + W\star\rho\right)\} & \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ \rho(t=0) = \rho_0 & \text{in} \quad \{0\}\times\mathbb{R}^n, \end{cases} \tag{109}$$

recently shown by Carillo, McCann and Villani in [9]. Here we consider the case where $V + W$ is uniformly convex and W convex, and the case when only $V + W$ is uniformly convex but the barycenter $b\left(\rho(t)\right)$ of any solution $\rho(t, x)$ of (109) is invariant in t. For a background and other cases of convergence to equilibrium for this equation, we refer to [9] and the references therein.

Corollary 5.1 *(*Trend to equilibrium*)*
Let $F:[0,\infty)\to I\!R$ be strictly convex, differentiable on $(0,\infty)$ and satisfies $F(0)=0$, $\lim_{x\to\infty}\frac{F(x)}{x}=\infty$, and $x\mapsto x^nF(x^{-n})$ is convex and non-increasing. Let V, $W:I\!R^n\to[0,\infty)$ be respectively C^2-confinement and interaction potentials with $D^2V\geq\lambda I$ and $D^2W\geq\nu I$, where $\lambda,\nu\in I\!R$. Assume that the initial probability density ρ_0 has finite total energy. Then

(i). If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and W is convex (i.e. $\nu\geq0$), then, for any solution ρ of (109), such that $\mathrm{H}_V^{F,W}(\rho(t))<\infty$, we have:

$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V)\leq e^{-2\lambda t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{110}$$

and

$$W_2(\rho(t),\rho_V)\leq e^{-\lambda t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda}}. \tag{111}$$

(ii). If $V+W$ is uniformly convex (i.e., $\lambda+\nu>0$) and if we assume that the barycenter $b(\rho(t))$ of any solution $\rho(t,x)$ of (109) is invariant in t, then, for any solution ρ of (109) such that $\mathrm{H}_V^{F,W}(\rho(t))<\infty$, we have:

$$\mathrm{H}_V^{F,W}(\rho(t)|\rho_V)\leq e^{-2(\lambda+\nu)t}\mathrm{H}_V^{F,W}(\rho_0|\rho_V), \tag{112}$$

and

$$W_2(\rho(t),\rho_V)\leq e^{-2(\lambda+\nu)t}\sqrt{\frac{2\mathrm{H}_V^{F,W}(\rho_0|\rho_V)}{\lambda+\nu}}. \tag{113}$$

Proof: Under the assumptions on F, V and W in Corollary 5.1, it is known (see [9], and references therein) that the total energy $\mathrm{H}_V^{F,W}$ – which is a Lyapunov functional associated with (109) – has a unique minimizer ρ_V defined by

$$\rho_V\nabla\left(F'(\rho_{_V})+V+W\star\rho_{_V}\right)=0\quad\text{a.e.}$$

Now, let ρ be a – smooth – solution of (109). We have the following energy dissipation equation

$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V)=-I_2(\rho(t)|\rho_V). \tag{114}$$

Combining (114) with (89), we have that

$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V)\leq-2\lambda\mathrm{H}_V^{F,W}(\rho(t)|\rho_V). \tag{115}$$

We integrate (115) over $[0,t]$ to conclude (110). (111) follows directly from (93) and (110).
To prove (112), we use (114) and (88) to have that

$$\frac{d}{dt}\mathrm{H}_V^{F,W}(\rho(t)|\rho_V)\leq-2(\lambda+\nu)\mathrm{H}_V^{F,W}(\rho(t)|\rho_V). \tag{116}$$

We integrate (116) over $[0,t]$ to conclude (112). As before, (113) is a consequence of (112) and (92).

Below, we apply Corollary 5.1 to obtain rates of convergence to equilibrium for some equations of the form (109) studied in the literature by many authors.

Examples:

- If $W = 0$ and $F(x) = x \ln x$ in which case (109) is the linear Fokker-Planck equation $\frac{\partial \rho}{\partial t} = \Delta\rho + \text{div}(\rho\nabla V)$, Corollary 5.1 gives an exponential decay in relative entropy of solutions of this equation to the Gaussian density $\rho_V = \frac{e^{-V}}{\sigma_V}$, $\sigma_V = \int_{R^n} e^{-V}\,\mathrm{d}x$, at the rate 2λ when $D^2V \geq \lambda I$ for some $\lambda > 0$, and an exponential decay in the Wasserstein distance, at the rate λ.

- If $W = 0$, $F(x) = \frac{x^m}{m-1}$ where $1 \neq m \geq 1 - \frac{1}{n}$, and $V(x) = \lambda\frac{|x|^2}{2}$ for some $\lambda > 0$, in which case (109) is the rescaled porous medium equation ($m > 1$), or fast diffusion equation ($1 - \frac{1}{n} \leq m < 1$), that is $\frac{\partial \rho}{\partial t} = \Delta\rho^m + \text{div}(\lambda x\rho)$, Corollary 5.1 gives an exponential decay in relative entropy of solutions of this equation to the Barenblatt-Prattle profile $\rho_V(x) = \left[\left(C + \frac{\lambda(1-m)}{2m}|x|^2\right)^{\frac{1}{m-1}}\right]^+$ (where $C > 0$ is such that $\int_{R^n} \rho(x)\,\mathrm{d}x = 1$) at the rate 2λ, and an exponential decay in the Wasserstein distance at the rate λ.

6 A remarkable duality

In this section, we apply Theorem 2.1 when $V = W = 0$, to obtain an intriguing duality between ground state solutions of some quasilinear PDEs and stationary solutions of Fokker-Planck type equations.

Corollary 6.1 *Let $\Omega \subset I\!R^n$ be open, bounded and convex, let $F : [0,\infty) \to I\!R$ be differentiable on $(0,\infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing. Let $\psi : I\!R \to [0,\infty)$ differentiable be chosen in such a way that $\psi(0) = 0$ and $|\psi^{\frac{1}{p}}(F' \circ \psi)'| = K$ where $p > 1$, and K is chosen to be 1 for simplicity. Then, for any Young function c with p-homogeneous Legendre transform c^*, we have the following inequality:*

$$\sup\{-\int_\Omega F(\rho) + c\rho; \rho \in \mathcal{P}_a(\Omega)\} \leq \inf\{\int_\Omega c^*(-\nabla f) - G_F \circ \psi(f); f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} \tag{117}$$

where $G_F(x) := (1-n)F(x) + nxF'(x)$.
Furthermore, equality holds in (117) if there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies

$$-(F' \circ \psi)'(\bar{f})\nabla\bar{f}(x) = \nabla c(x) \ \ a.e. \tag{118}$$

Moreover, $\bar{f}$ solves

$$\begin{array}{ll} \mathrm{div}\{\nabla c^*(-\nabla f)\} - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f) \cdot \nu = 0 & \text{on } \partial\Omega, \end{array} \tag{119}$$

for some $\lambda \in I\!R$, while $\bar{\rho}$ is a stationary solution of

$$\begin{array}{ll} \frac{\partial \rho}{\partial t} = \mathrm{div}\{\rho \nabla \left(F'(\rho) + V\right)\} & \text{in } (0,\infty) \times \Omega \\ \rho \nabla \left(F'(\rho) + V\right) \cdot \nu = 0 & \text{on } (0,\infty) \times \partial\Omega. \end{array} \tag{120}$$

Proof: Assume that c^* is p-homogeneous, and let $Q''(x) = x^{\frac{1}{q}} F''(x)$. Let

$$J(\rho) := -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy$$

and

$$\tilde{J}(\rho) := -\int_\Omega (F + nP_F)(\rho(x))dx + \int_\Omega c^*(-\nabla(Q'(\rho(x)))dx.$$

Equation (48) (where we use $V = W = 0$, and then $\lambda = \nu = 0$) then becomes

$$J(\rho_1) \leq \tilde{J}(\rho_0) \tag{121}$$

for all probability densities ρ_0, ρ_1 on Ω such that $\mathrm{supp}\, \rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$. If $\bar{\rho}$ satisfies

$$-\nabla(F'(\bar{\rho}(x))) = \nabla c(x) \text{ a.e.},$$

then equality holds in (121), and $\bar{\rho}$ is an extremal of the variational problems

$$\sup\{J(\rho);\ \rho \in \mathcal{P}_a(\Omega)\} = \inf\{\tilde{J}(\rho); \rho \in \mathcal{P}_a(\Omega), \mathrm{supp}\, \rho \subset \Omega, P_F(\rho) \in W^{1,\infty}(\Omega)\}.$$

In particular, $\bar{\rho}$ is a solution of

$$\begin{array}{ll} \mathrm{div}\{\rho \nabla(F'(\rho) + c)\} = 0 & \text{in } \Omega \\ \rho \nabla(F'(\rho) + c) \cdot \nu = 0 & \text{on } \partial\Omega. \end{array} \tag{122}$$

Suppose now $\psi : I\!R \to [0,\infty)$ differentiable, $\psi(0) = 0$ and that $\bar{f} \in C_0^\infty(\Omega)$ satisfies $-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x)$ a.e. Then equality holds in (121), and $\bar{f}$ and $\bar{\rho} = \psi(\bar{f})$ are extremals of the following variational problems

$$\inf\{I(f);\ f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\} = \sup\{J(\rho); \rho \in \mathcal{P}_a(\Omega)\}$$

where

$$I(f) = \tilde{J}(\psi(f)) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla(Q' \circ \psi(f))).$$

If now ψ is such that $|\, \psi^{\frac{1}{p}}(F' \circ \psi)'\,| = 1$, then $|\,(Q' \circ \psi)'\,| = 1$ and

$$I(f) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla f)),$$

because c^* is p-homogeneous. This proves (117). The Euler-Lagrange equation of the variational problem

$$\inf\Big\{\int_\Omega c^*(-\nabla(f)) - [F\circ\psi + nP_F\circ\psi](f);\ \int_\Omega \psi(f)=1\Big\}$$

reads as

$$\begin{array}{ll} \operatorname{div}\{\nabla c^*(-\nabla f)\} - (G_F\circ\psi)'(f) = \lambda\psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f)\cdot\nu = 0 & \text{on } \partial\Omega \end{array} \tag{123}$$

where $\lambda \in I\!\!R$ is a Lagrange multiplier, and $G(x) = (1-n)F(x) + nxF'(x)$. This proves (119). To prove that the maximizer $\bar\rho$ of

$$\sup\{-\int_\Omega (F(\rho)+c\rho)\ \mathrm{d}x;\ \rho\in\mathcal{P}_a(\Omega)\}$$

is a stationary solution of (120), we refer to [19] and [22].

Now, we apply Corollary 6.1 to the functions $F(x) = x\ln x, \psi(x) = |\,x\,|^p$ and $c(x) = (p-1)|\,\mu x\,|^q$, with $\mu > 0$ and $c^*(x) = \frac{1}{p}\Big|\frac{x}{\mu}\Big|^p$ and $\frac{1}{p}+\frac{1}{q}=1$, to derive a duality between stationary solutions of Fokker-Planck equations, and ground state solutions of some semi-linear equations. We note here that the condition $|\,\psi^{\frac{1}{p}}(F'\circ\psi)\,| = K$ holds for $K = p$. We obtain the following:

Corollary 6.2 *Let $p > 1$ and let q be its conjugate ($\frac{1}{p}+\frac{1}{q}=1$). For all $f \in W^{1,p}(I\!\!R^n)$, such that $\|\,f\,\|_p = 1$, any probability density ρ such that $\int_{R^n}\rho(x)|x|^q dx < \infty$, and any $\mu > 0$, we have*

$$J_\mu(\rho) \le I_\mu(f), \tag{124}$$

where

$$J_\mu(\rho) := -\int_{R^n}\rho\ln(\rho)\ dy - (p-1)\int_{R^n}|\,\mu y\,|^q\rho(y)\ dy,$$

and

$$I_\mu(f) := -\int_{R^n}|\,f\,|^p\ln(|\,f\,|^p) + \int_{R^n}\Big|\,\frac{\nabla f}{\mu}\,\Big|^p - n.$$

Furthermore, if $h \in W^{1,p}(I\!\!R^n)$ is such that $h \ge 0$, $\|\,h\,\|_p = 1$, and

$$\nabla h(x) = -\mu^q x|\,x\,|^{q-2}h(x) \quad a.e.,$$

then

$$J_\mu(h^p) = I_\mu(h).$$

Therefore, h (resp., $\rho = h^p$) is an extremum of the variational problem:

$$\sup\{\,J_\mu(\rho) : \rho\in W^{1,1}(I\!\!R^n),\ \|\,\rho\,\|_1 = 1\} = \inf\{\,I_\mu(f) : f\in W^{1,p}(I\!\!R^n), \|\,f\,\|_p = 1\}.$$

It follows that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, i.e., h is a solution of

$$\mu^{-p}\Delta_p f + pf|\, f\,|^{p-2}\ln(|\, f\,|) = \lambda f|\, f\,|^{p-2}, \tag{125}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^p$ is a stationary solution of the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \Delta u + \operatorname{div}(p\mu^q|x|^{q-2}xu). \tag{126}$$

We can also apply Corollary 6.1 to recover the duality associated to the Gagliardo-Nirenberg inequalities obtained recently in [11].

Corollary 6.3 *Let $1 < p < n$, and $r \in \left(0, \frac{np}{n-p}\right]$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $f \in W^{1,p}(I\!\!R^n)$ such that $\|\, f\,\|_r = 1$, for any probability density ρ and for all $\mu > 0$, we have*

$$J_\mu(\rho) \leq I_\mu(f) \tag{127}$$

where

$$J_\mu(\rho) := -\frac{1}{\gamma - 1}\int_{R^n} \rho^\gamma - \frac{r\gamma\mu^q}{q}\int_{R^n} |\, y\,|^q \rho(y)(y)\, dy,$$

and

$$I_\mu(f) := -\left(\frac{1}{\gamma - 1} + n\right)\int_{R^n} |\, f\,|^{r\gamma} + \frac{r\gamma}{p\mu^p}\int_{R^n} |\,\nabla f\,|^p.$$

Furthermore, if $h \in W^{1,p}(I\!\!R^n)$ is such that $h \geq 0$, $\|\, h\,\|_r = 1$, and

$$\nabla h(x) = -\mu^q x|\, x\,|^{q-2} h^{\frac{r}{p}}(x) \quad a.e.,$$

then

$$J_\mu(h^r) = I_\mu(h).$$

Therefore, h (resp., $\rho = h^r$) is an extremum of the variational problems

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(I\!\!R^n),\ \|\, \rho\,\|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(I\!\!R^n), \|\, f\,\|_r = 1\}.$$

Proof: Again, the proof follows from Corollary 6.1, by using now $\psi(x) = |\, x\,|^r$ and $F(x) = \frac{x^\gamma}{\gamma - 1}$, where $1 \neq \gamma \geq 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right]$. Indeed, for this value of γ, the function F satisfies the conditions of Corollary 6.1. The Young function is now $c(x) = \frac{r\gamma}{q}|\, \mu x\,|^q$, that is, $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\left|\, \frac{x}{\mu}\,\right|^p$, and the condition $|\, \psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ holds with $K = r\gamma$.
Moreover, if $h \geq 0$ satisfies (118), which is here,

$$-\nabla h(x) = \mu^q x|\, x\,|^{q-2} h^{\frac{r}{p}}(x) \text{ a.e.},$$

then h is extremal in the minimization problem defined in Corollary 6.3.

As above, we also note that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, that is, h is a solution of

$$\mu^{-p}\Delta_p f + \left(\frac{1}{\gamma-1}+n\right) f|\, f\,|^{r\gamma-2} = \lambda f|\, f\,|^{r-2}, \tag{128}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^r$ is a stationary solution of the evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u^\gamma + \mathrm{div}(r\gamma\mu^q|x|^{q-2}xu). \tag{129}$$

Example: In particular, when $\mu = 1, p = 2, \gamma = 1 - \frac{1}{n}$ and then $r = 2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, then Corollary 6.3 yields a duality between solutions of (128), which here the Yamabe equation:

$$-\Delta f = \lambda f|\, f\,|^{2^*-2},$$

(where λ is the Lagrange multiplier due to the constraint $\|\, f\,\|_{2^*} = 1$), and stationary solutions of (129), which is here the rescaled fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \mathrm{div}\left(\frac{2n-2}{n-2}xu\right).$$

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