Computation at the Heart of Mathematics:
Celebrating the Work of David Boyd, Recipient of the 2005 CRM-Fields Prize

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One of the great themes of modern number theory is how analysis and algebra often give us the same information by somewhat different means, for example by a formula with an algebraic interpretation on one side and an analytic interpretation on the other.[1] There is one such formula that every mathematician has been exposed to in their education, but has only recently been seriously interpreted in this way, and that is Jensen's theorem in complex analysis: This tells us that for a function f which is analytic on a closed disk of radius r, the average value of $\log|f(z)|$ on the boundary of the disk can be determined precisely in terms of $f(0)$ and the zeros of f inside this disk. In the particular case of an irreducible polynomial f with leading coefficient a_0, and the unit disk, this leads to the formula

$$M(f) := |a_0| \prod_{\alpha : f(\alpha)=0} \max\{1, |\alpha|\} = \exp\left(\int_0^1 \log|f(e^{2i\pi\theta})| d\theta\right).$$

There are a host of results which show that this *Mahler measure* is "natural", and there are several intriguing open questions. Foremost is Lehmer's 1934 conjecture suggesting that $M(f) \geq 1 + \delta$ for some fixed $\delta > 0$ for any polynomial $f(x) \in \mathbb{Z}[x]$ other than x and the cyclotomic polynomials; in fact computational evidence suggests that the lower bound is given by $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. Much of this evidence has been given by Boyd and his collaborators who have also settled certain special cases of Lehmer's conjecture (most notably, Smyth showed that it is true for non-reciprocal polynomials). There are certain special classes of numbers that one expects to have particularly small measure, for example Pisot numbers, which are real algebraic integers greater than 1, all of whose conjugates have absolute value less than 1, and Salem numbers, where the conjugates all have absolute value less than *or equal to* 1. Boyd's nicest work in this area was to show that Salem's construction of Salem numbers does in fact give them all [5], as well as his understanding of Pisot numbers in the neighbourhood of a limit point [6]. I should also mention his 1977 result that there are Pisot sequences which satisfy no linear recurrence [4]. Lehmer's conjecture has recently been surveyed in this newsletter by Vaaler [14].

One can generalize the notion of Mahler measure to multivariable polynomials, so that for $P \in \mathbb{Z}[x_1, \ldots, x_n]$ we define

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(e^{2i\pi\theta_1}, \ldots, e^{2i\pi\theta_n})| d\theta_1 \ldots d\theta_n,$$

(note that $m(f) = \log M(f)$ when $n = 1$). We think of this as an integral on the n-dimensional torus $(\mathbb{R}/\mathbb{Z})^n$, where the co-ordinates vary independently. Boyd recognized that one can approximate the co-ordinates varying independently by replacing each θ_i by a suitable multiple of θ; in

[1] For examples, we see this in the Birch-Swinnerton Dyer conjecture, the Tate and Stark conjectures, in Wiles' proof of the Fermat's Last Theorem, at the heart of the Langlands' program, . . .

particular $m(1+x+y) = \lim_{n\to\infty} m(1+x+x^n)$. In 1981 Chris Smyth observed that the value of $m(1+x+y)$ is quite extraordinary; it is not some arbitrary real number, nor is it the product of the roots of some polynomial like in 1-dimension, but rather $m(1+x+y)$ is the value of an L-function at a particular point, specifically $L'(1,(-3/.))$. This one example inspired a host of questions about the values of $m(P)$; for example if we replace 3 by any other prime $\equiv -1 \pmod 4$ can one find P that yields an analogous formula? Or are there extensions of this to other types of characters and zeta functions? There are a few direct generalizations proven (mostly by Ray, and by Boyd and Rodriguez-Villegas) but a lot still remains mysterious. For a long while Smyth and Ray's results had remained splendid but seemingly ad hoc, in that no one had discovered why there should be such results.

Motives have been one of the great generalizations of modern arithmetic geometry, and they showed much early promise of providing a more complete understanding of key concepts, yet arguably have failed to deliver as much as had been hoped for. However it was in Deninger's research in this area that he developed the first viewpoint to account for these special Mahler measure values. His original work did not easily yield new examples, but it did tell one how to look and in a new way. In particular he predicted that there should be various $P \in \mathbb{Z}[x,y]$ with $m(P) = rL'(E,0)$ for some non-zero rational number r, where E is the (elliptic) curve determined by the equation $P(x,y) = 0$. Following Deninger's lead, Boyd did a vast amount of calculations to find many examples (in his paper he gives these examples with, in each case, r a rational with small numerator and denominator, with the identity confirmed up to 24 decimal places!). Boyd determined under what conditions P should be expected to satisfy a relation $m(P) = rL'(E,0)$:[2] the outcome is uncanny in that it seems tailor-made for subsequent researchers to be able to use (well-known) conjectures of Bloch–Beilinson to predict relations of this form (and sometimes prove them). His work [9] on this is an astonishing blend of experimentation and intuition.

Fernando Rodriguez-Villegas proved several of the formulas in Boyd's paper, and then together they have a couple of important papers. The first [12] is perhaps the ultimate generalization of Chris Smyth's result about $m(1+x+y)$ though it barely scratches the surface of what they have proved (see [13]). The value $L'(1,(-3/.))$ and appropriate generalizations appear as the volume of a hyperbolic manifold, the upper half space of hyperbolic 3-space, modulo certain discrete, torsion-free subgroups of $PSL(2,\mathbb{C})$. Such hyperbolic manifolds can be decomposed as a sum of "ideal" tetrahedra whose volumes can each be given by the Bloch-Wigner dilogarithm function evaluated at a certain point, a connection used by Thurston and then Neumann-Zagier in their work on Dehn surgery.

The A-polynomial is a remarkable invariant of a one-cusped hyperbolic 3-manifold whose zero locus basically describes the representations of its fundamental group into $SL(2,\mathbb{C})$. Manipulating the above construction appropriately, Boyd showed how the Mahler measure of any irreducible factor of A is given by a sum of Bloch-Wigner dilogarithm values, a remarkable result (see [10, 11]). Although notoriously difficult to compute, Boyd has come up with new ways to determine the A-polynomial in certain special but important cases leading him to some beautiful new results (for example for the A-polynomial of periodic knots) discovered, of course, experimentally. Understanding these connections has obsessed David over the last decade: he tells me that he has twenty-seven 192 page notebooks of computations (and their interpretation)!

Boyd has an extraordinary breadth of mathematical interests, making important contributions

[2] He also allows P to define a genus 2 curve whose Jacobian splits into two factors, one of which is E.

to areas of analysis, number theory, and geometry. Some of his earliest papers were on packings: In 1973 he showed [3] that the Hausdorff dimension of the residual set of the Apollonian packing is exactly the same as the exponent of convergence, a result that has been quite influential recently. Likewise Boyd's results [1, 2] on higher dimensional packings were a little ahead of their time and were not followed up by others until very recently. I particularly like a more recent result that came out of understanding a question on the heights of factors of polynomials which has practical applications to symbolic computation: Given a polynomial $f(x) \in \mathbb{Z}[x]$ of degree n, provide a good bound for the size of the coefficients of any possible factor g of f. That the (mean square of the) coefficients of g is $\leq 2^n$ times that of f is easy, and 2 had been replaced by $(1+\sqrt{5})/2$ in the literature; what Boyd did was to find the best possible constant [7], which *of course* turned out to be the Mahler measure of some two-variable polynomial! Such questions were at the time also of importance in "effective nullstellensatz" and Boyd produced a string of important results (eg. [8]).

Canada has a proud tradition of experimental mathematicians using great ingenuity, tenacity and stamina to extract deep insights from data (John McKay, who won this award in 2003, is another example). Often the final published work does not adequately reflect the difficulty of producing the data, nor how much is learnt in the process, though experts in the field appreciate the difference made. In the case of David Boyd's work one sees, time and again, that he has made valuable insights aided by extensive computation, creating new research themes which are subsequently developed internationally, so that he is a worthy laureate of the 2005 CRM-Fields prize.

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