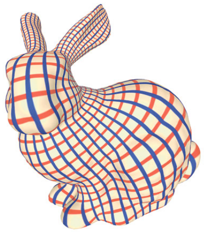
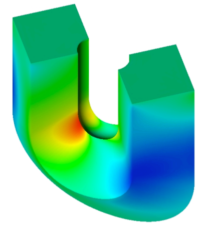


Isogeometric Analysis



T.J.R. Hughes

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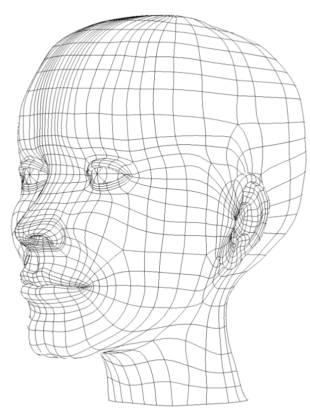
Co-authors: Y. Bazilevs, V. Calo, J.A. Cottrell, H. Gomez Diaz, A. Reali, G. Sangalli, J. Zhang

PIMS ~ Syncrude Lecture
University of Alberta, January 18, 2008

Outline

- Isogeometric analysis
- NURBS
- Structural vibrations
- Wave propagation
- Phase field modeling
- Fluid-structure interaction
- Cardiovascular modeling
- T-Splines
- Conclusions



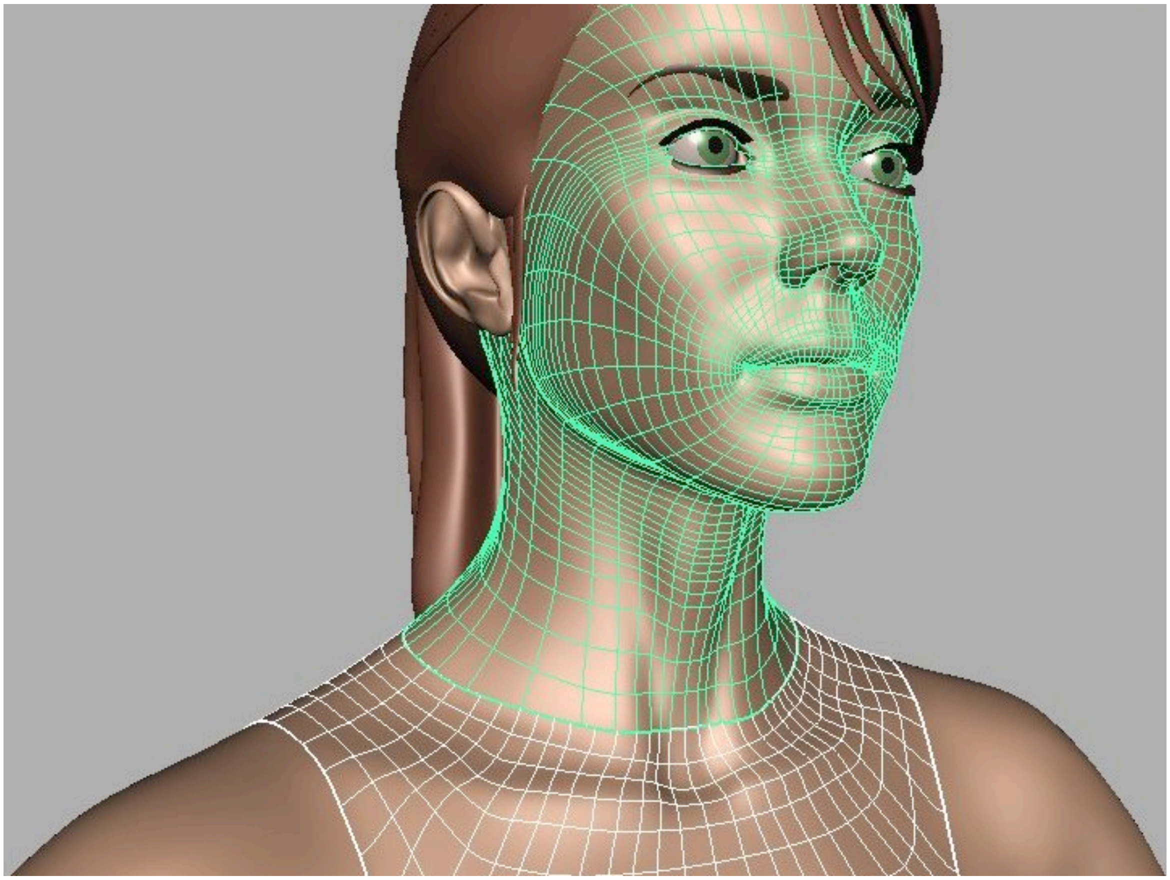
Isogeometric Analysis

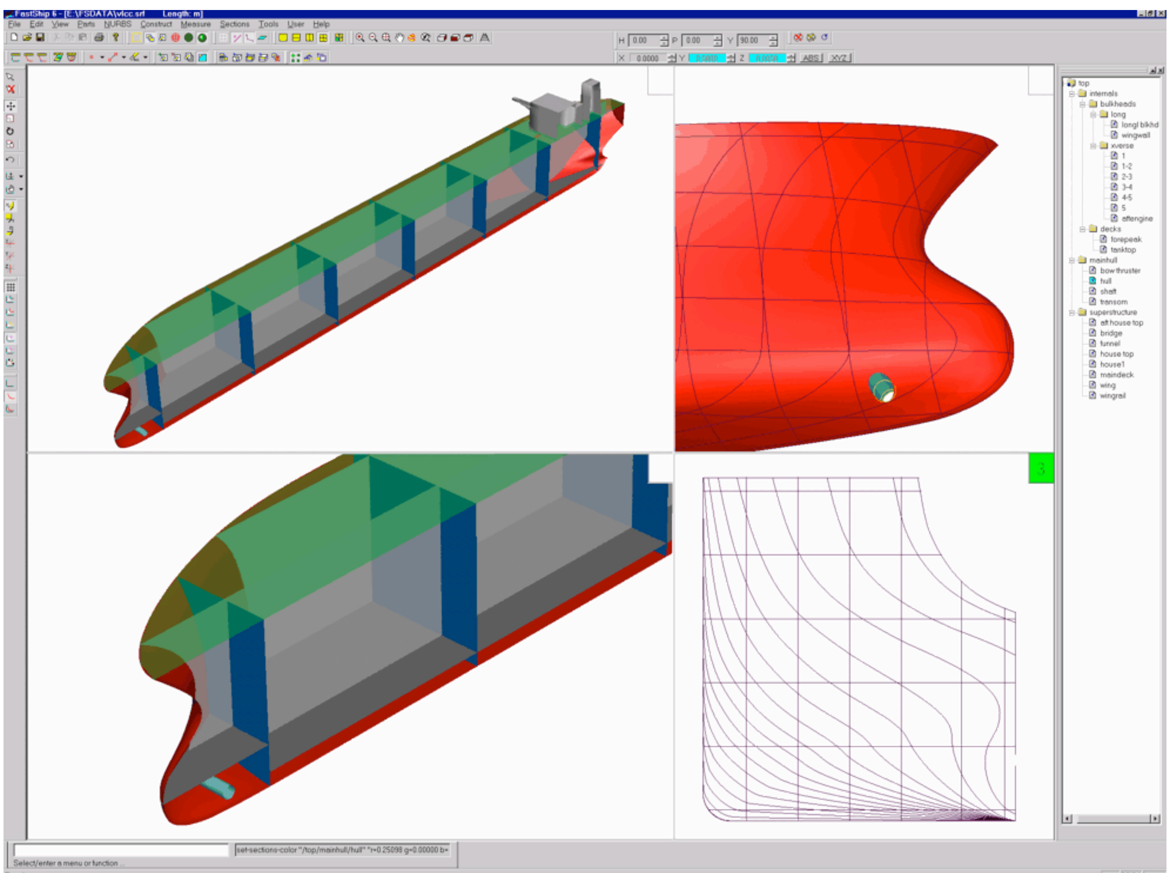
- Based on technologies (e.g., NURBS) from *computational geometry* used in:
  - Design
  - Animation
  - Graphic art
  - Visualization

- Includes standard FEA as a special case, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Superior approximation properties
  - *Integration* of design and analysis


1997 SYMPOSIUM ON INTERACTIVE 3D GRAPHICS
Proceedings 1997 Symposium on Interactive 3D Graphics


Source 2,000,000 polys




FastShip 6 - [E:\FSDATA\vlcc.srf Length: m]
File Edit View Parts NURBS Construct Measure Sections Tools User Help
top
internals
bulkheads
long
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wingwall
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1-2
2-3
3-4
4-5
5
aftengine
decks
forepeak
tanktop
mainhull
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house top
house1
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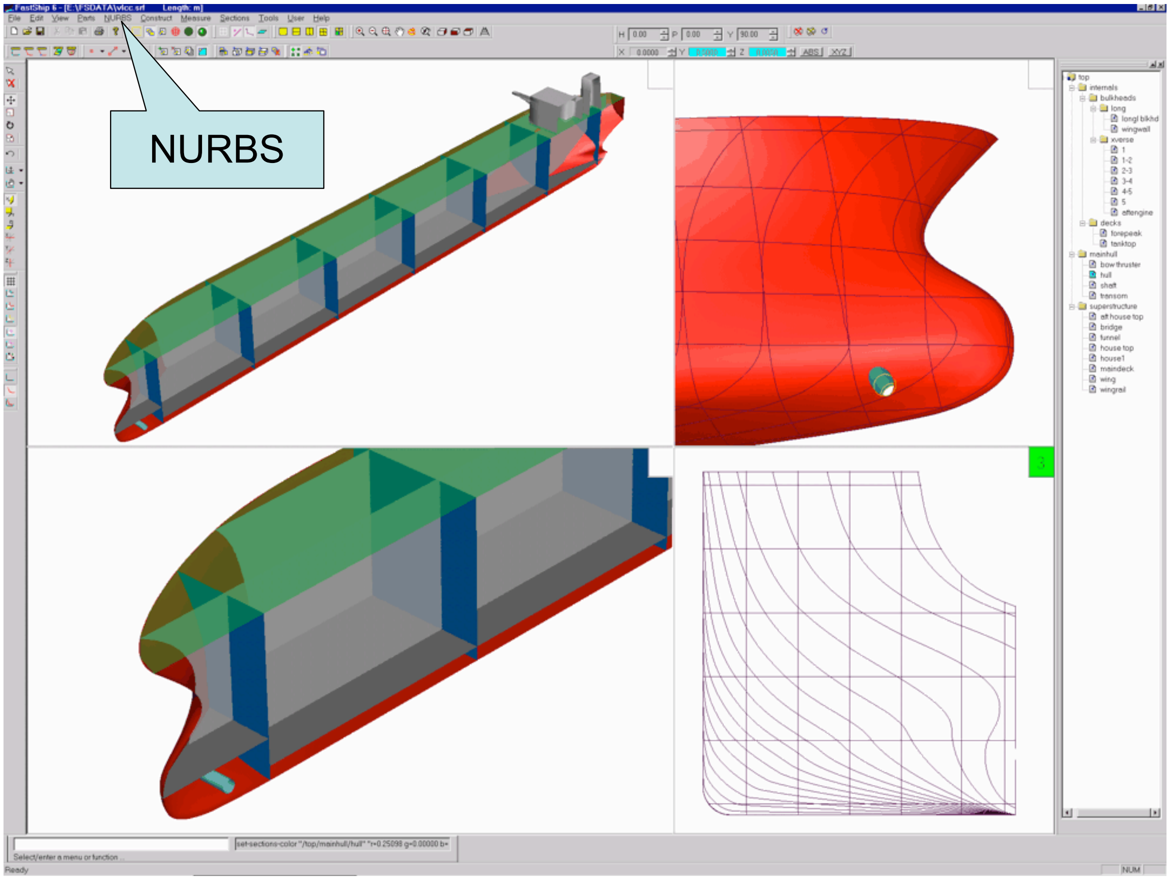
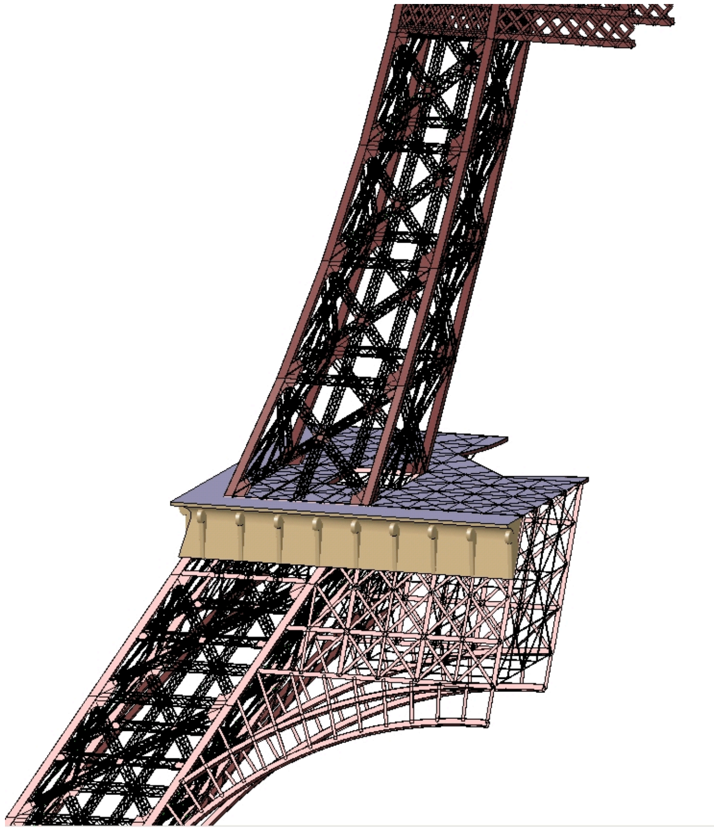
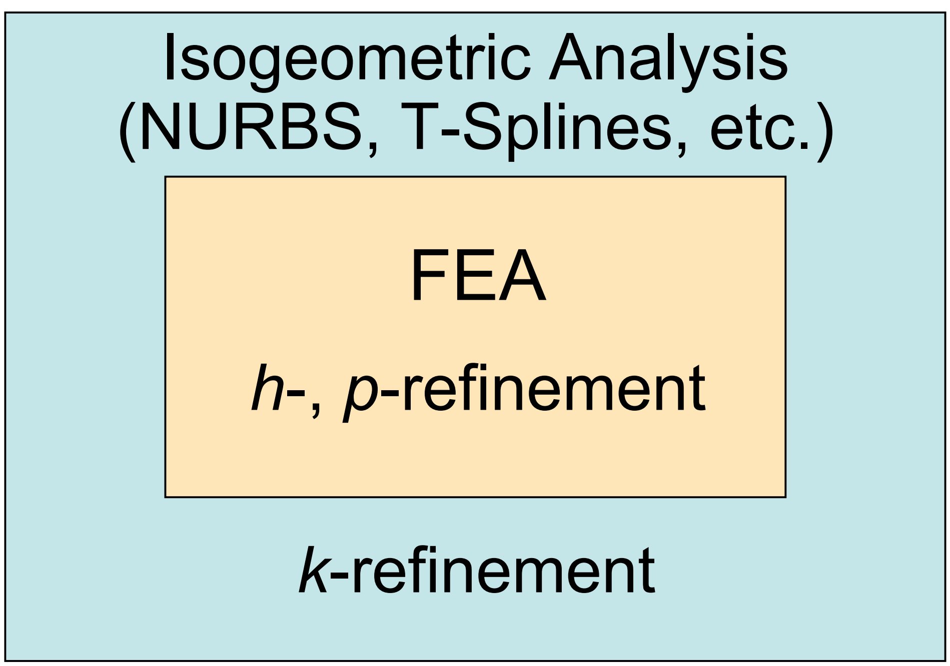
wing
wingrail
Ready
NUM


NURBS
File Edit View Parts NURBS Construct Measure Sections Tools User Help
Ready
NUM

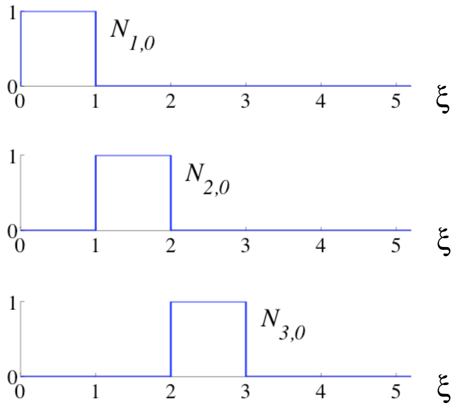


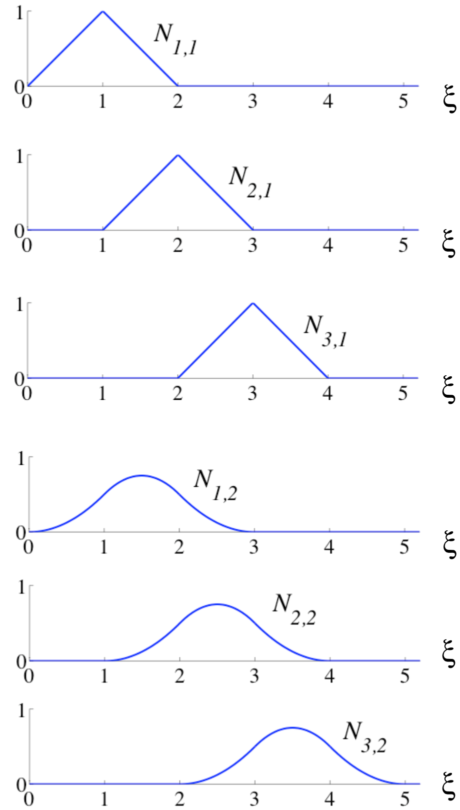

Isogeometric Analysis
(NURBS, T-Splines, etc.)
FEA
h-, *p*-refinement
k-refinement

B-Splines

B-spline Basis Functions

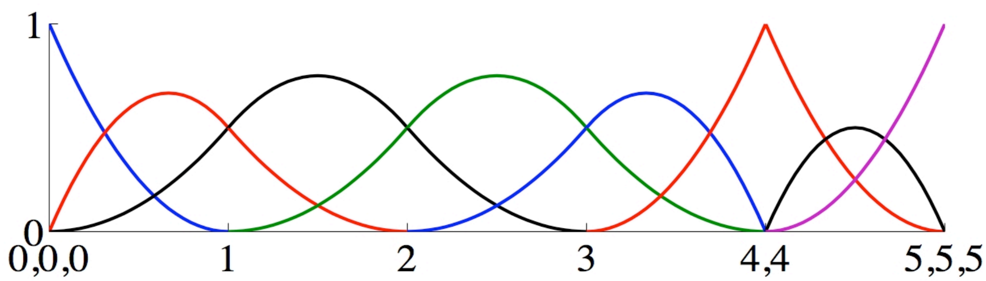
- $$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$
- $$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) \;+$$
$$\frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$





B-spline basis functions of order 0, 1, 2 for a *uniform knot vector:*

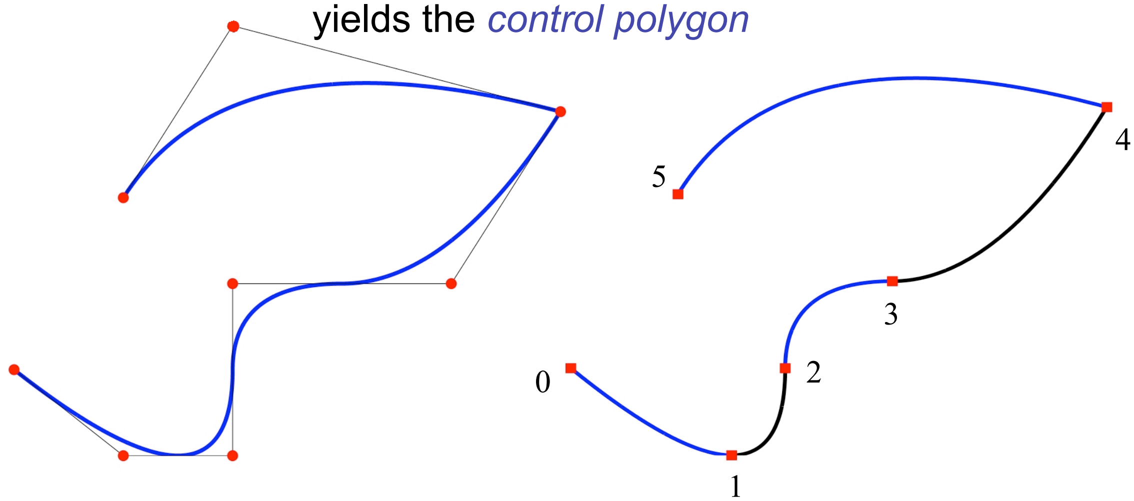
$$\Xi = \{0,1,2,3,4,\ldots\}$$



Quadratic (p=2) basis functions for an *open, non-uniform knot vector:*

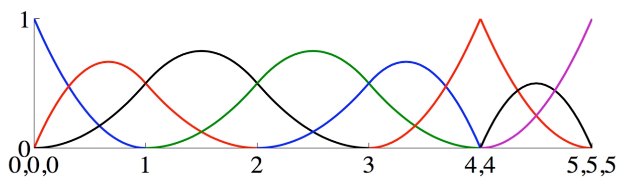
$$\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$$

Linear interpolation of control points yields the *control polygon*

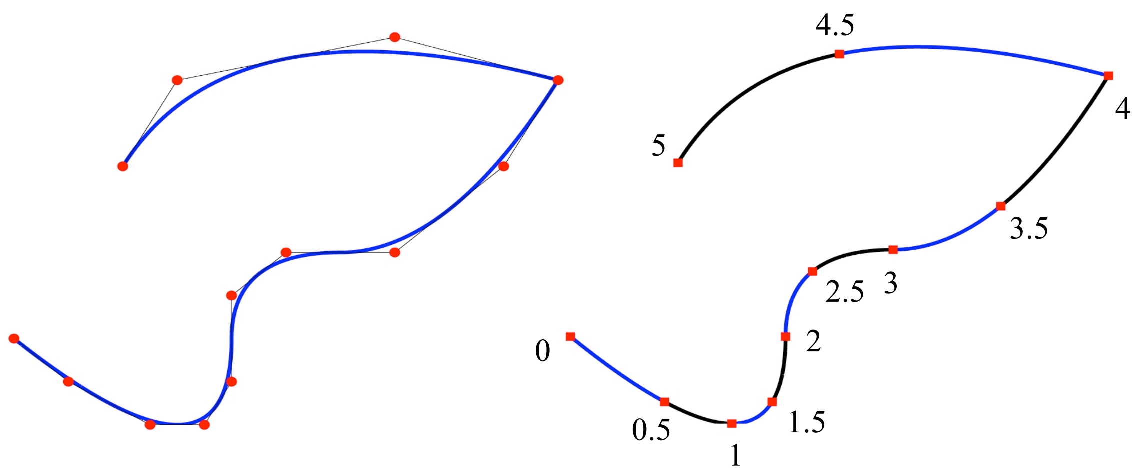


● - control points ■ - knots

Quadratic basis

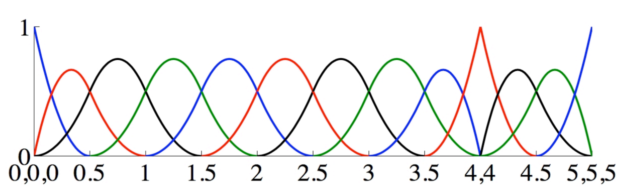


h-refined Curve

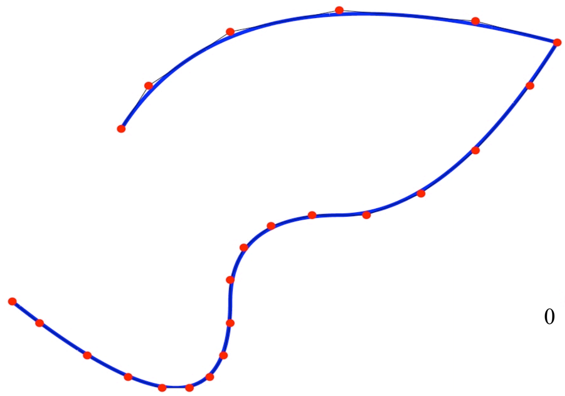


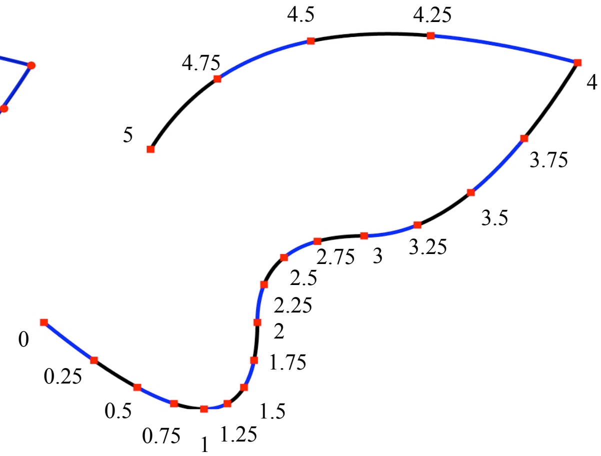
● - control points ■ - knots

Quadratic basis



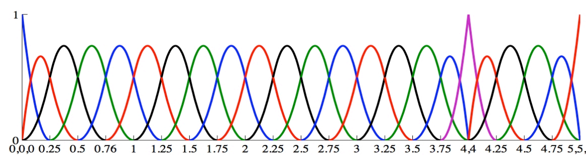
Further *h*-refined Curve



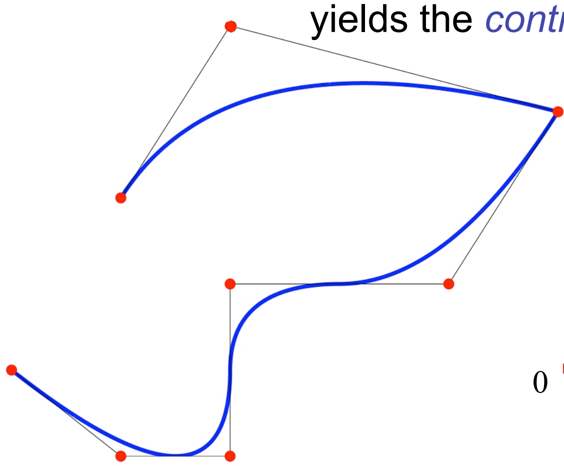


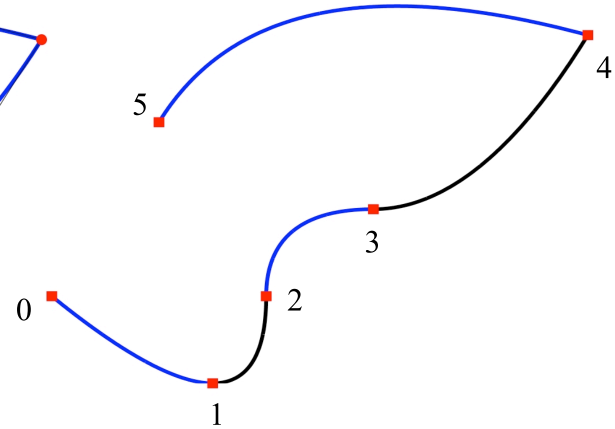
● - control points ■ - knots

Quadratic basis



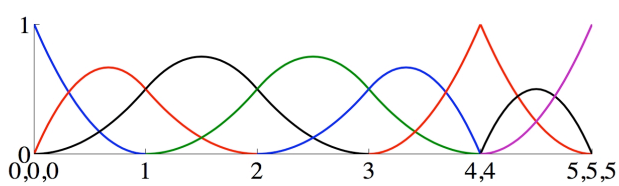
Linear interpolation of control points yields the *control polygon*



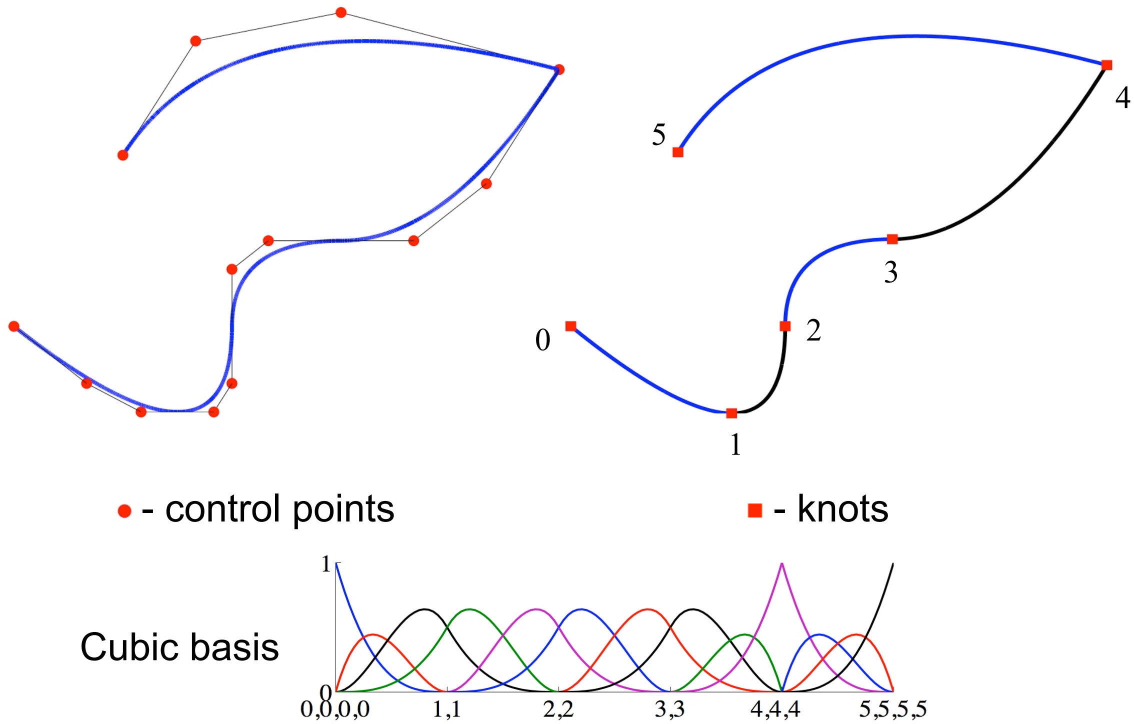


● - control points ■ - knots

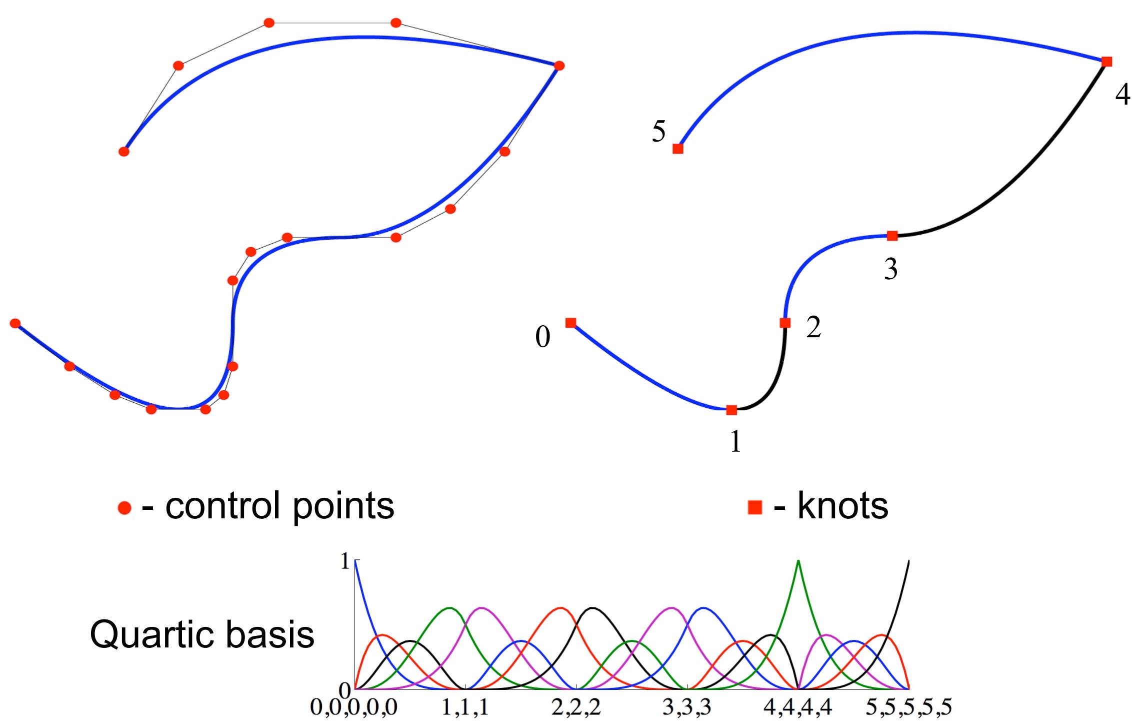
Quadratic basis



Cubic *p*-refined Curve

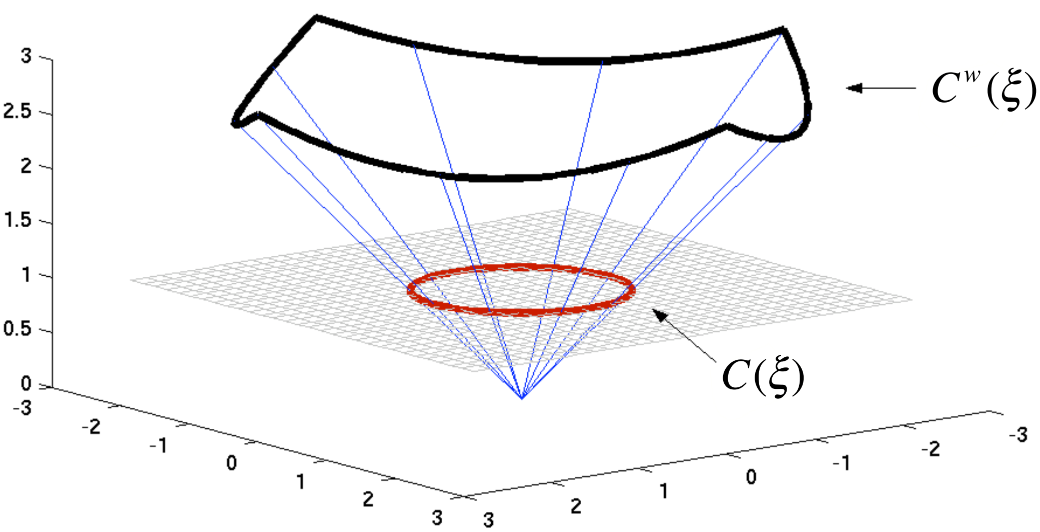


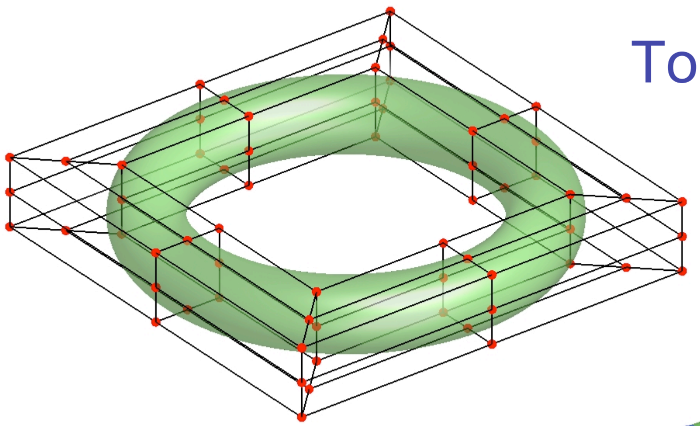
Quartic *p*-refined Curve



Non-Uniform Rational B-splines

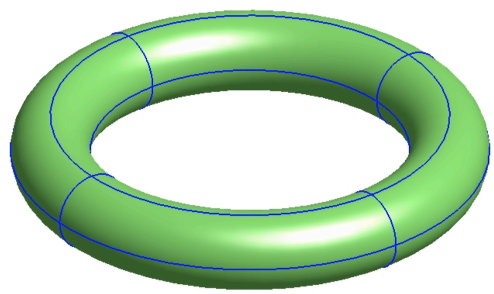
Circle from 3D Piecewise Quadratic Curves



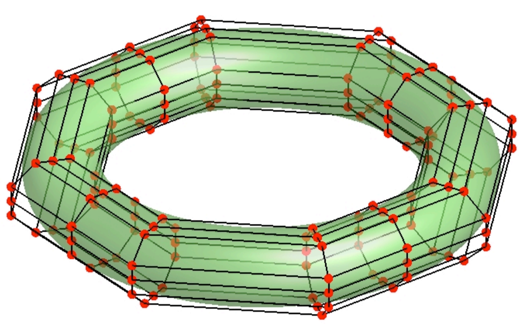


Control net

Toroidal Surface

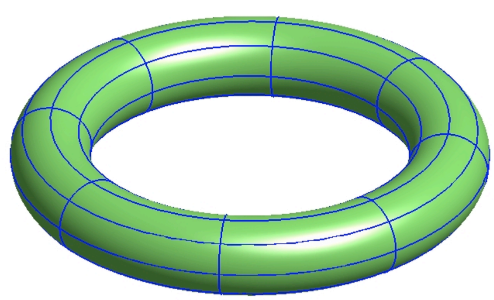


Mesh



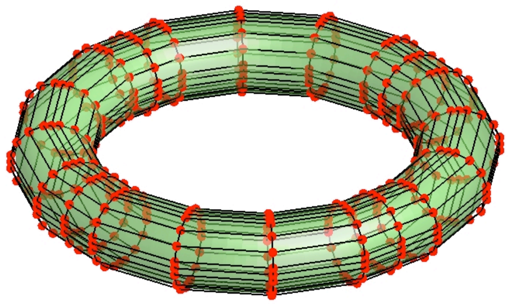
Control net

h-refined Surface

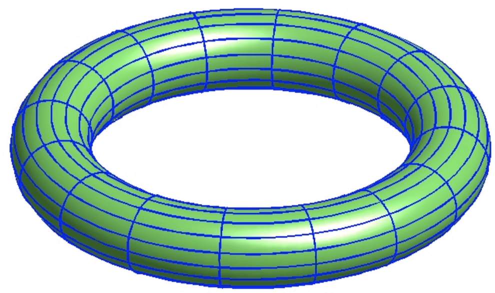


Mesh

Further *h*-refined Surface

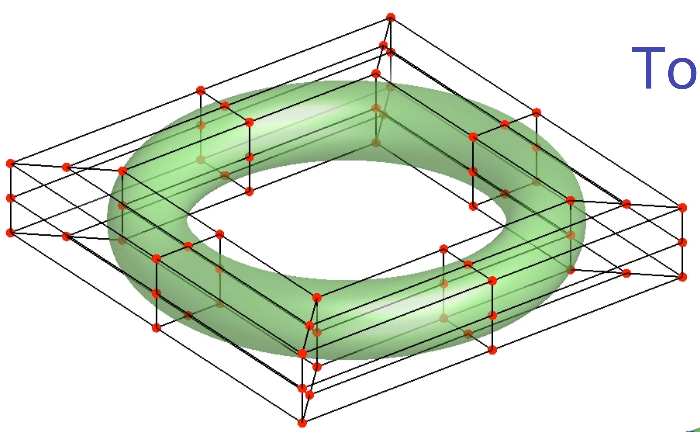


Control net

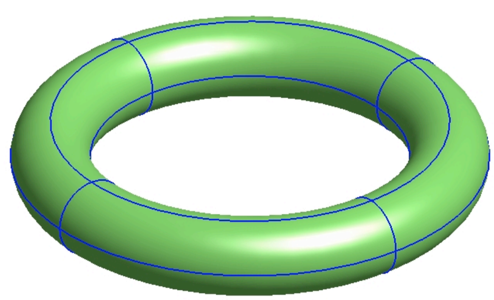


Mesh

Toroidal Surface

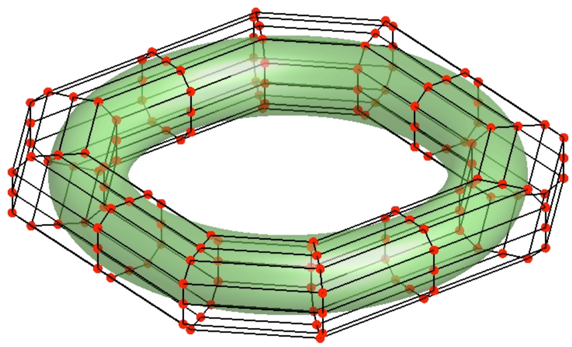


Control net

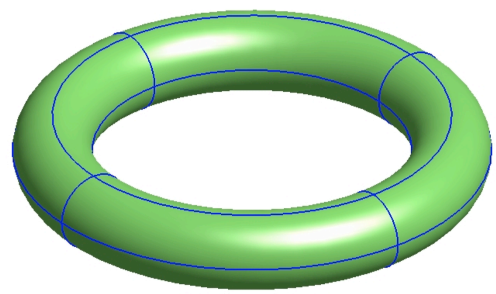


Mesh

Cubic *p*-refined Surface

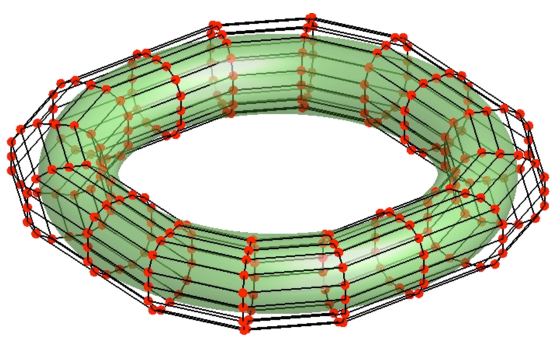


Control net

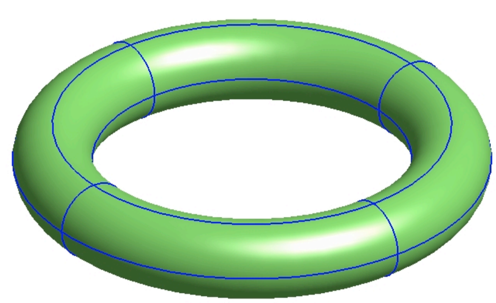


Mesh

Quartic *p*-refined Surface



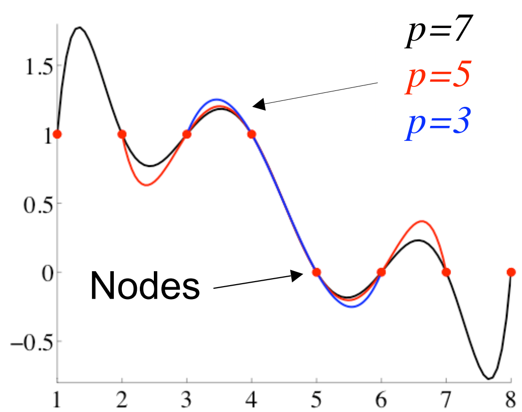
Control net



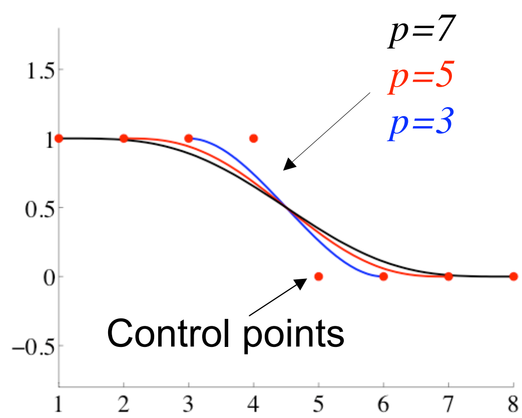
Mesh

Variation Diminishing Property

Lagrange polynomials



NURBS

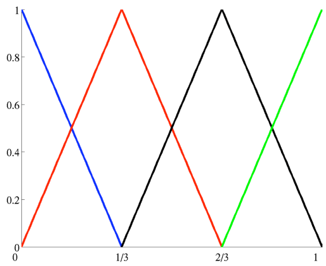


Finite Element Analysis and Isogeometric Analysis

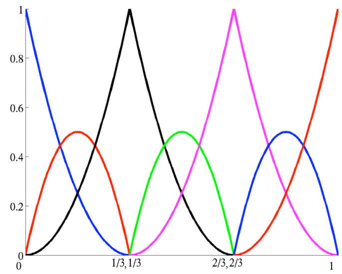
▪ Compact support
▪ Partition of unity
▪ Affine covariance
▪ Isoparametric concept
▪ Patch tests satisfied

Three Quadratic Elements

Knot insertion →

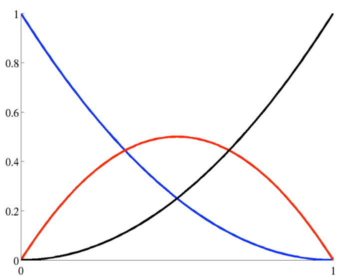


Order elevation →

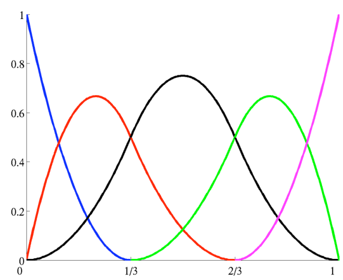


p-refinement

Order elevation →



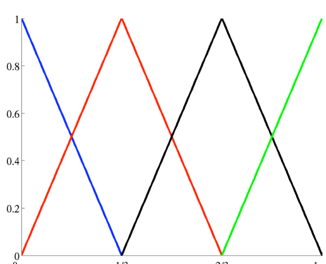
Knot insertion →



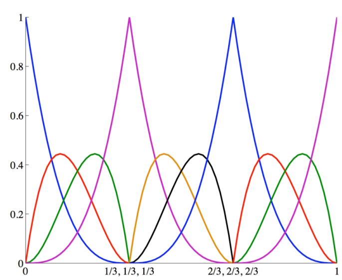
k-refinement

Three Cubic Elements

Knot insertion →

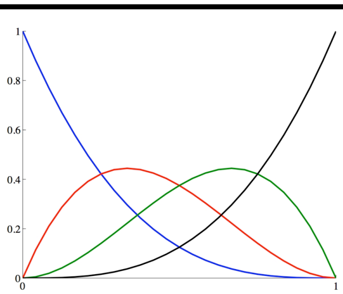


Order elevation →

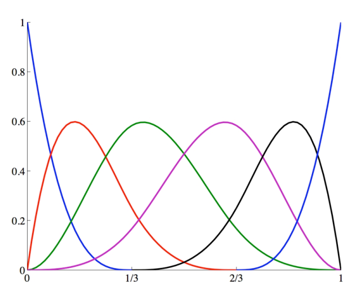


p-refinement

Order elevation →

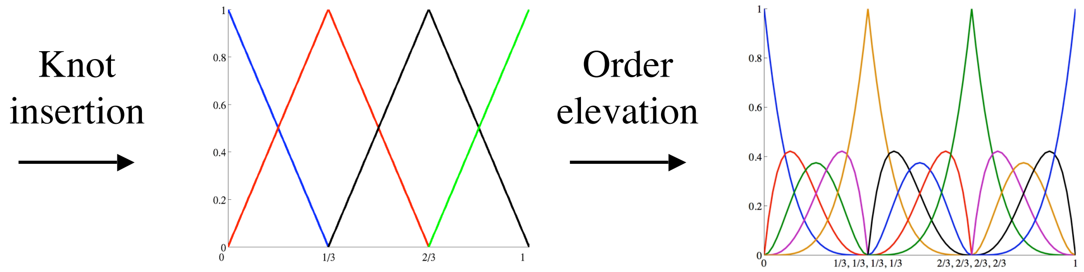


Knot insertion →

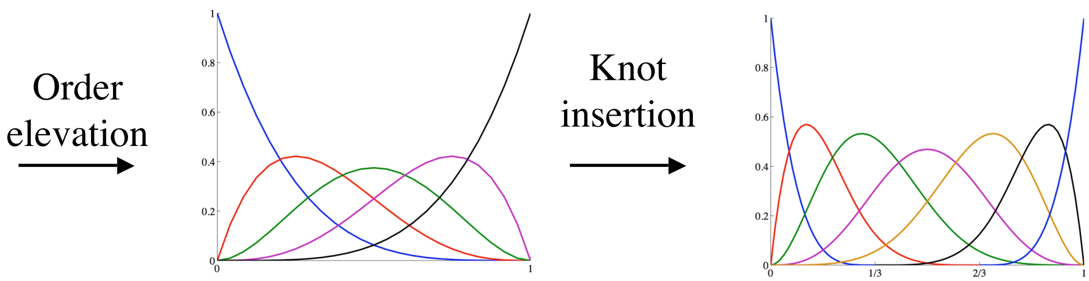


k-refinement

Three Quartic Elements



p-refinement

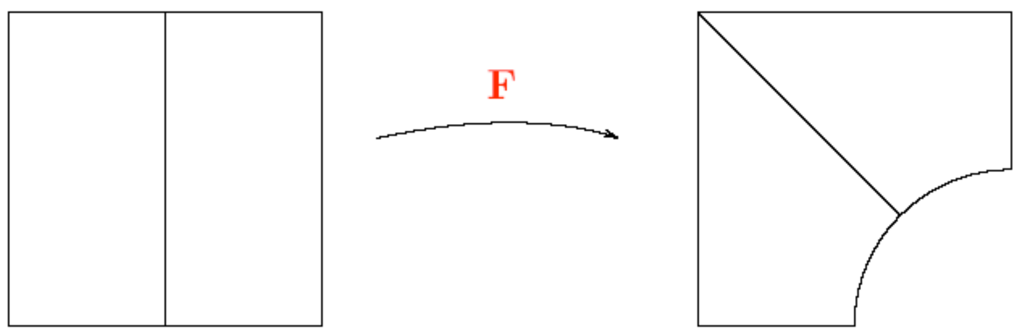


k-refinement

Mathematical Theory of *h*-refinement

Coarsest Discretization

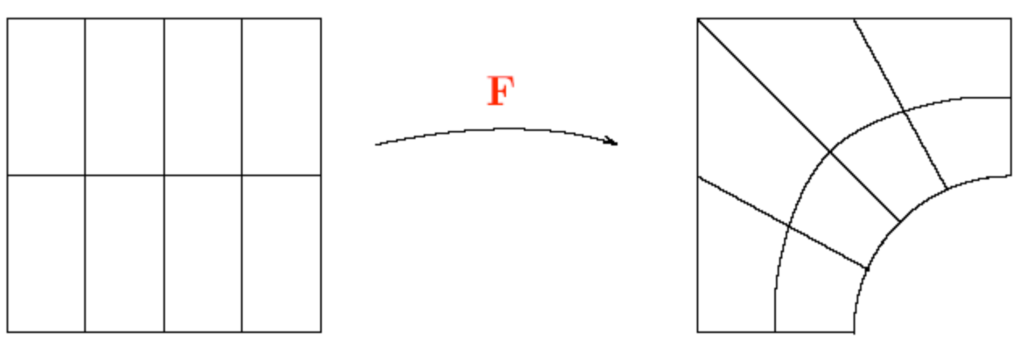
Patch $(0,1)^2$ Physical domain



$\left\{ \frac{w_i N_i}{w} \right\}_{i=1,\ldots,n_0}$ $\mathbf{F}, w$ – fixed $\left\{ \frac{w_i N_i}{w} \circ \mathbf{F}^{-1} \right\}_{i=1,\ldots,n_0}$

First *h*-refinement

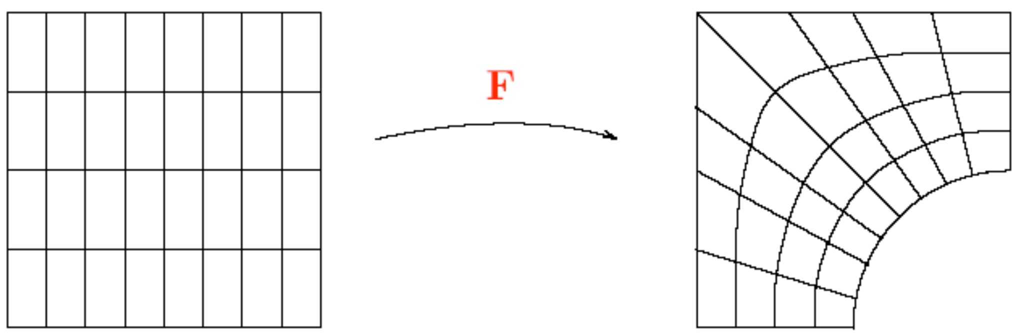
Patch $(0,1)^2$ Physical domain



$\left\{ \frac{w_i N_i}{w} \right\}_{i=1,\ldots,n_1}$ $\mathbf{F}, w$ – fixed $\left\{ \frac{w_i N_i}{w} \circ \mathbf{F}^{-1} \right\}_{i=1,\ldots,n_1}$

Second h-refinement

Patch $(0,1)^2$ Physical domain



$\left\{\frac{w_i N_i}{w}\right\}_{i=1,\ldots,n_2}$ $\mathbf{F}, w$ – fixed $\left\{\frac{w_i N_i}{w}\circ \mathbf{F}^{-1}\right\}_{i=1,\ldots,n_2}$

Approximation with NURBS

Theorem

Let $k, l,$ be the integer indices such that $0 \le k \le l \le p+1$. Let $u \in H^l(\Omega)$, then

$$\sum_{K\in \mathrm{K}_h} \left| u - \Pi_{V_h} u \right|^2_{H^k(K)} \le C \sum_{K\in \mathrm{K}_h} h_K^{2(l-k)} \sum_{i=0}^{l} \left\| \nabla \mathbf{F} \right\|^{2(i-l)}_{L^\infty\left(\mathbf{F}^{-1}(K)\right)} |u|^2_{H^i(K)}$$

Positive constant, depends on p, shape of Ω (but not its size), and shape regularity of the mesh.

Factors which render error estimate dimensionally consistent.

Error Estimates

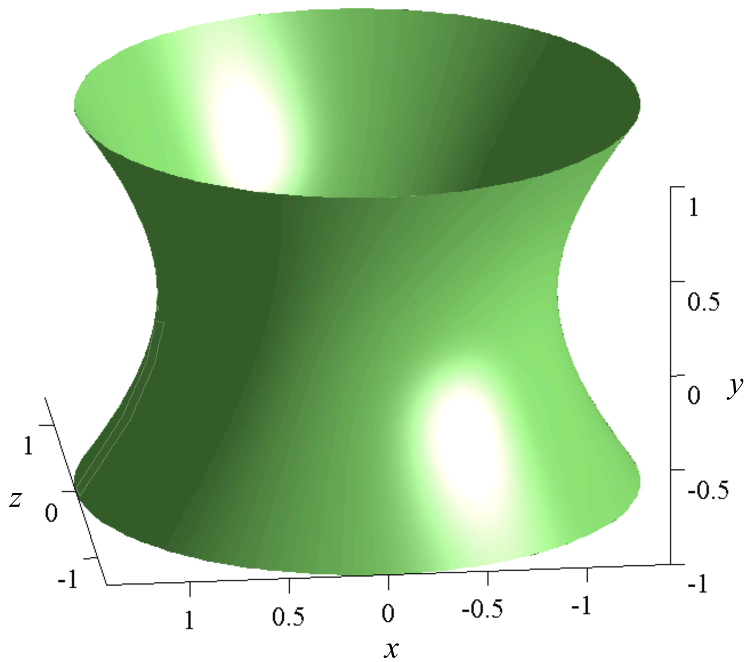
- Strongly elliptic problems:
  - Elasticity, structures
- Stabilized/multiscale methods:
  - Advection-diffusion
  - Incompressible elasticity, Stokes flow
- BB-stable mixed elements:
  - Incompressible elasticity, Stokes flow
- Numerical tests confirm and go beyond theoretical results

Isogeometric Structural Analysis

- Isoparametric NURBS elements exactly represent all *rigid body motions* and *constant strain states*

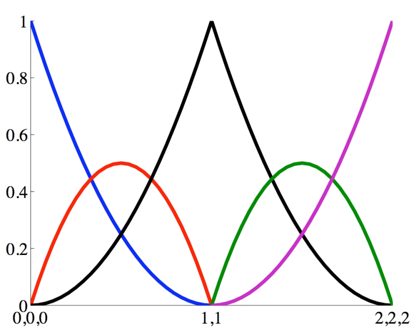
Hyperboloidal Shell

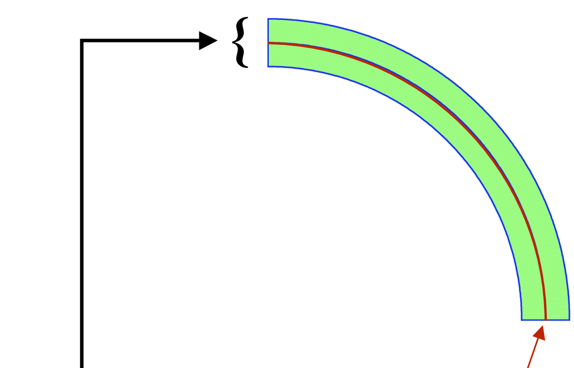
- Mid-surface:
  - $x^2 + z^2 - y^2 = 1$
  - $-1 \leq y \leq 1$
- $R/t = 10^3$
- Fixed at the top and bottom
- Loading:
  - $p = p_0 \cos 2\theta$



Thickness Discretization

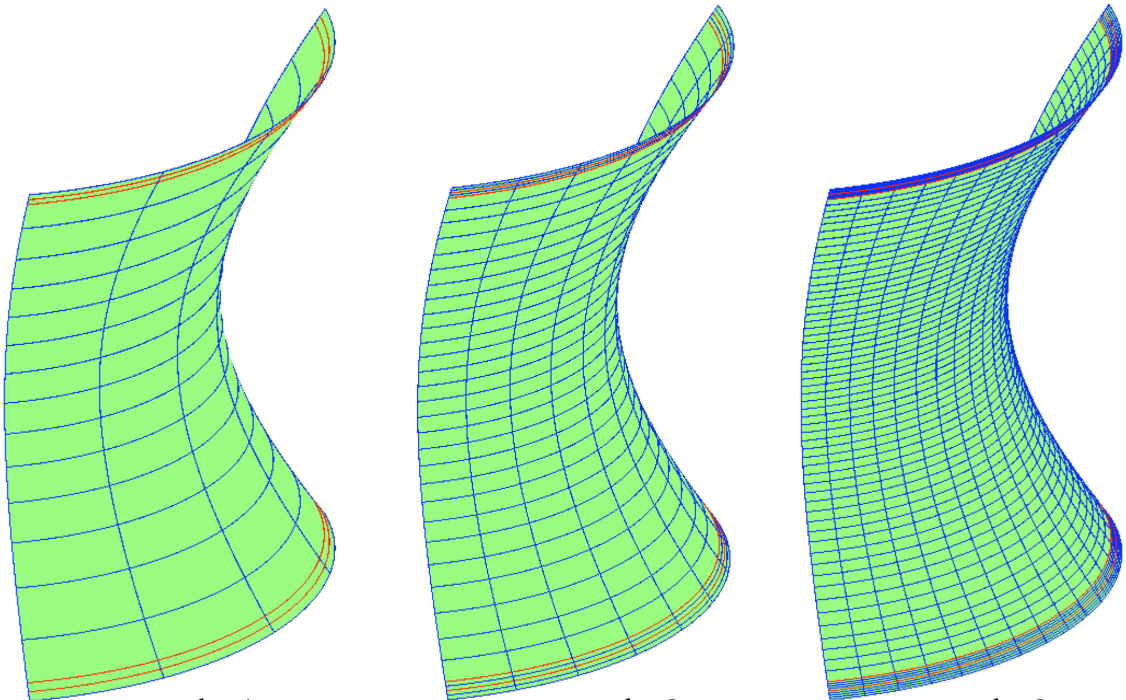
Rational quadratic basis functions through the thickness





The NURBS surface defining the mid-surface is geometrically exact

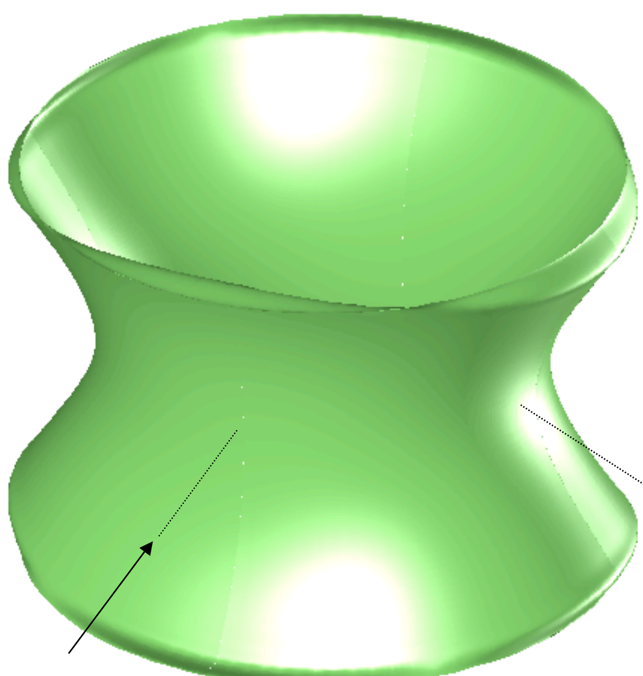
Surface Discretization



Mesh 1 Mesh 2 Mesh 3

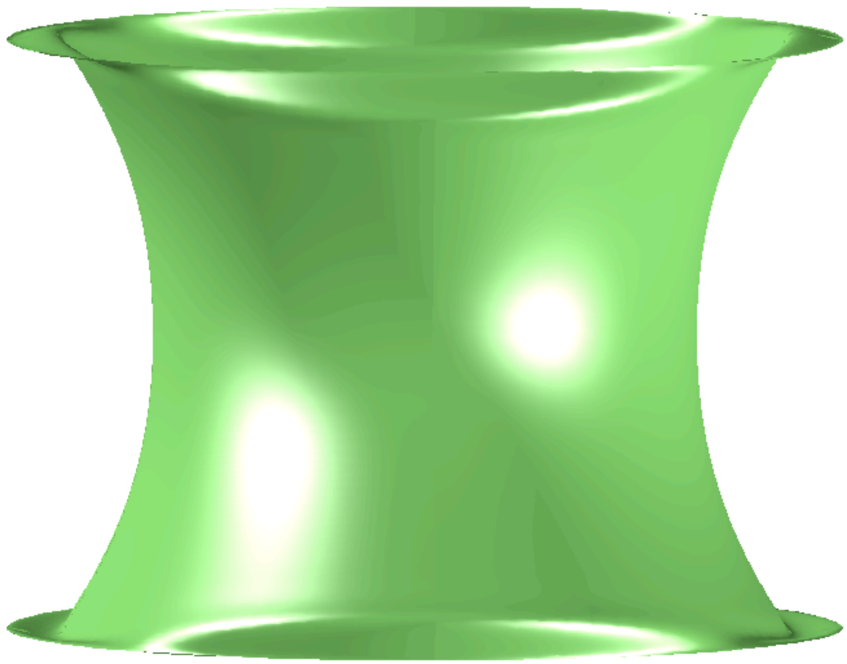
Deformed Shell

(displacement amplification factor of 10)



View 1

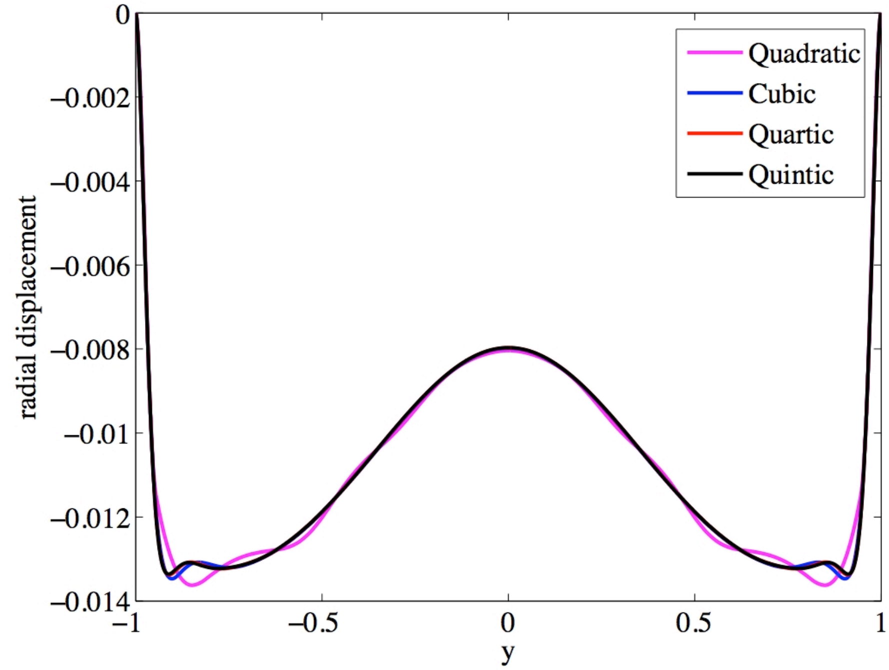
(displacement amplification factor of 10)



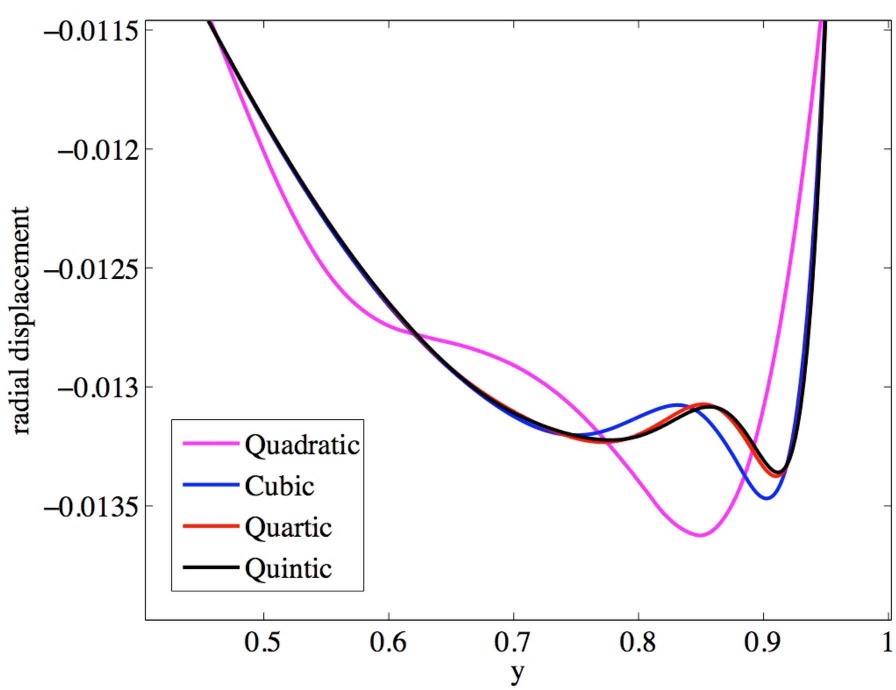
View 2

(displacement amplification factor of 10)

Radial Displacement at Compression Lobe (Mesh 3)

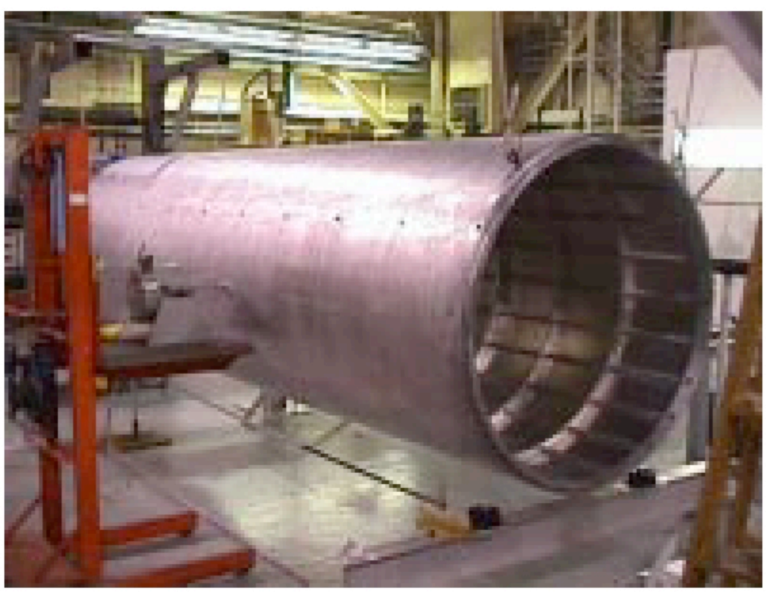


Detail of Radial Displacement at Compression Lobe (Mesh 3)

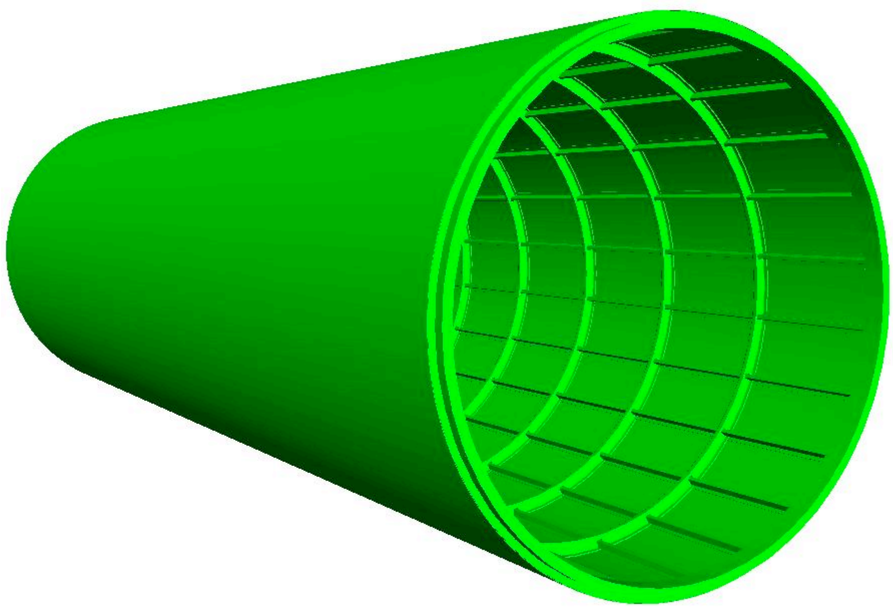


Isogeometric Vibration Analysis

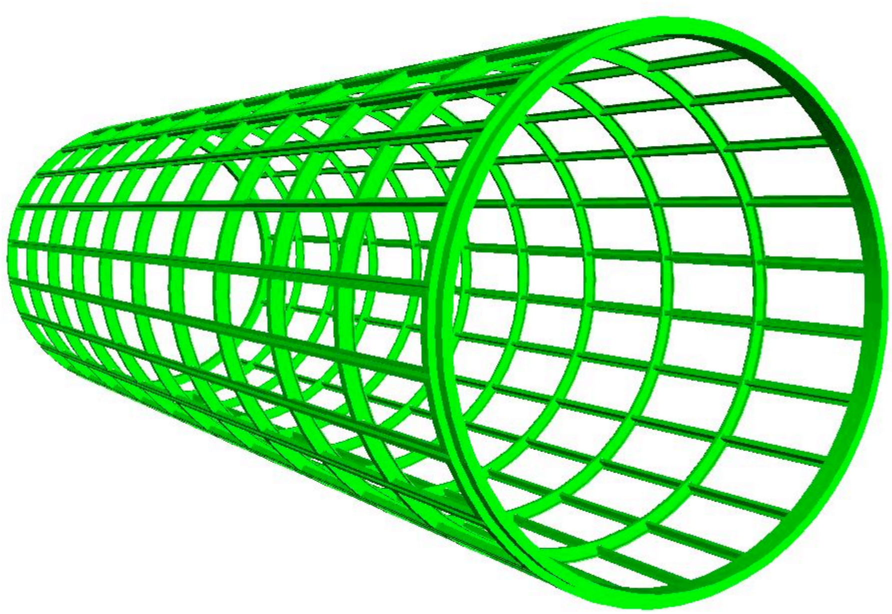
NASA Aluminum Testbed Cylinder (ATC)

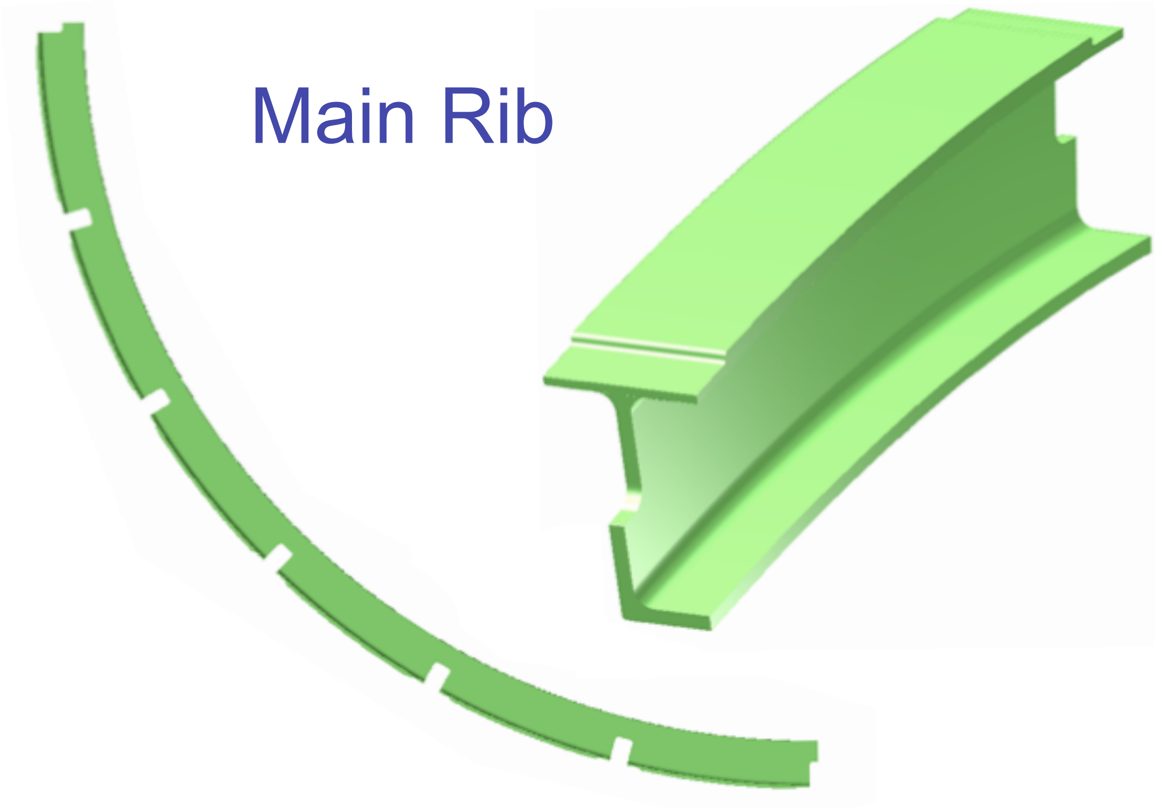


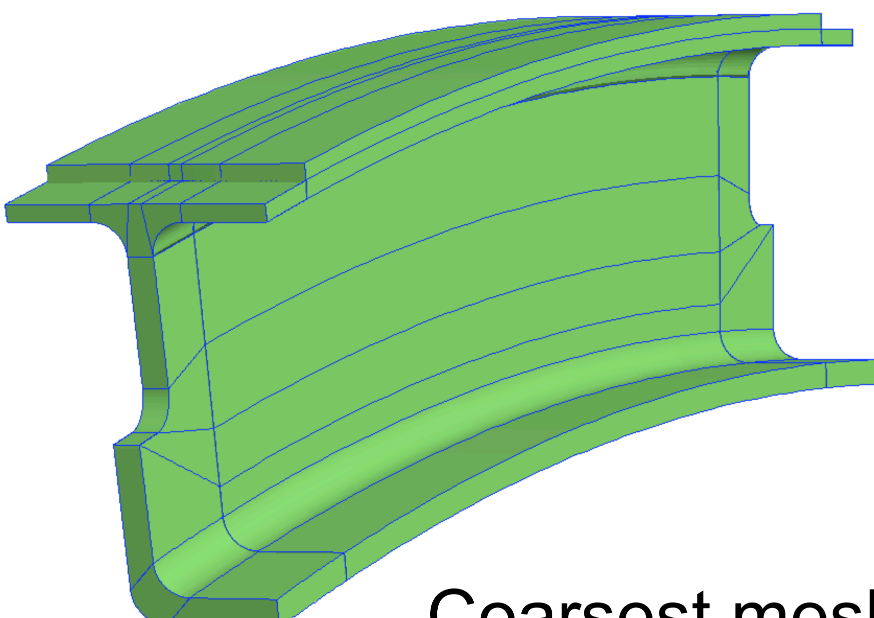
NASA ATC Frame and Skin



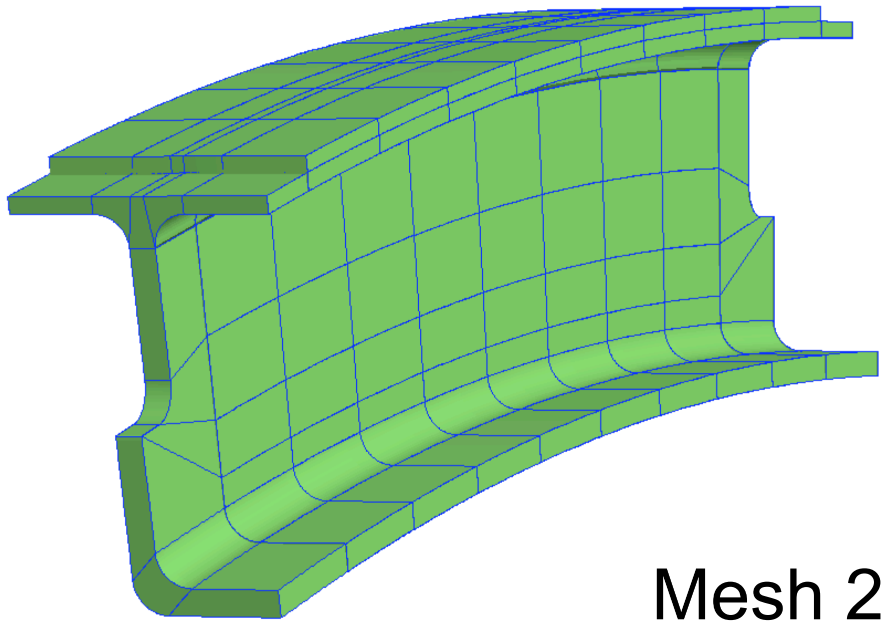
NASA ATC Frame



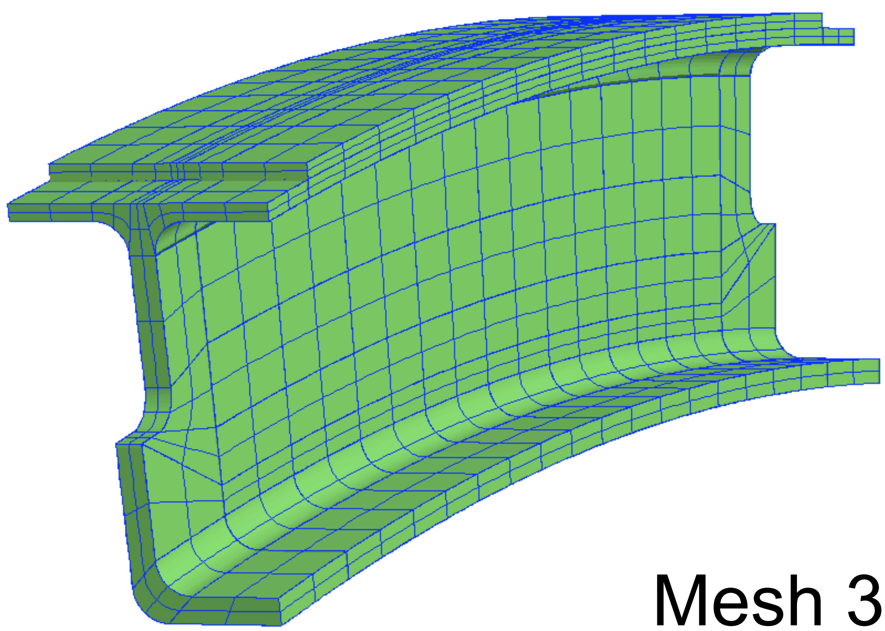




Coarsest mesh
15° segment of main rib

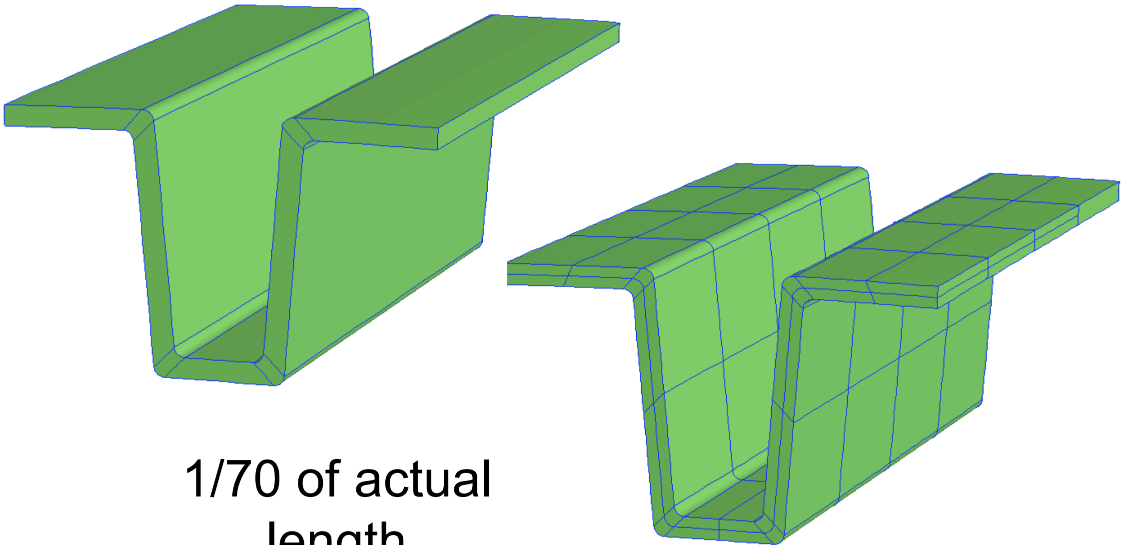


Mesh 2



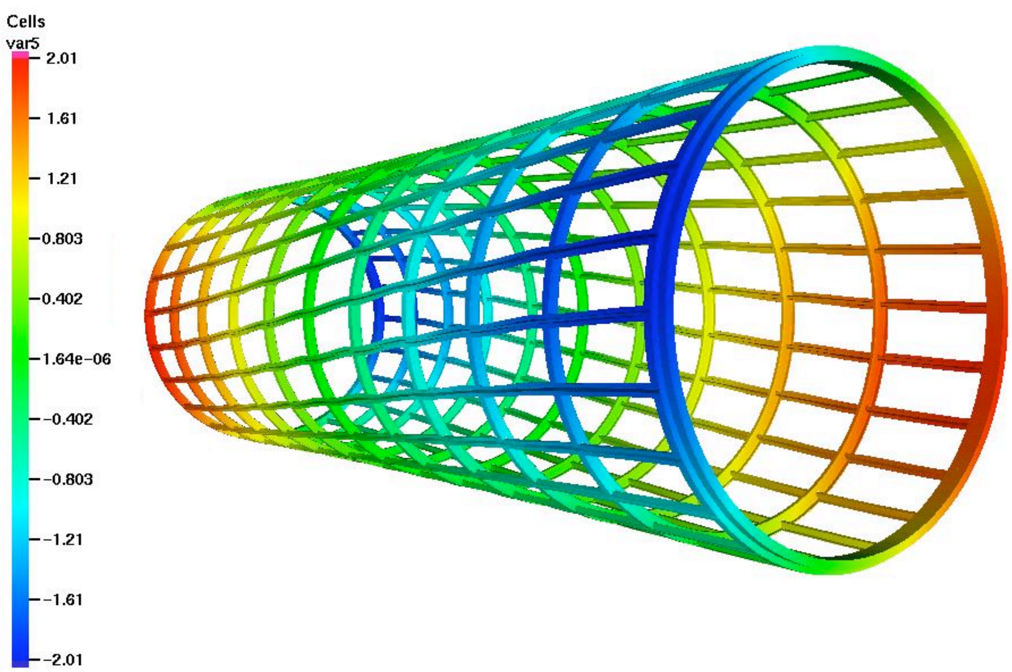
Mesh 3

Longitudinal Stringer Sample Meshes

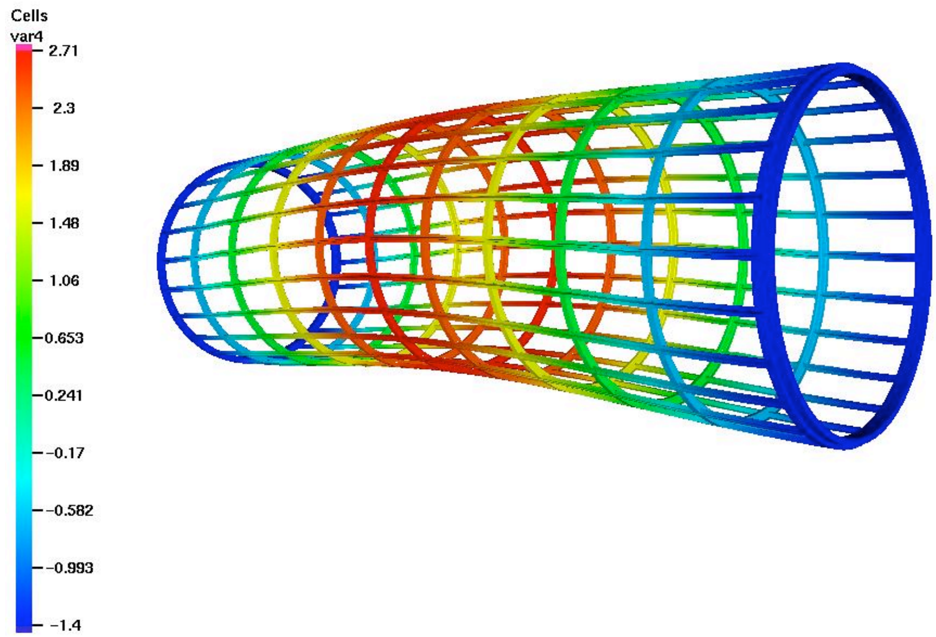


1/70 of actual length

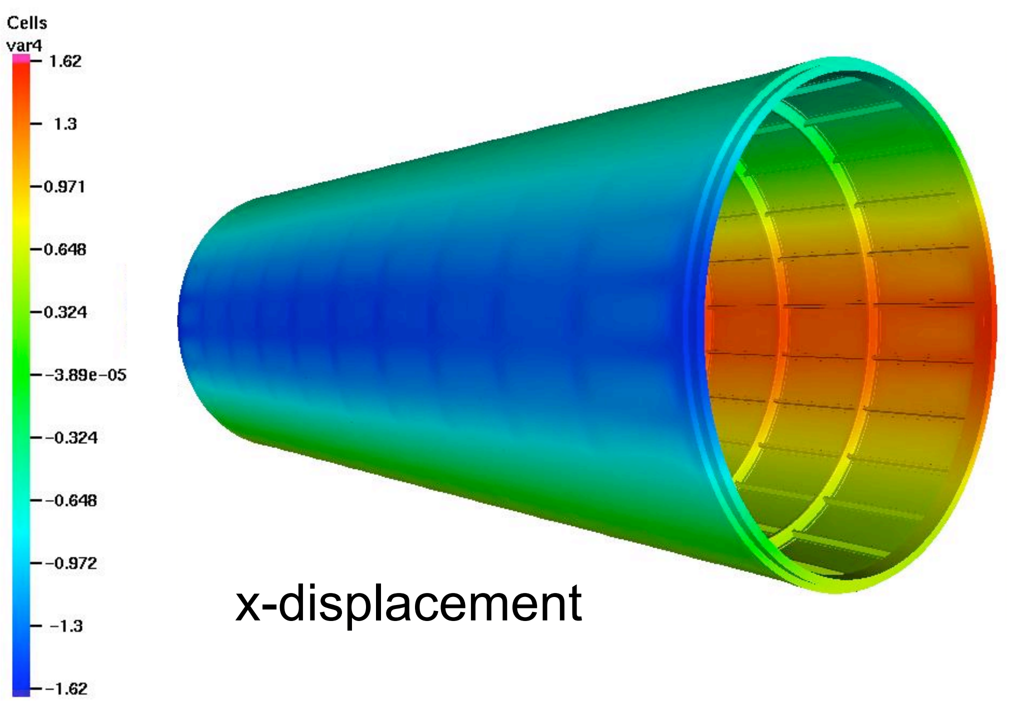
First Torsion Mode



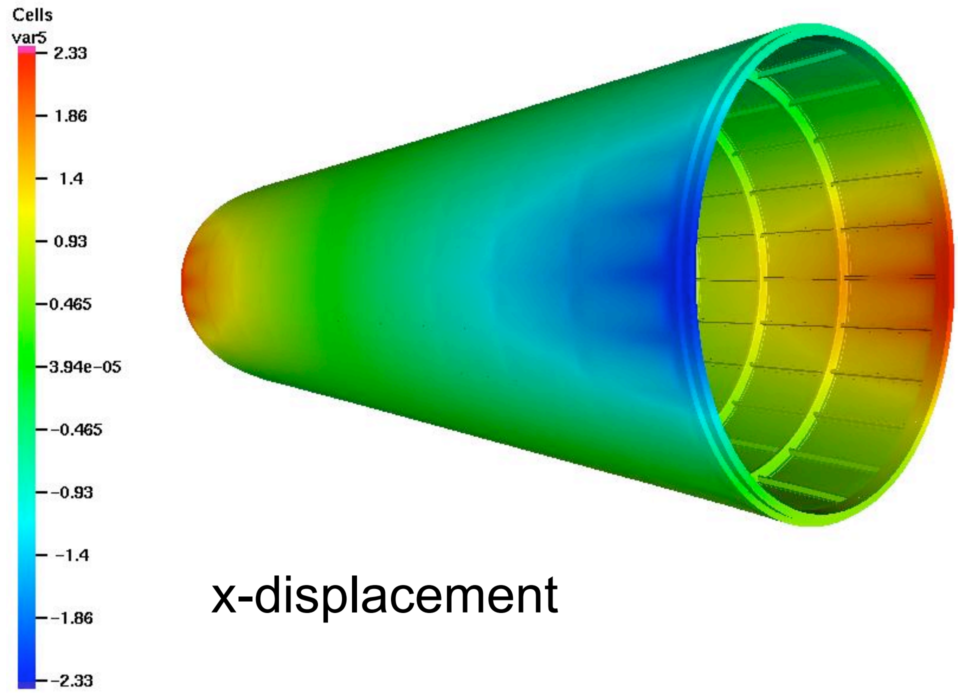
First Bending Mode



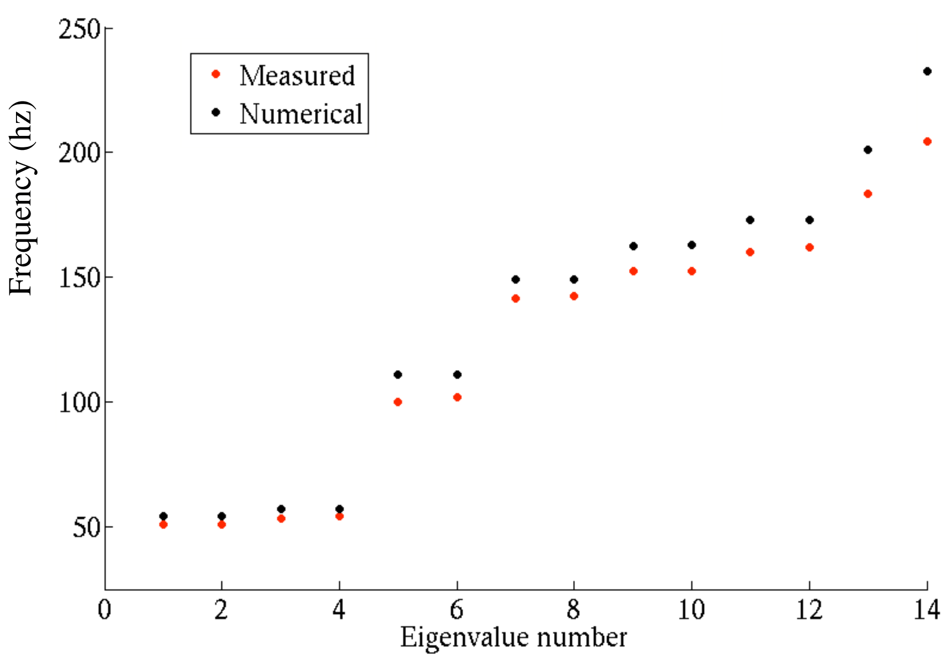
First Rayleigh Mode



First Love Mode



ATC Frame and Skin



Vibration of a Finite Elastic Rod with Fixed Ends

Problem:

$$\begin{cases} u_{,xx} + \omega^2 u = 0 \quad \text{for} \quad x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

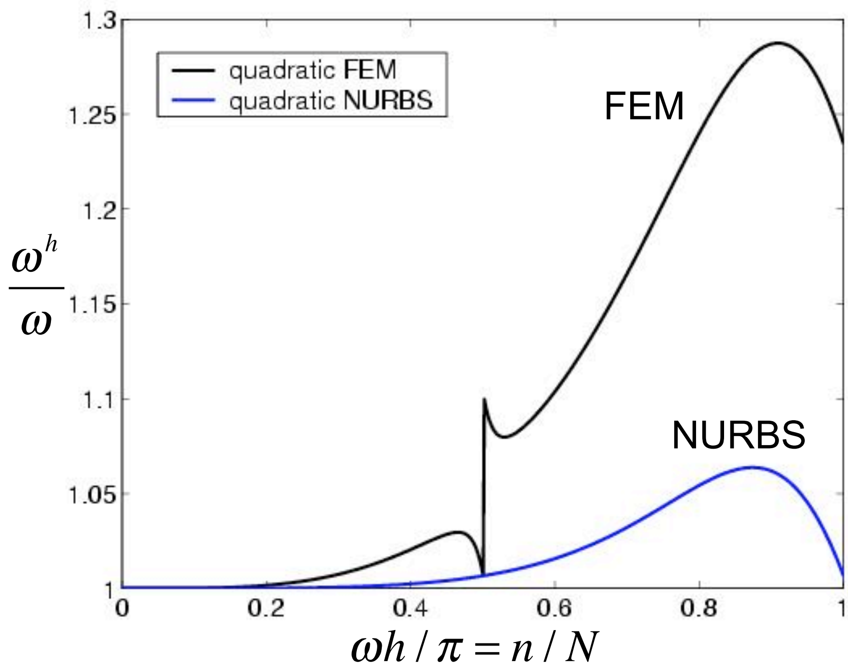
Natural frequencies:

$$\omega_n = n\pi, \quad \text{with} \quad n = 1, 2, 3, \ldots$$

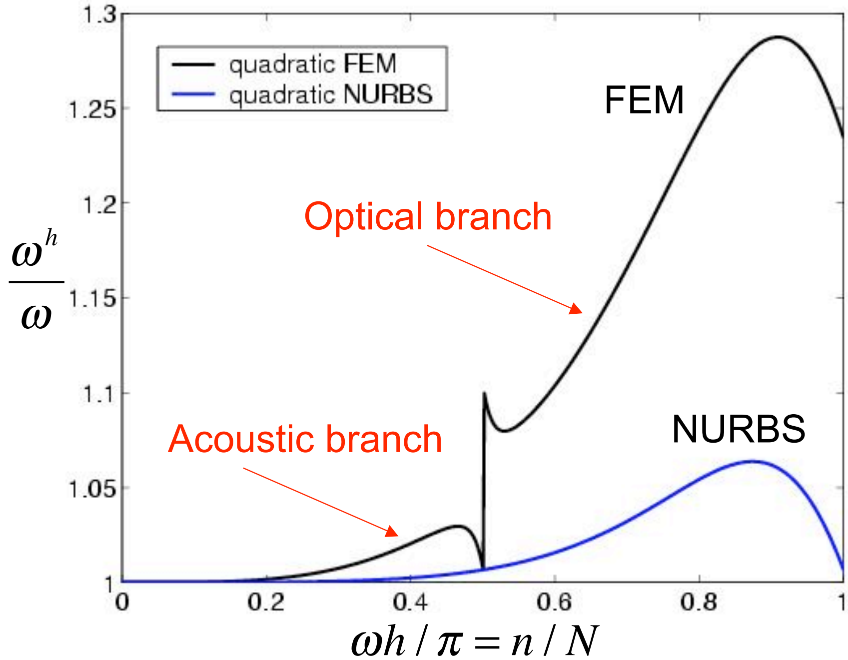
Frequency errors:

$$\omega_n^h / \omega_n$$

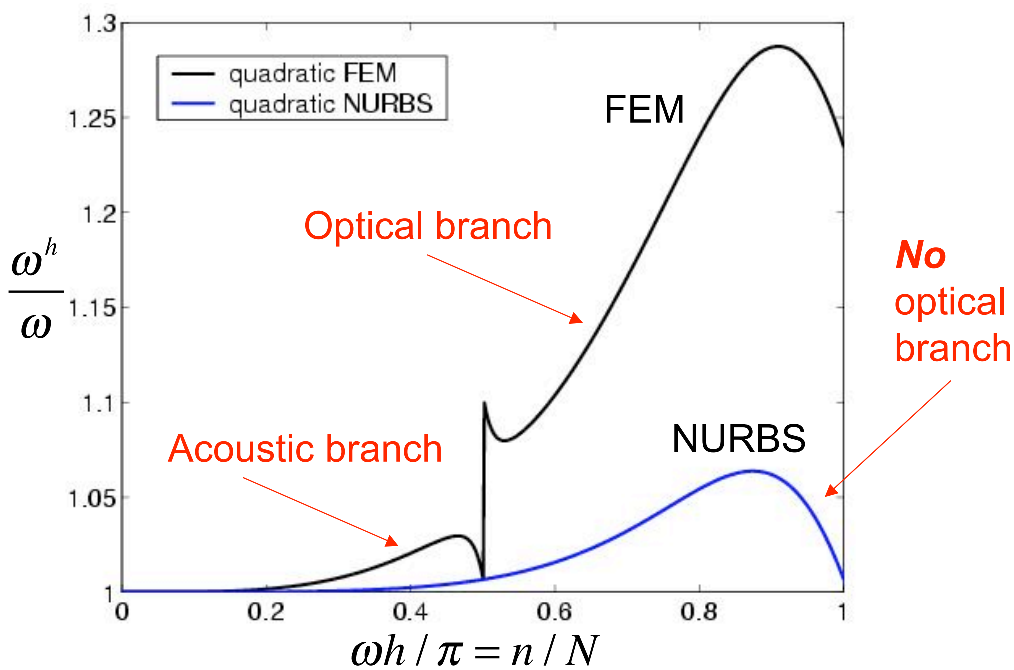
Comparison of FEM (*p*-refinement) and NURBS (*k*-refinement) Frequency Errors



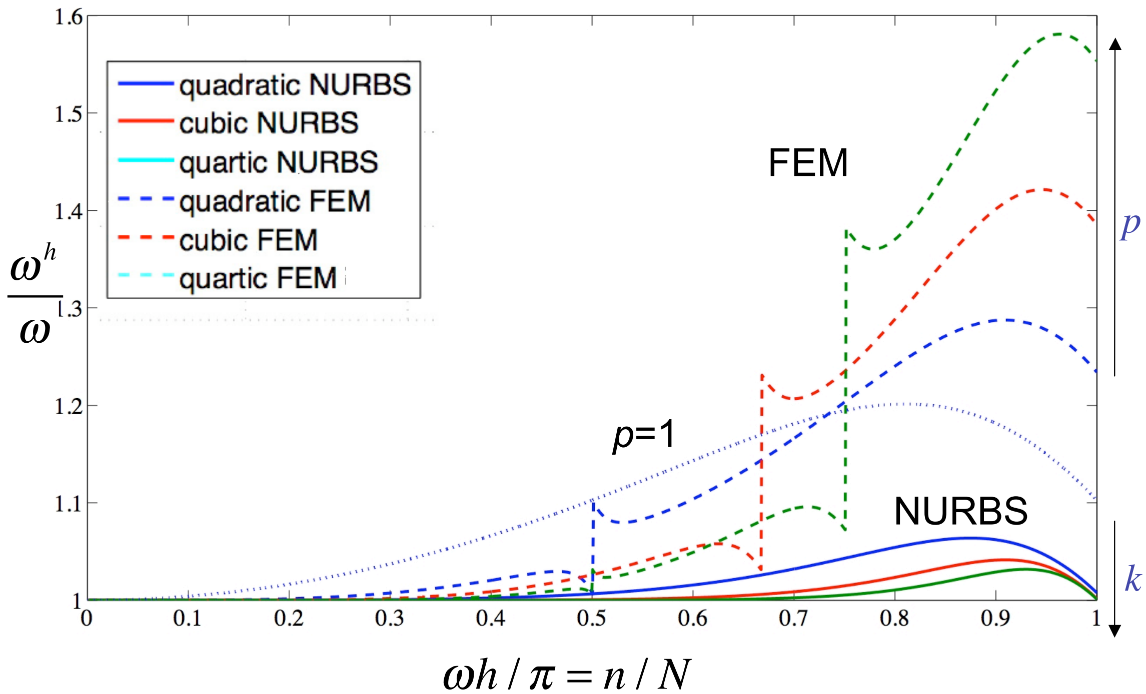
Comparison of FEM (*p*-refinement) and NURBS (*k*-refinement) Frequency Errors



Comparison of FEM (*p*-refinement) and NURBS (*k*-refinement) Frequency Errors



Comparison of FEM (p-refinement) and NURBS (k-refinement) Frequency Errors



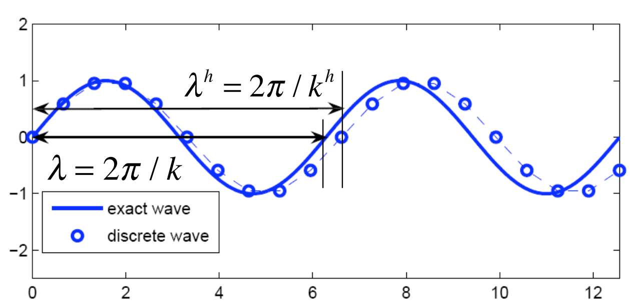
Wave Propagation in an Infinite Domain

Helmholtz equation:

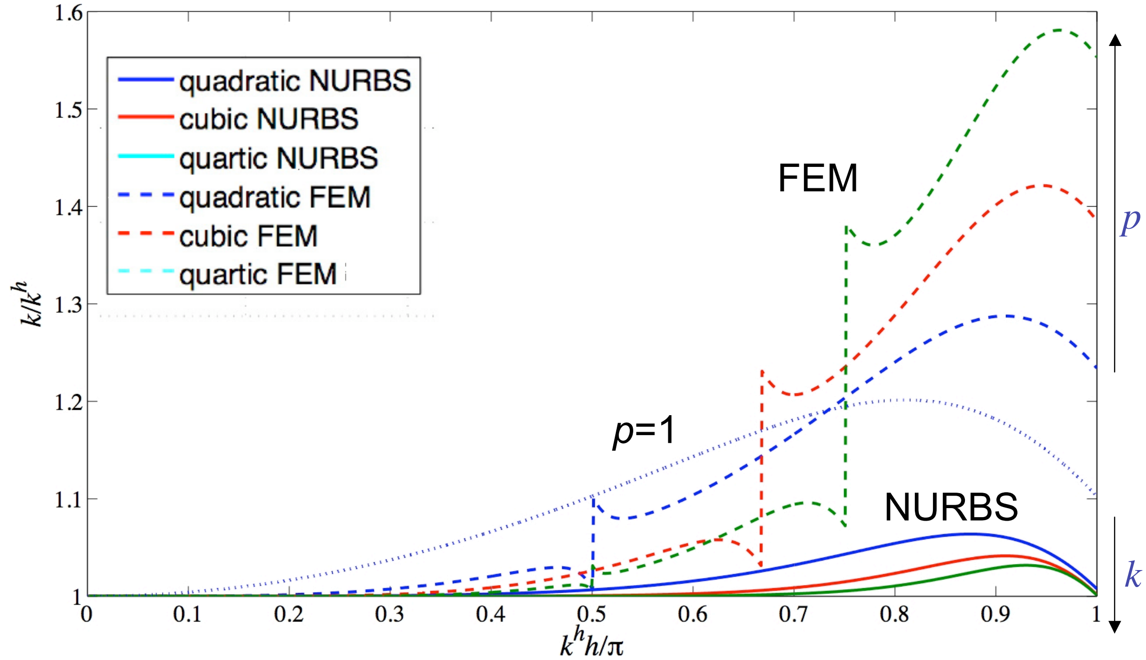
$$u_{,xx} + k^2 u = 0 \quad \text{for} \quad x \in (-\infty, +\infty)$$

Wave number: k

Phase error: k / k^h



Helmholtz Equation Phase Error



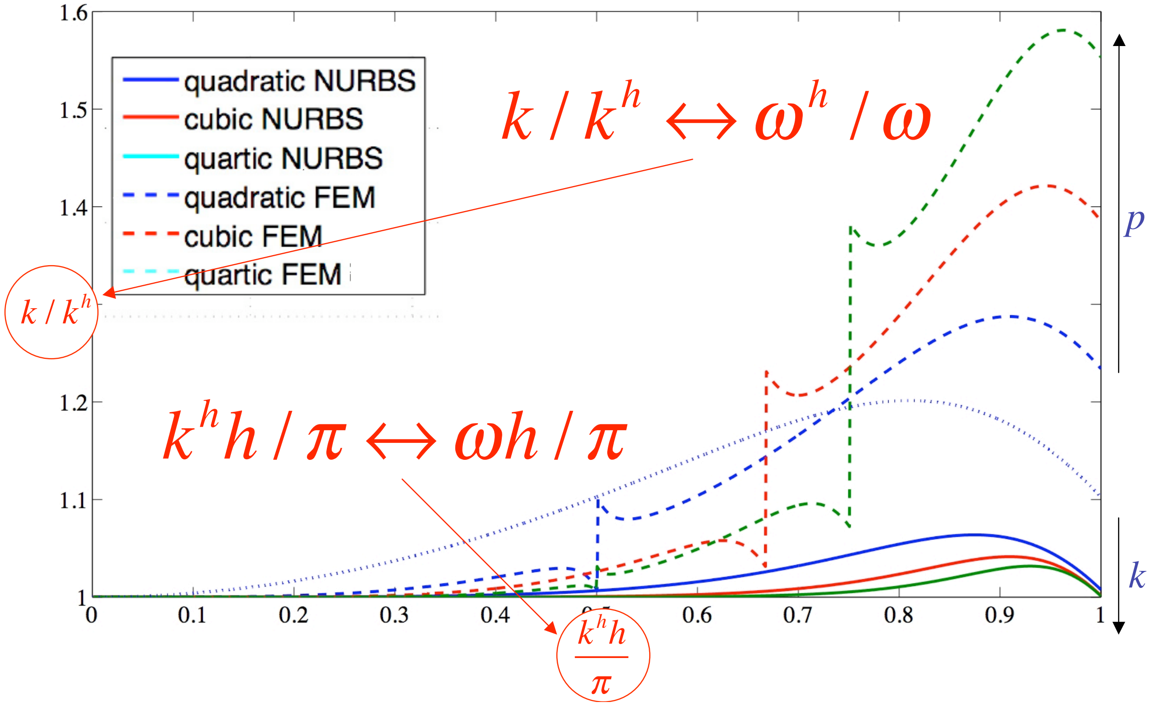
Duality Principle

- Relationship between wave propagation in an *infinite* domain and vibration of a *finite* structure
- Frequency errors and phase errors are related by a change of variables:

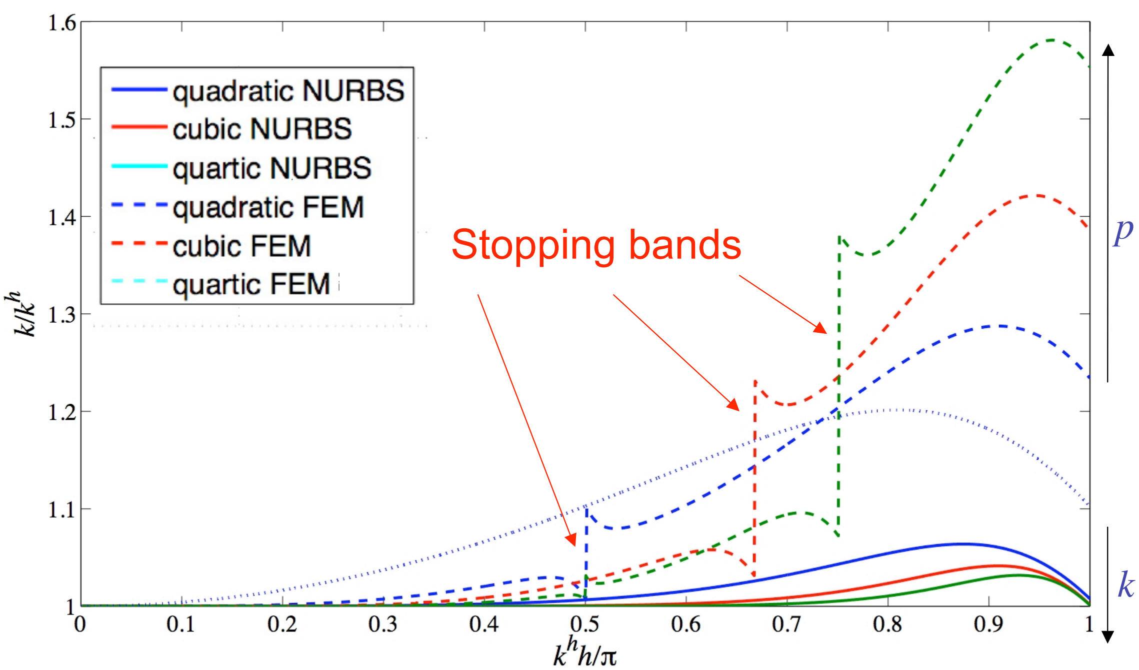
$$k / k^h \leftrightarrow \omega^h / \omega$$

$$k^h h / \pi \leftrightarrow \omega h / \pi$$

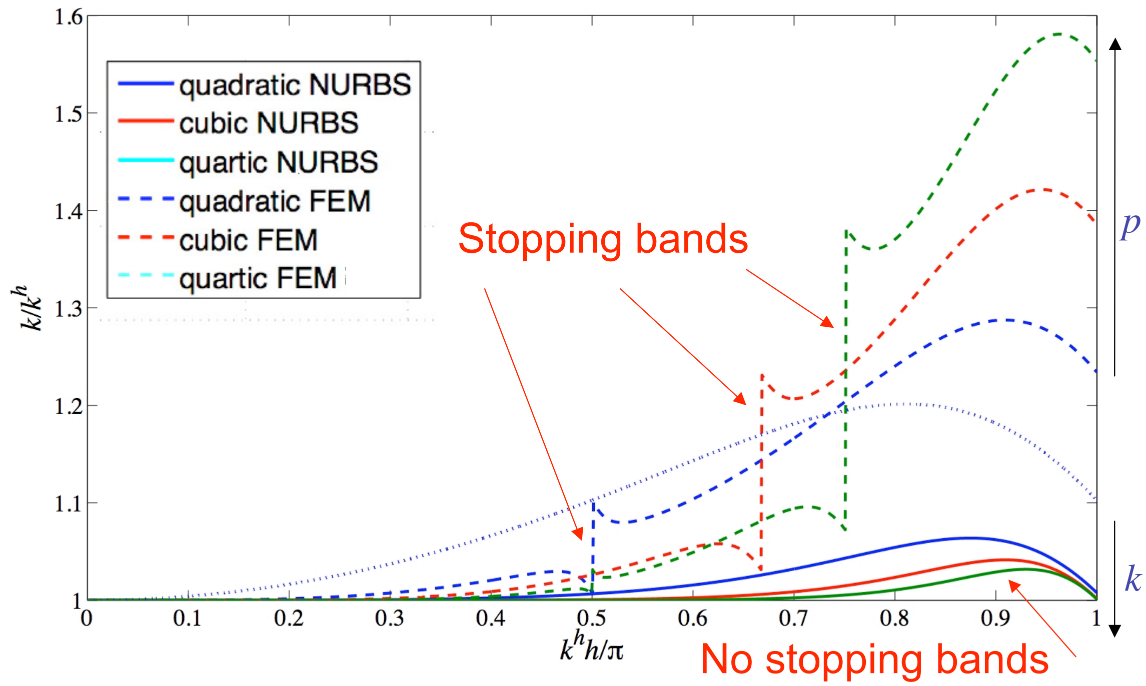
Duality of Frequency and Phase Errors



Helmholtz Equation Phase Error



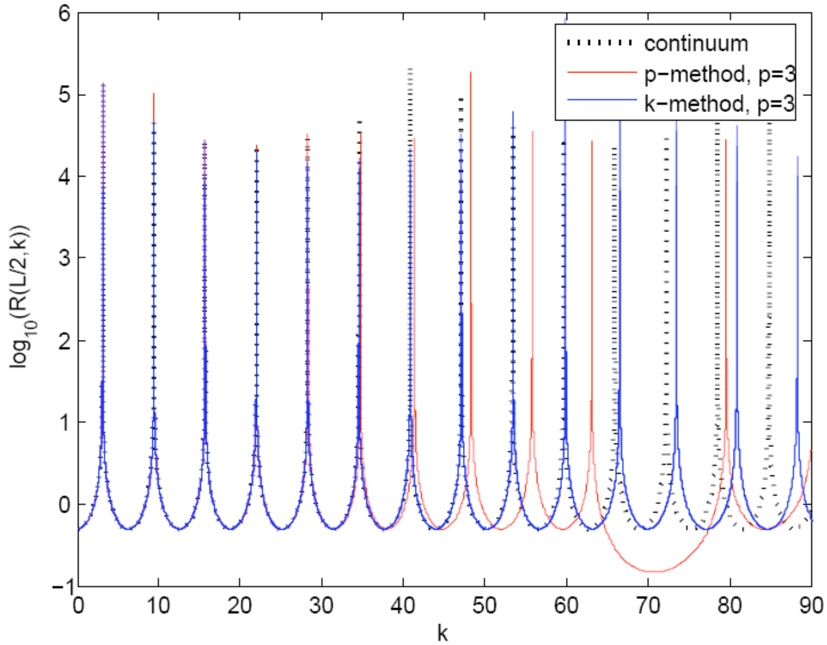
Helmholtz Equation Phase Error



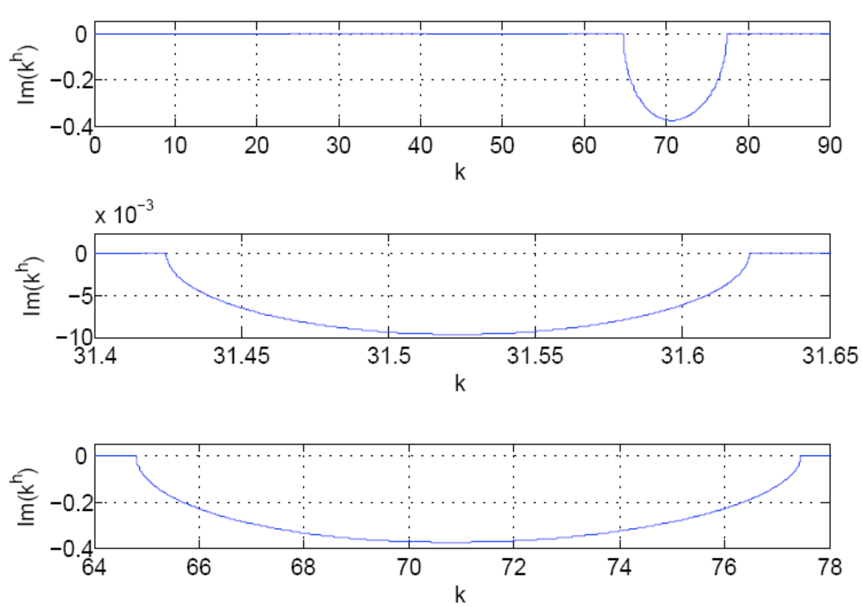
Helmholtz equation in 1D,
Dirichlet boundary conditions

(31 control points for p = 3)

Response spectrum for $p = 3$ at $x = L/2$

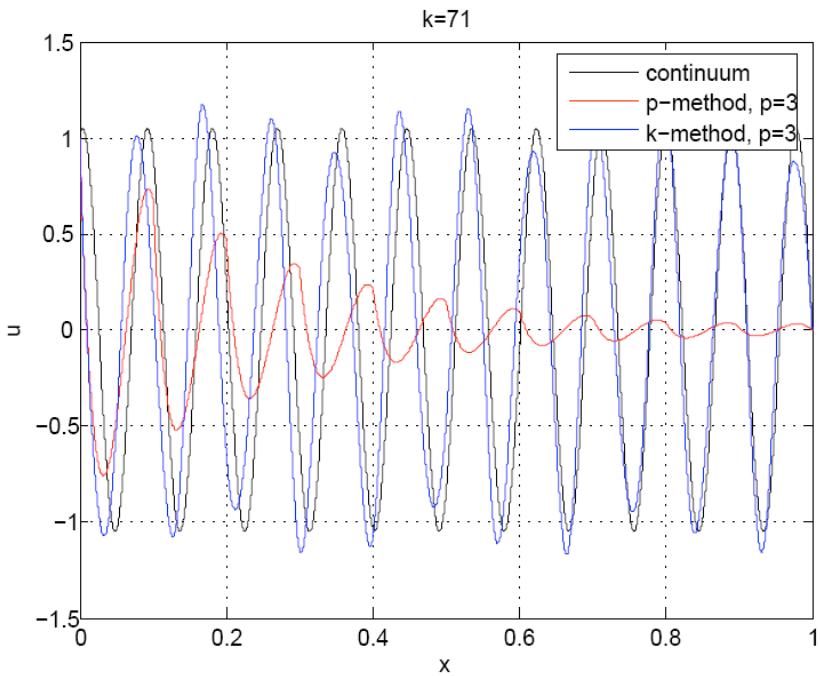


p-method stopping band for $p = 3$



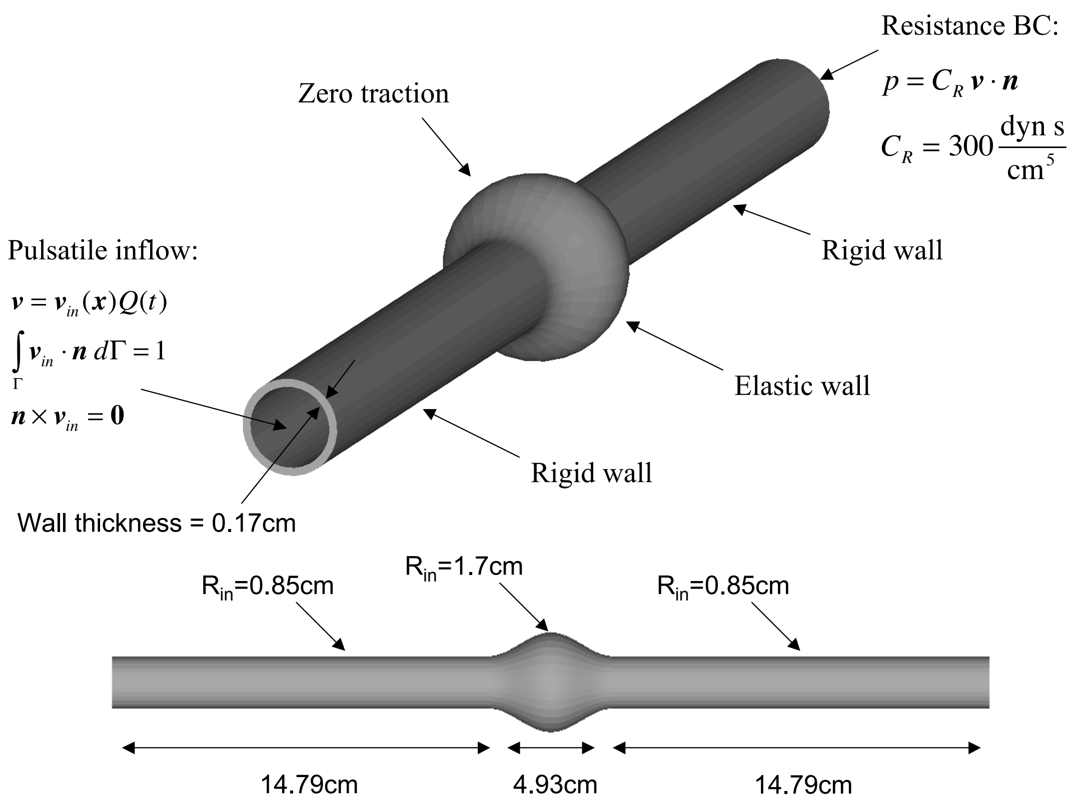
k = 71; p = 3

(inside the 2nd p-method stopping band)

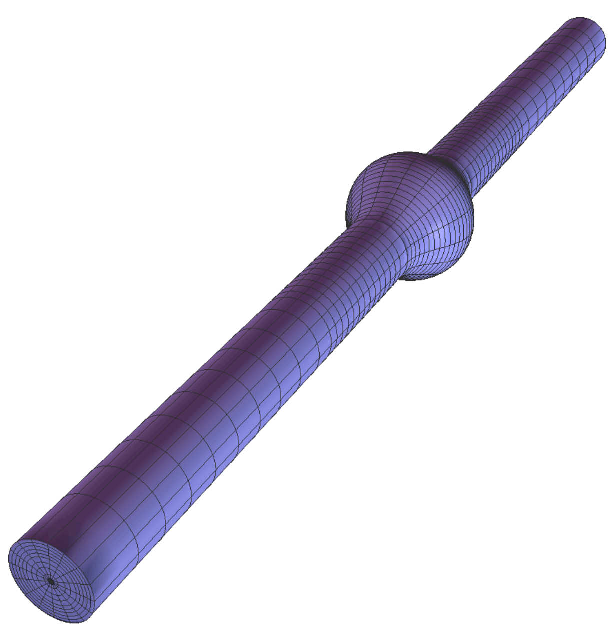


Fluid-Structure Interaction

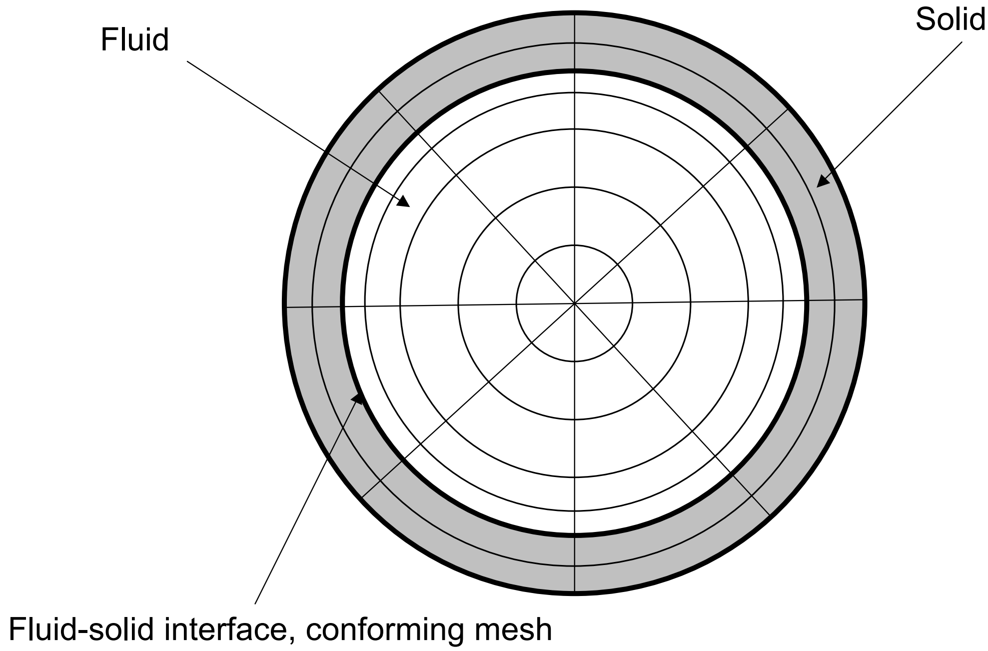
- Incompressible viscous fluids (*ALE description*) and nonlinear solids (*Lagrangian description*)
- Residual-based *variational multiscale formulation*, applicable to laminar and turbulent flows
- Both fluid and solid may undergo *large motions*
- Geometry and kinematics are *fully compatible* across fluid-structure interfaces
- Strongly coupled, *monolithic* solution algorithm


Resistance BC:
$p = C_R \boldsymbol{v} \cdot \boldsymbol{n}$
$C_R = 300 \frac{\text{dyn s}}{\text{cm}^5}$
Zero traction
Pulsatile inflow:
$\boldsymbol{v} = \boldsymbol{v}_{in}(\boldsymbol{x})Q(t)$
$\int_\Gamma \boldsymbol{v}_{in} \cdot \boldsymbol{n}\, d\Gamma = 1$
$\boldsymbol{n} \times \boldsymbol{v}_{in} = \boldsymbol{0}$
Rigid wall
Elastic wall
Rigid wall
Wall thickness = 0.17cm
R_{in}=0.85cm
R_{in}=1.7cm
R_{in}=0.85cm
14.79cm
4.93cm
14.79cm

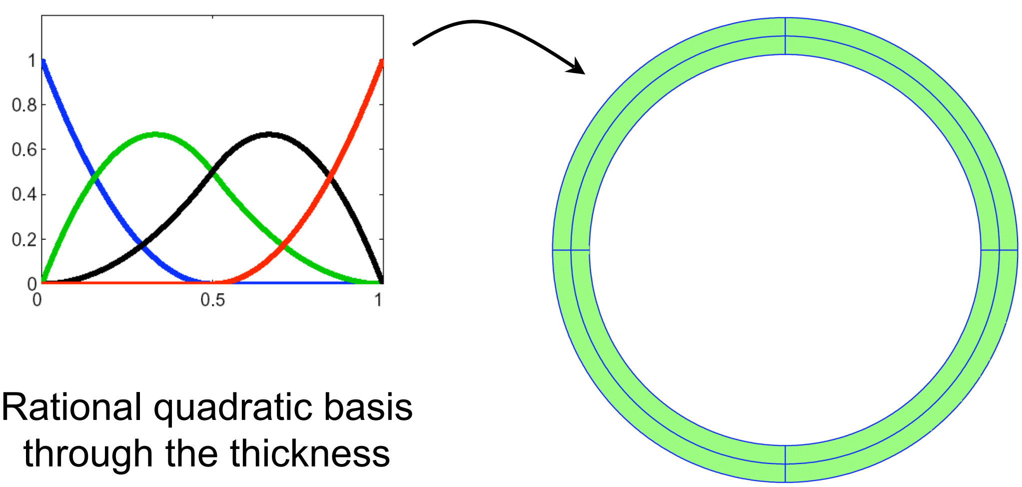
Mesh



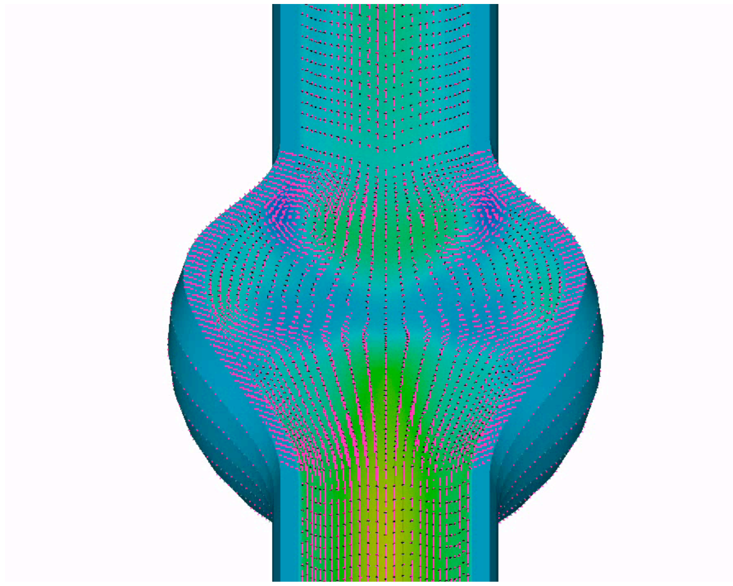
Cross-section Schematic

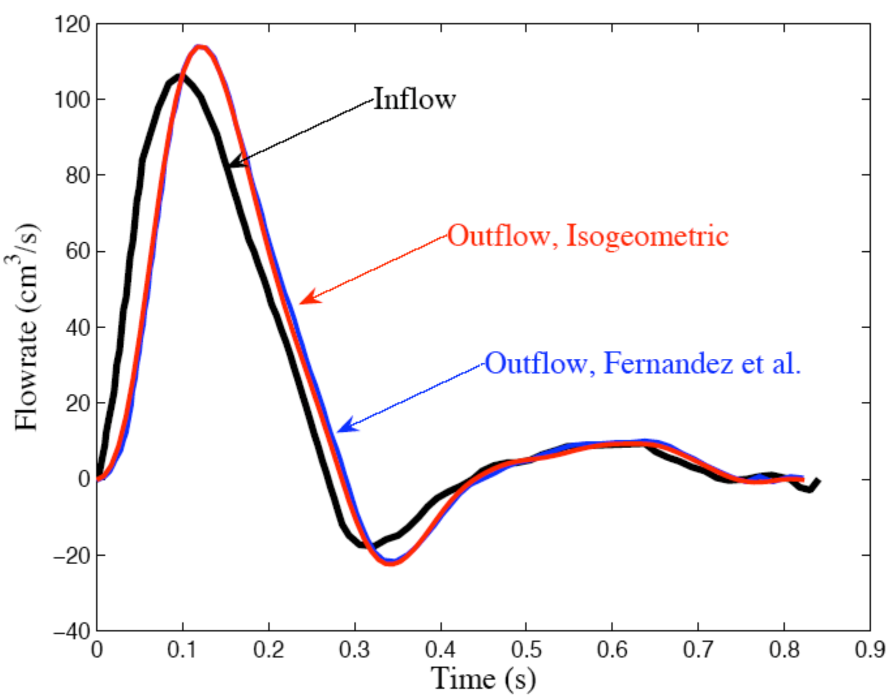


Through-thickness discretization



Rational quadratic basis through the thickness

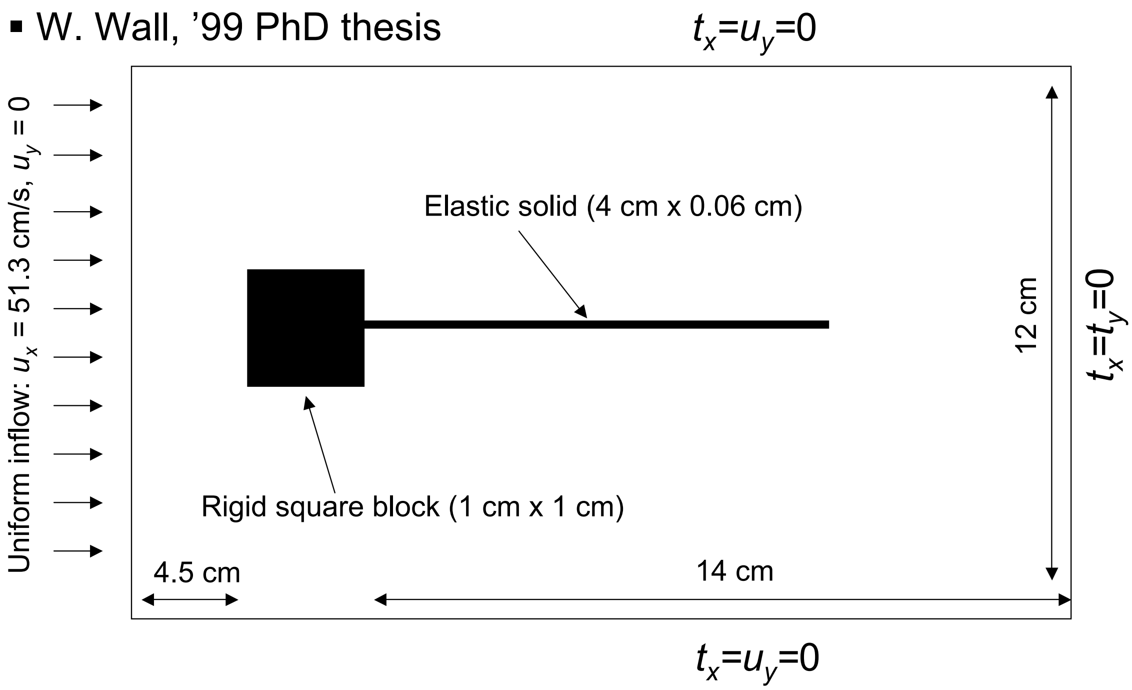




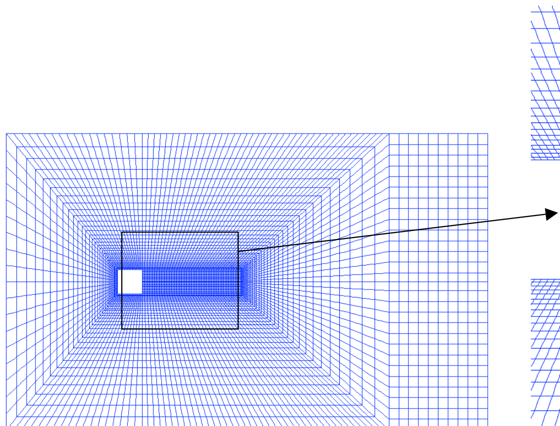
Inflow
Outflow, Isogeometric
Outflow, Fernandez et al.
Flowrate (cm^3/s)
Time (s)

Re = 100 flow over a rigid block with an elastic beam

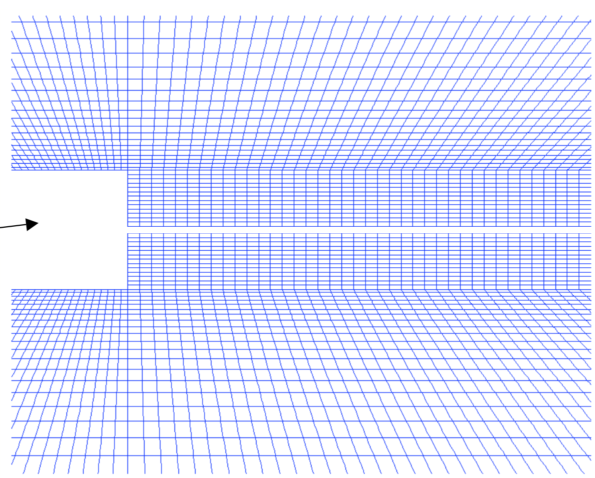
- W. Wall, '99 PhD thesis



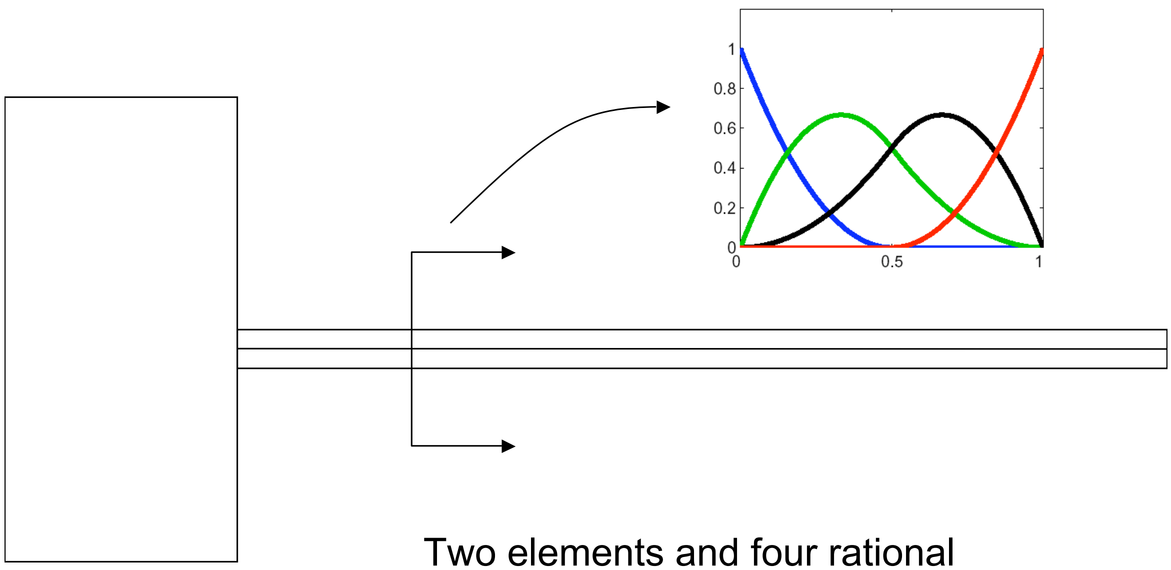
Computational Mesh



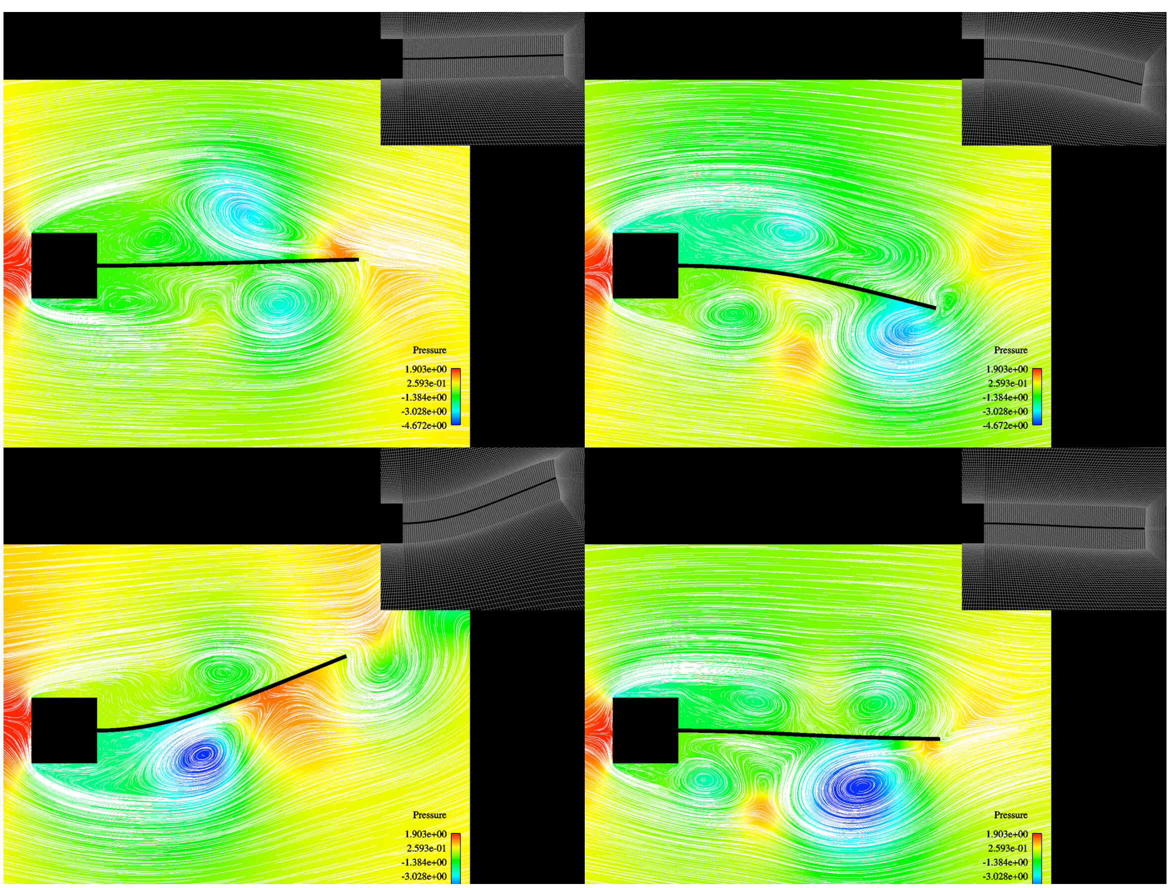
(a) Entire domain

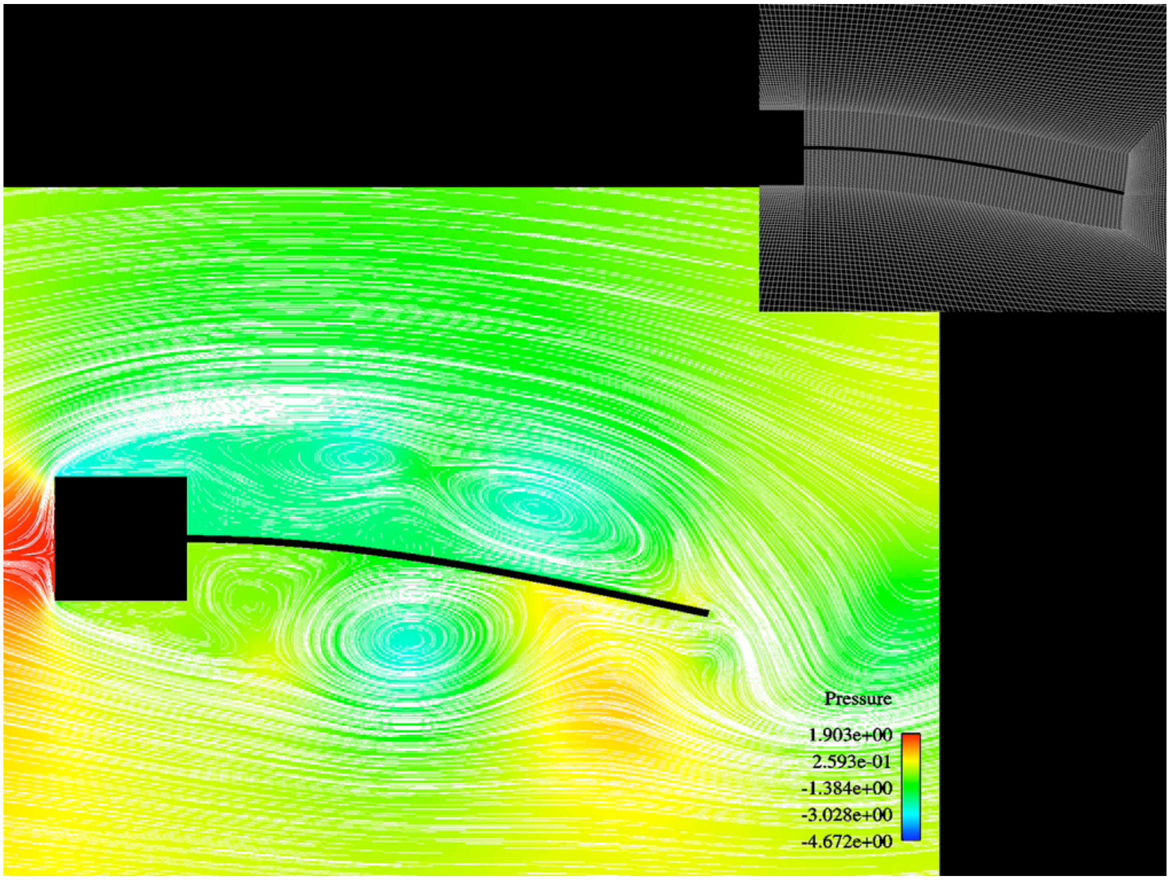


(b) Fluid domain near the block

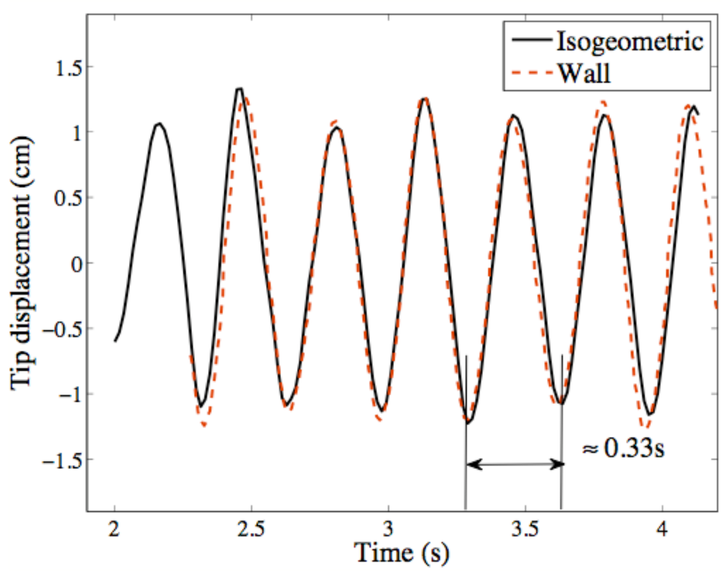


Two elements and four rational quadratic basis functions through the thickness of the bar

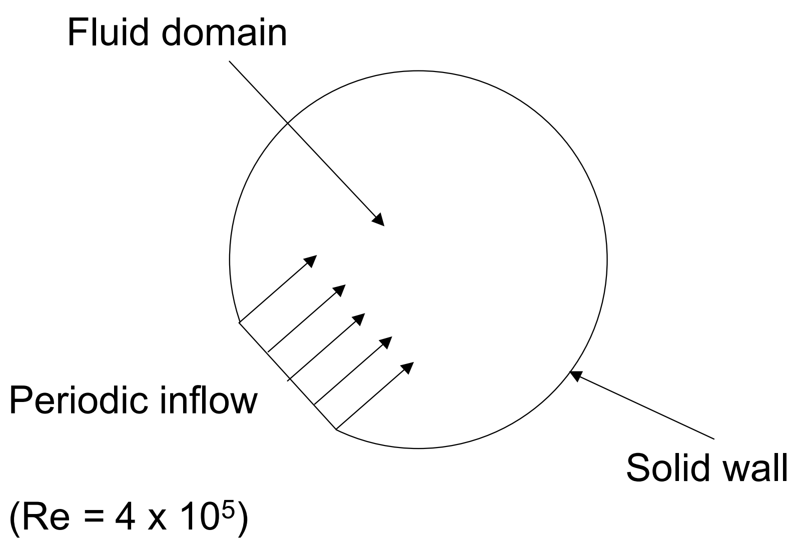




Tip Displacement of Beam



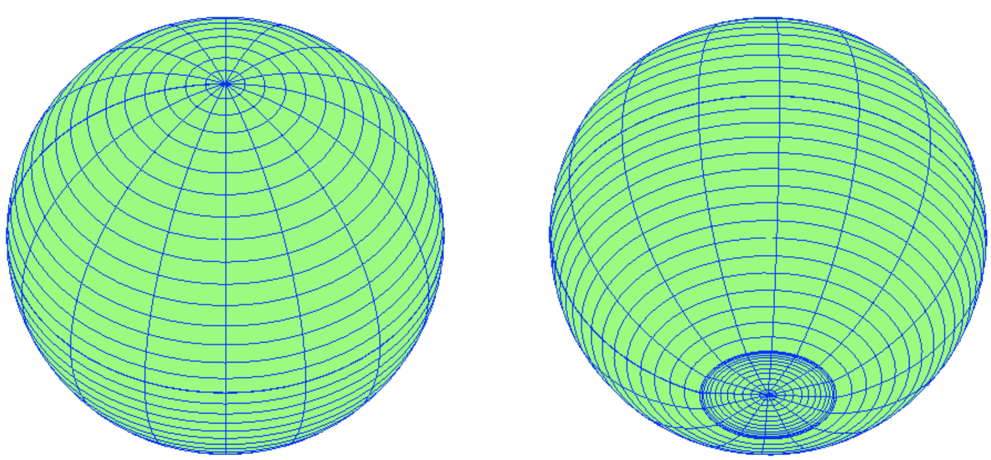
Balloon Containing an Incompressible Fluid



From Wall ’06, Tezduyar ’07

Balloon Containing an Incompressible Fluid

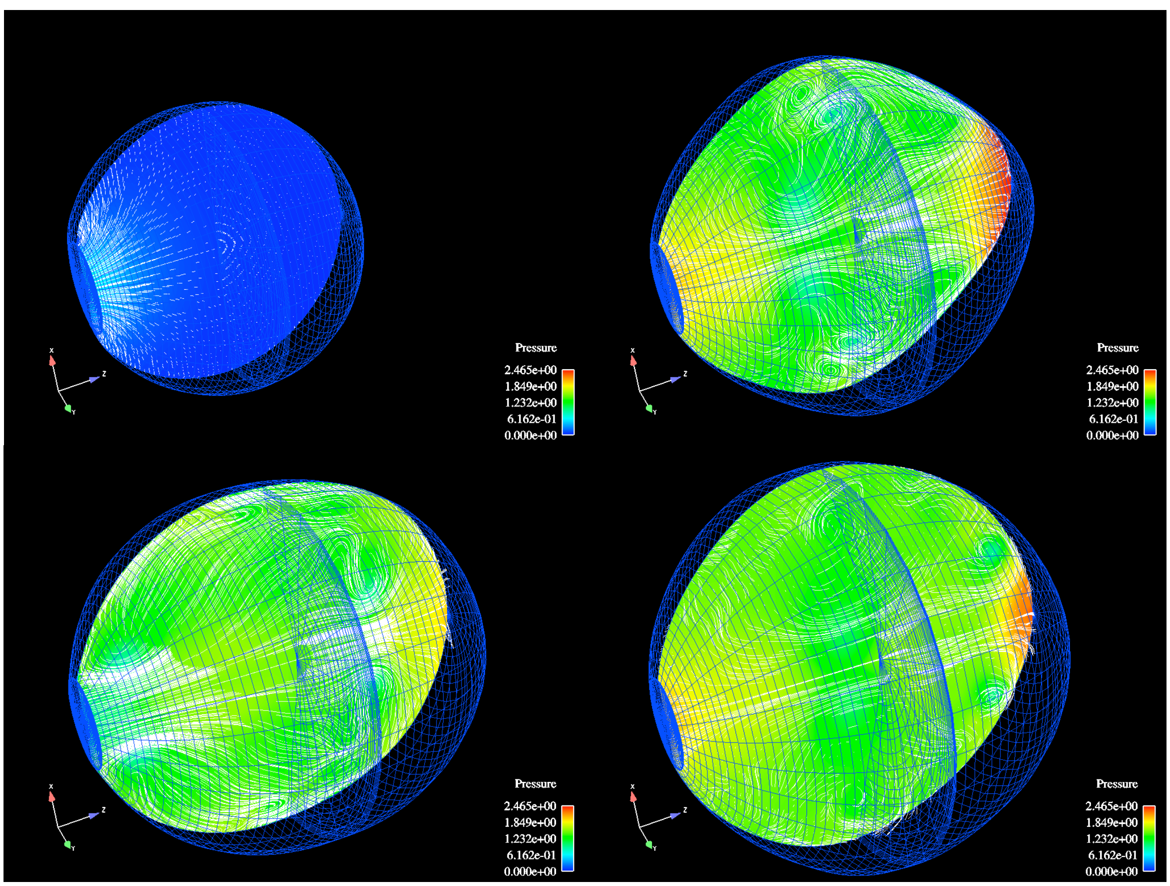
- Quadratic NURBS for both solid and fluid
- Boundary layer meshing

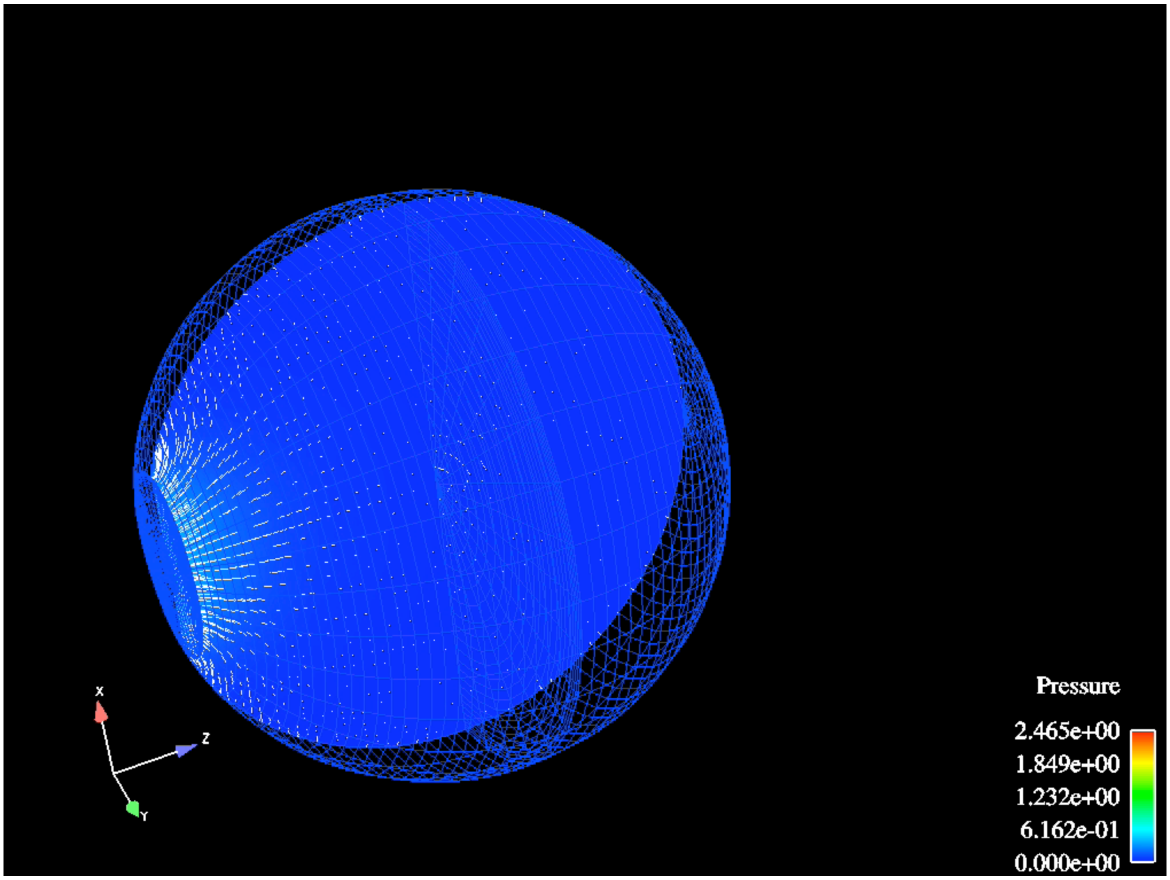


(a) Top view (b) Bottom view

Balloon Containing an Incompressible Fluid

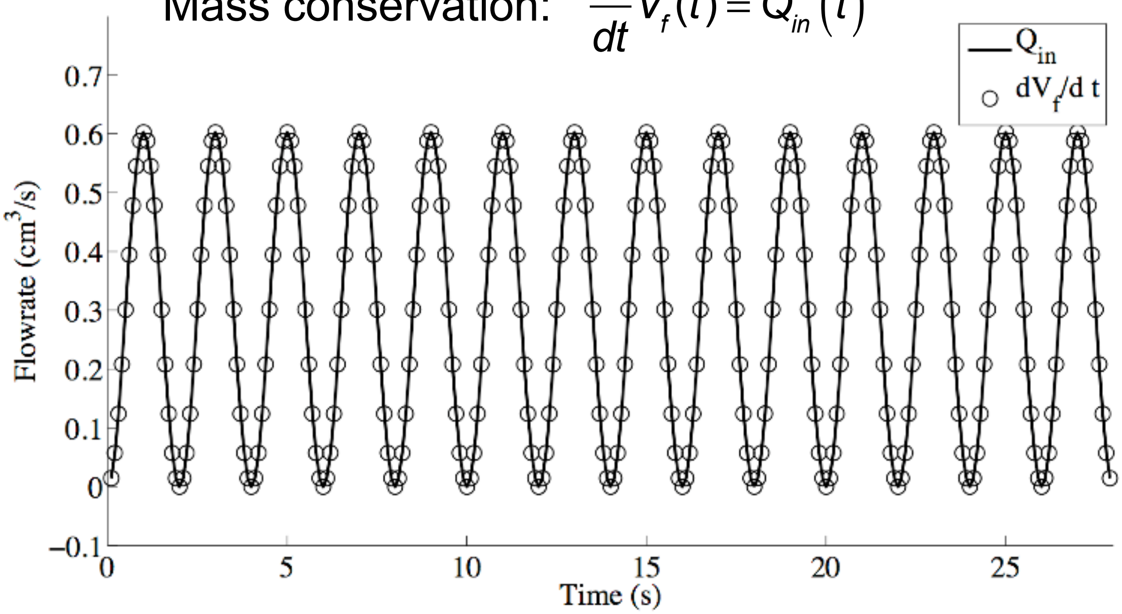
- Staggered algorithms:
  - Fluid domain geometry is defined by motion of the solid, which does not account for fluid *incompressibility*
  - Calculations *fail* unless special procedures are devised
- Strongly coupled, monolithic algorithms *work well*



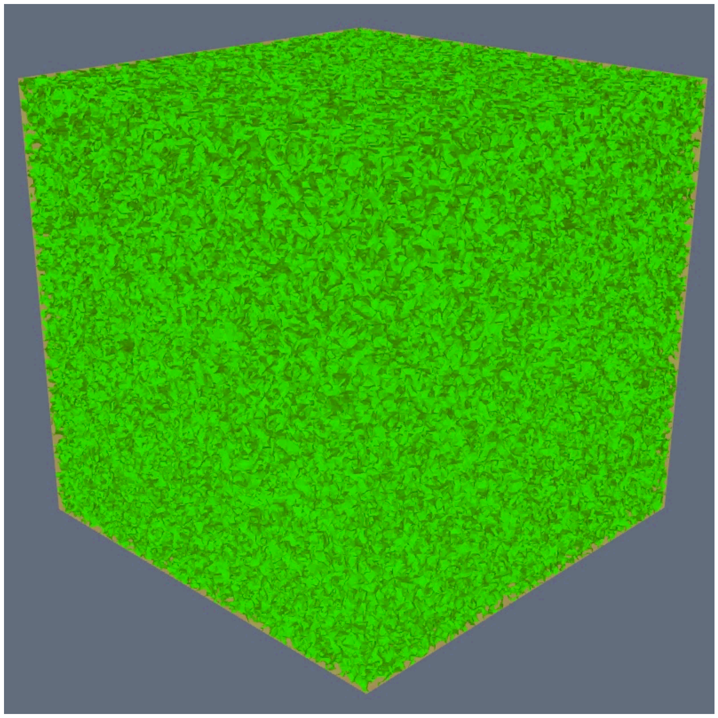


Balloon Containing an Incompressible Fluid

Mass conservation: $\frac{d}{dt}V_f(t) = Q_{in}(t)$



Phase Field Modeling: C^1 Quadratics NURBS

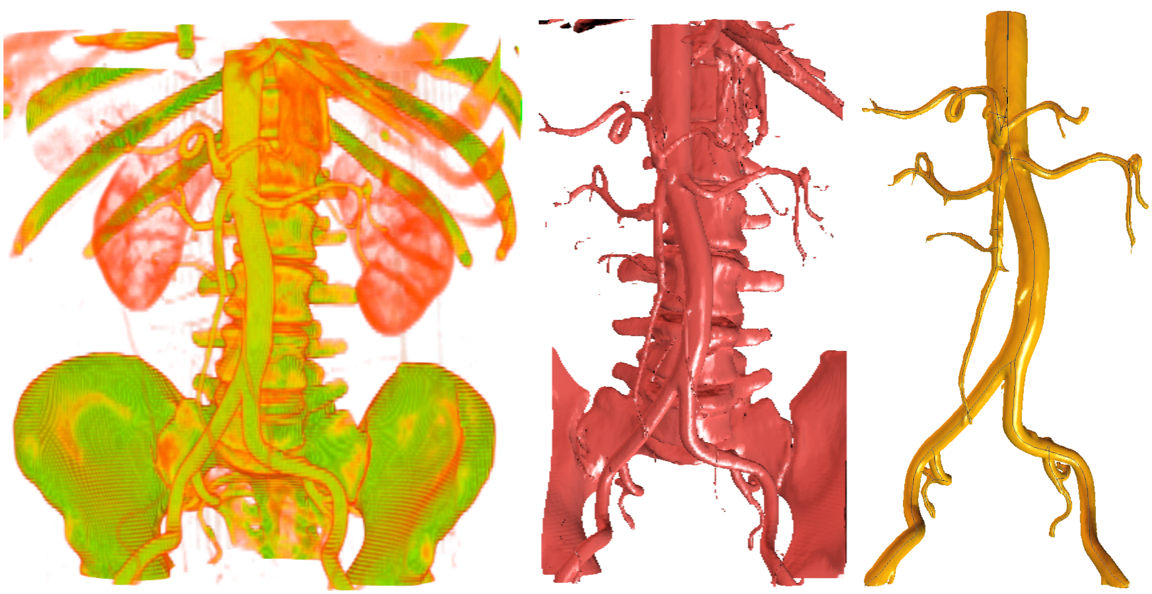


Cardiovascular Research

- Patient-specific mathematical models of major arteries and the heart
- Cardiovascular Modeling Toolkit
  - Abdominal aorta
  - LVADs: Left Ventricular Assist Devices (R. Moser)
  - Aneurysms
  - Vulnerable plaques and drug delivery systems
  - Hearts

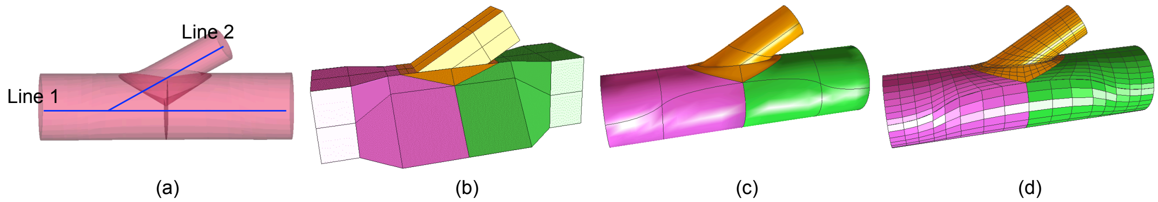
Medical Imaging: Computed Tomography (CT)

Abdominal Aorta

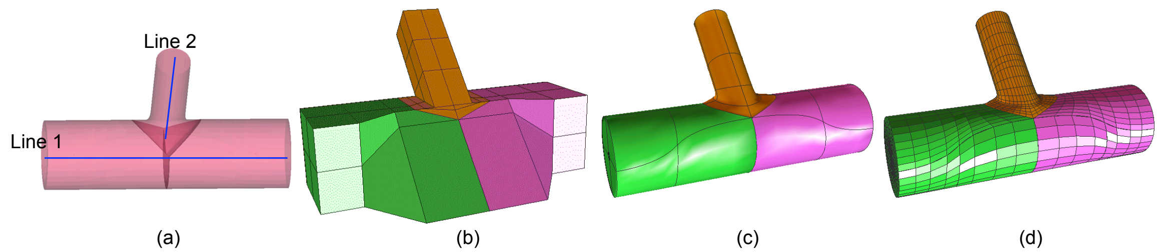


(a) Volume rendering (b) Isocontouring (c) Surface model & path

Bifurcation Templates

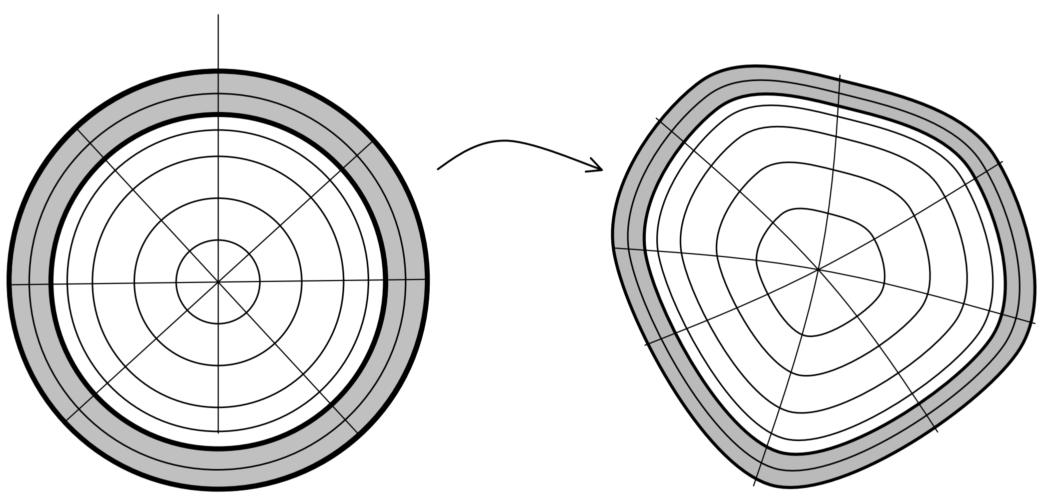


Bifurcation Case 3

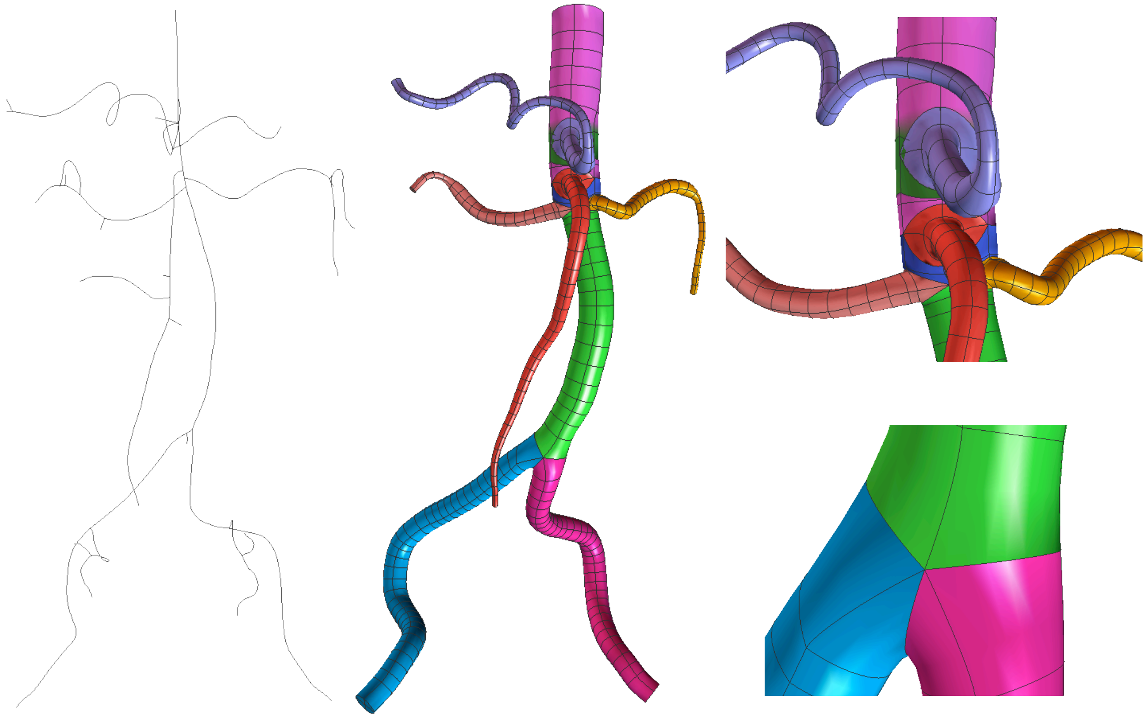


Bifurcation Case 4

Mapping onto a patient-specific arterial cross-section

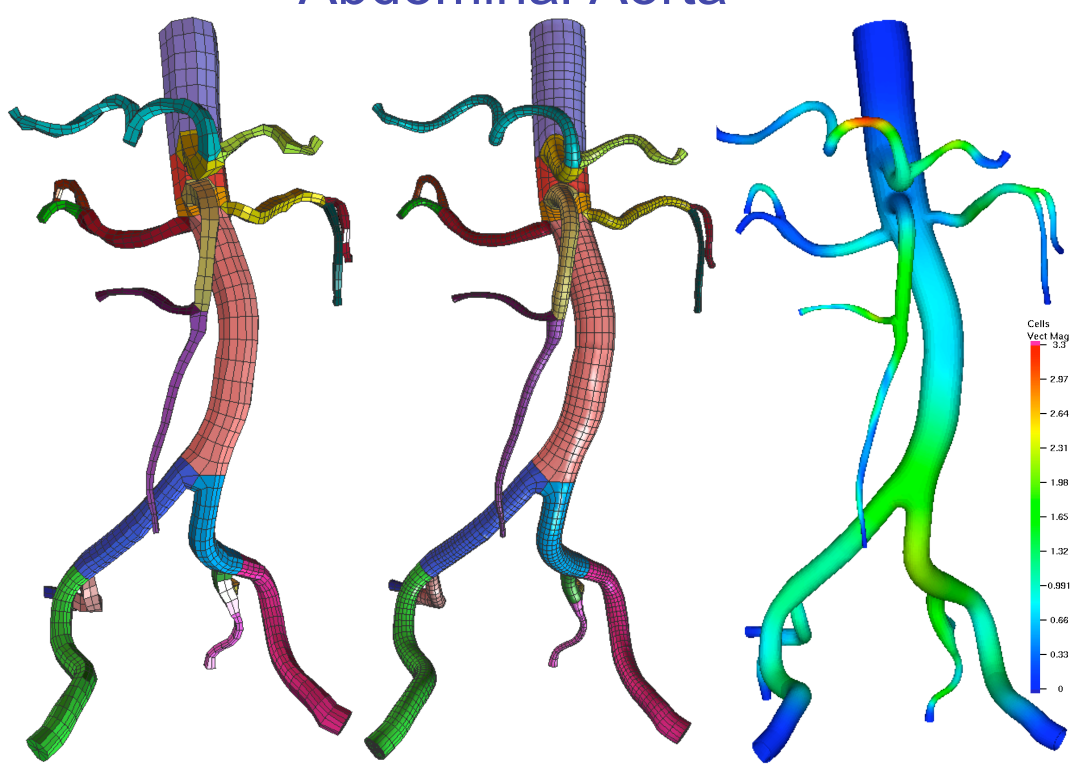


Abdominal Aorta



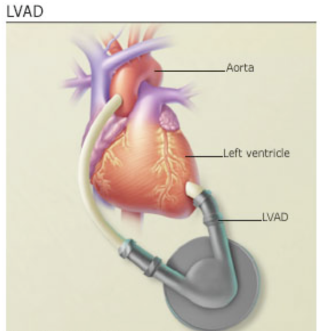
Path NURBS hex mesh Detail

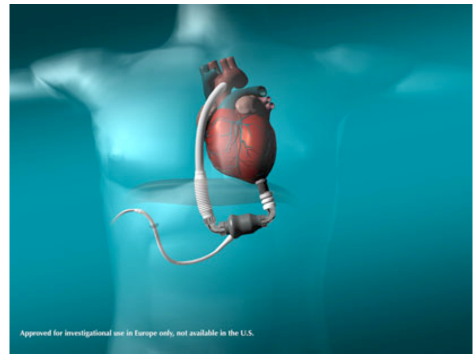
Abdominal Aorta

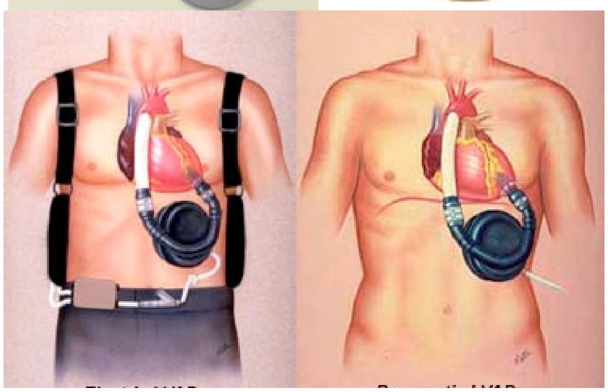


(d) Control mesh (e) Solid NURBS (f) Simulation results

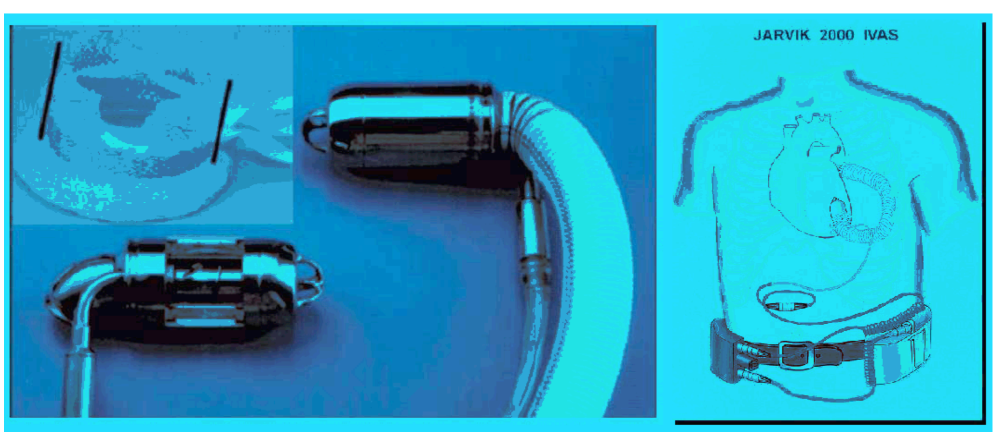
Left Ventricular Assist Devices (LVADs) with Ascending Aortic Distal Anastomosis







Jarvik 2000 and Schematic of Descending Aortic Distal Anastomosis

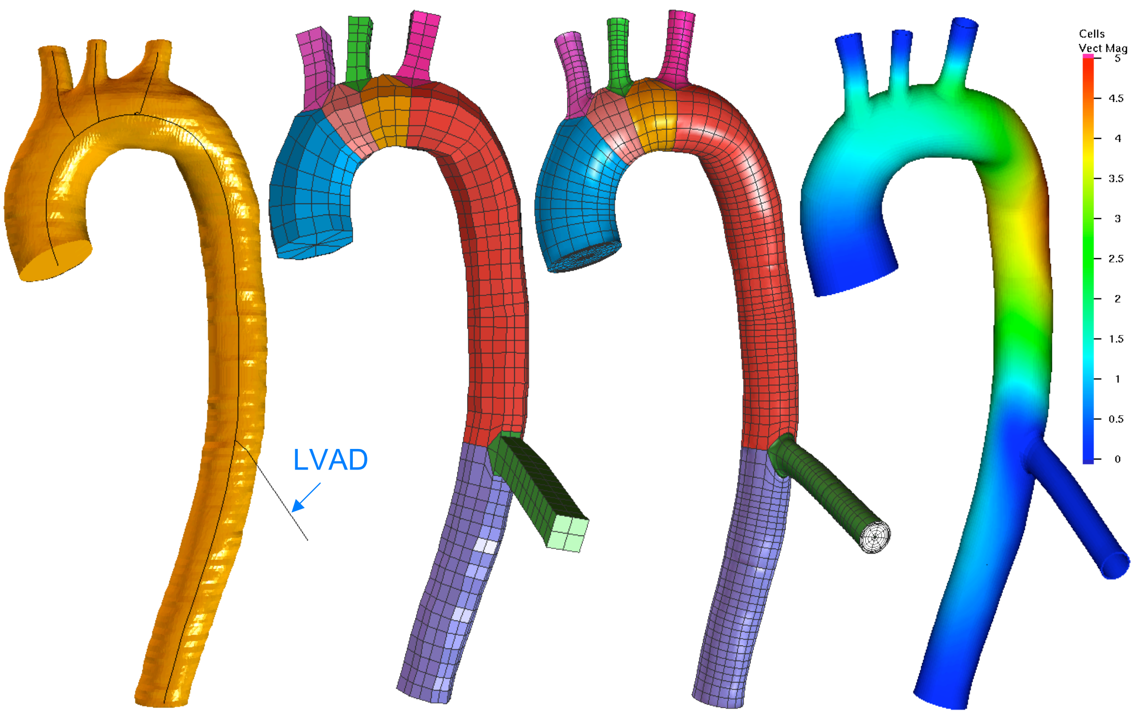


Comparison of Ascending and Descending Aortic Anastamoses

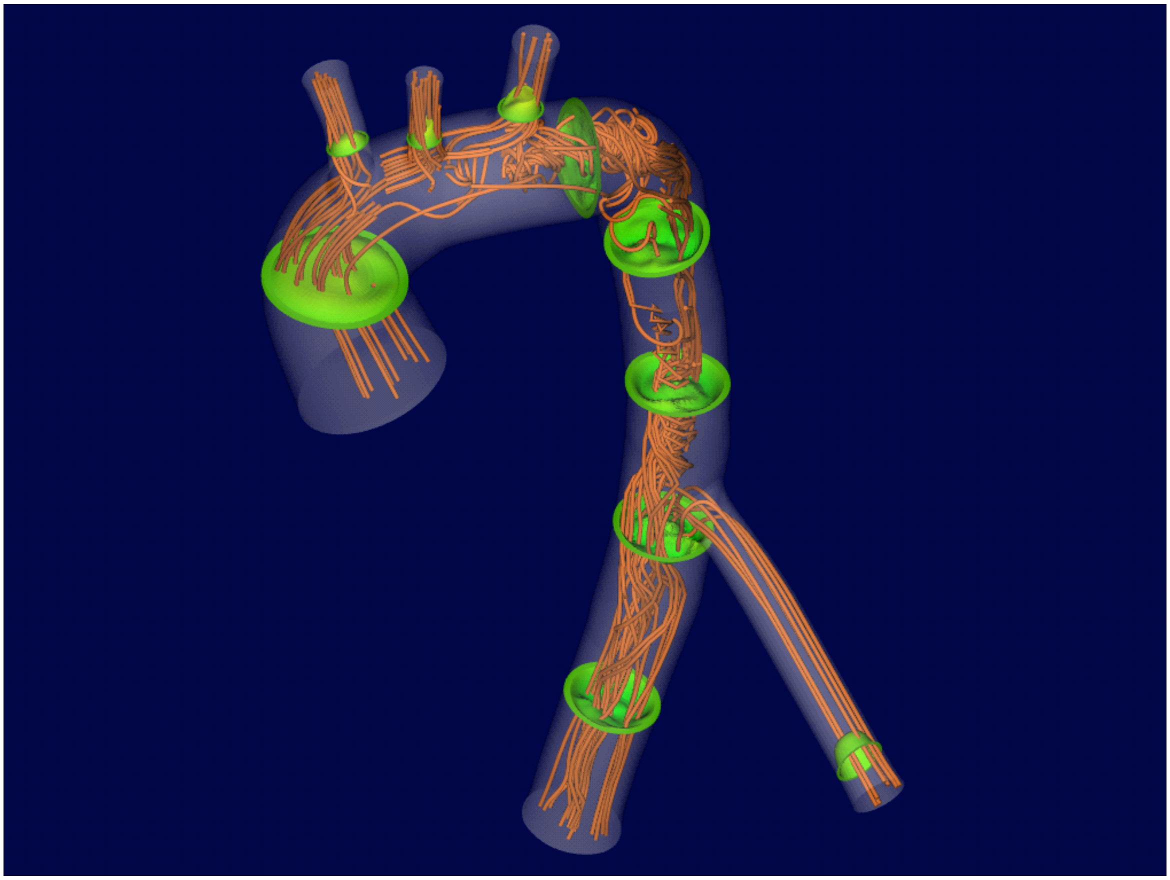
Post-operative complications	Group A: Ascending Aortic Graft	Group B: Descending Aortic Graft
Myocardial infarction:	0/9	4/26
Thrombus in aortic root:	0/9	3/26
Right ventricular failure due to infarction:	0/9	2/26
Accelerated carotid occlusion:	0/9	1/26
Multiple cerebral infarcts:	0/9	1/26

Ref. Texas Heart Institute

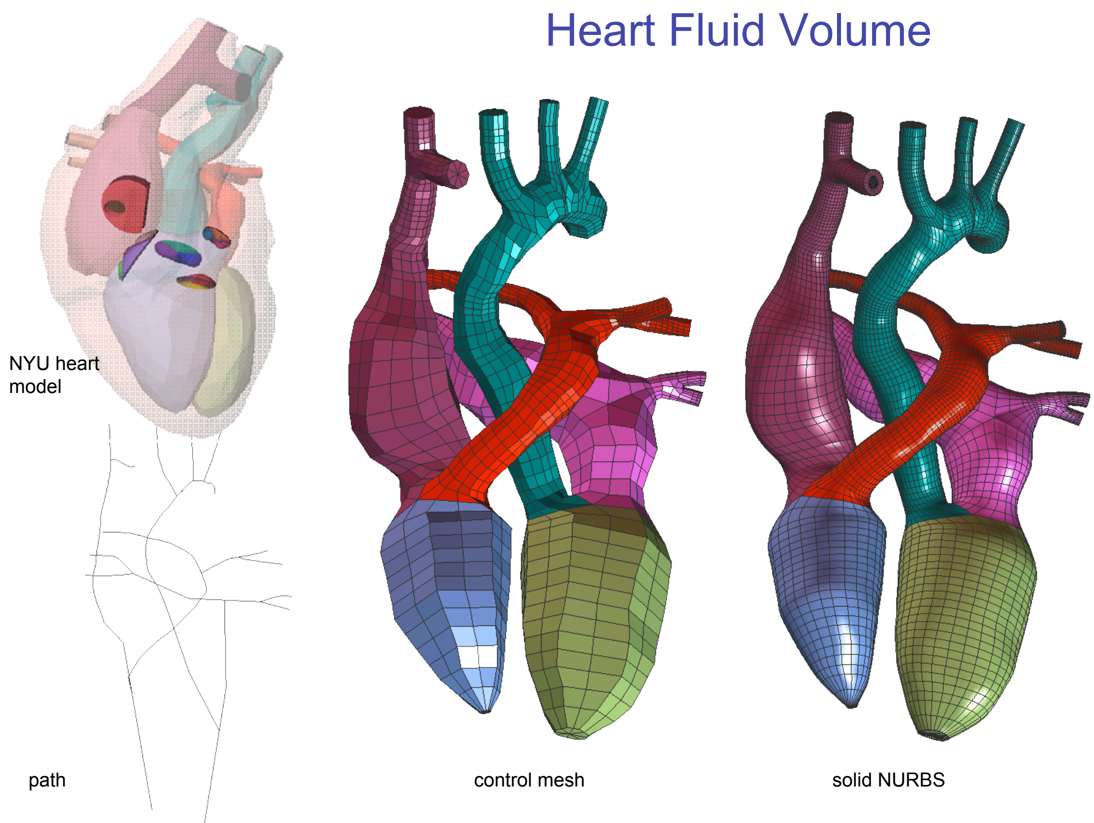
Thoracic Aorta

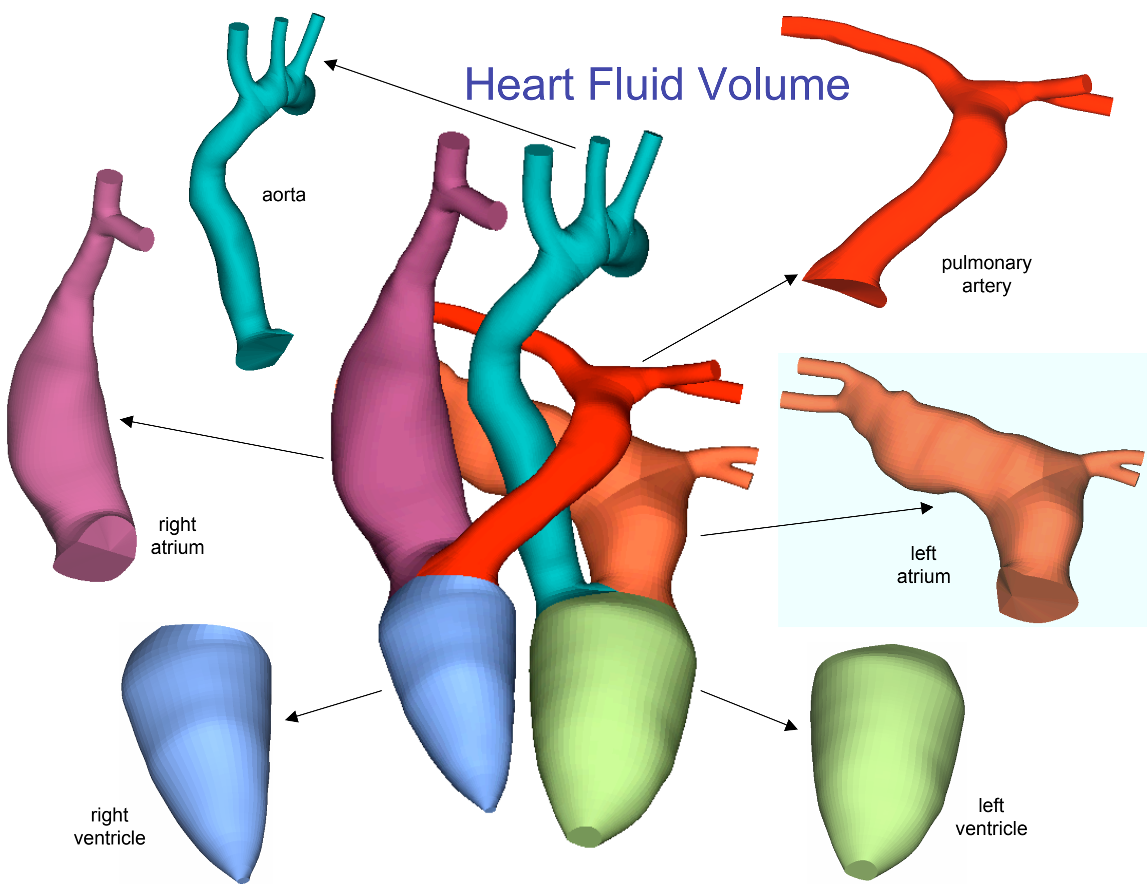


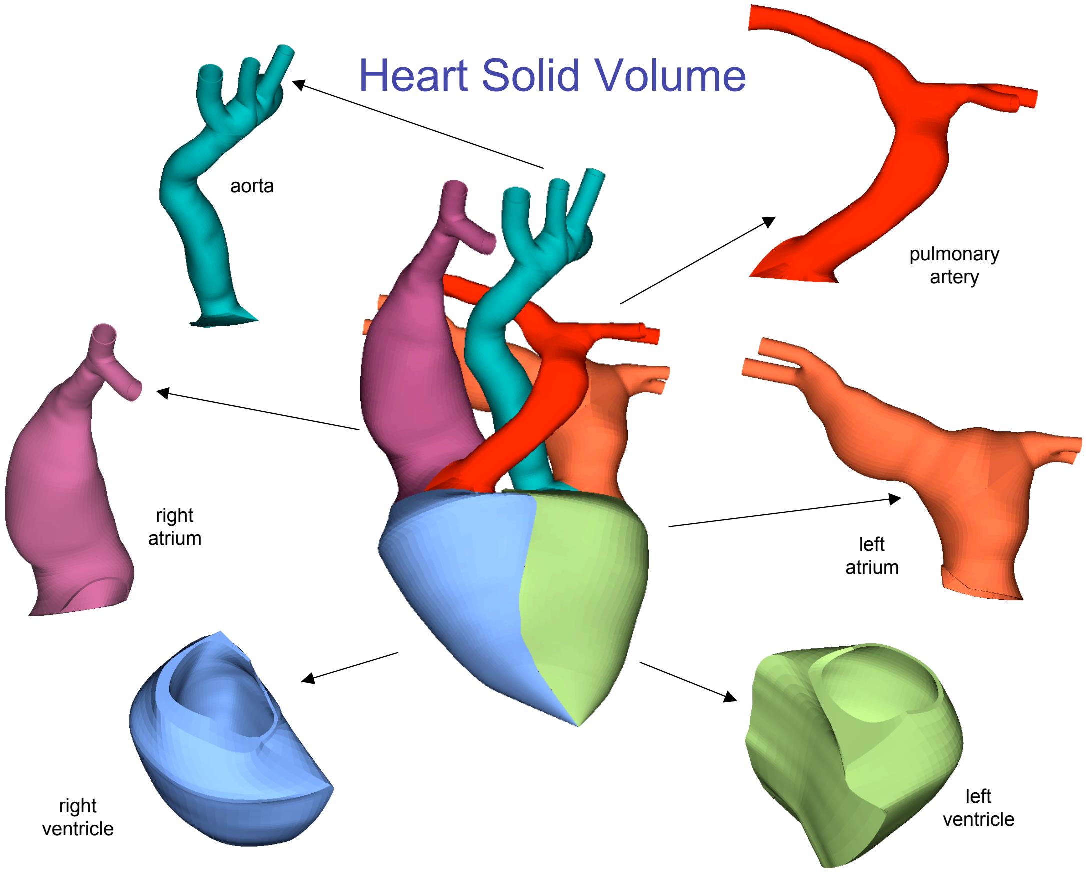
(a) Surface model and path (b) Control mesh (c) Solid NURBS (d) Simulation results



Heart Modeling Toolkit

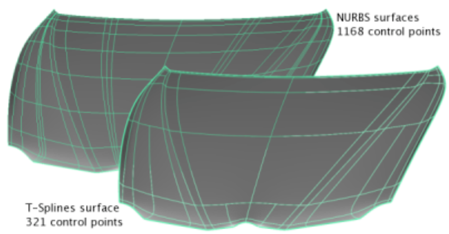

Heart Fluid Volume
NYU heart model
path
control mesh
solid NURBS

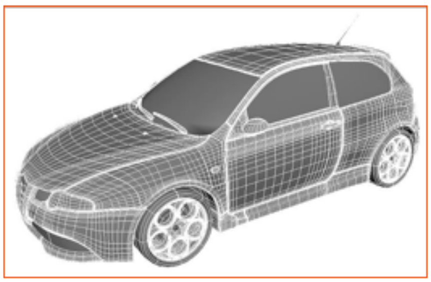

Heart Fluid Volume
aorta
pulmonary artery
right atrium
left atrium
right ventricle
left ventricle

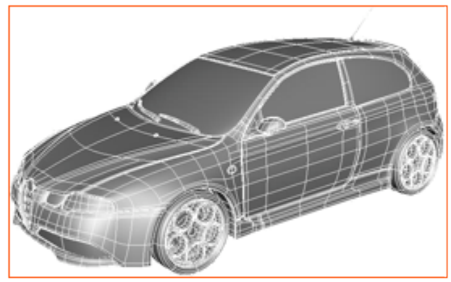


Unstructured NURBS Mesh (T. Sederberg, T-Splines)

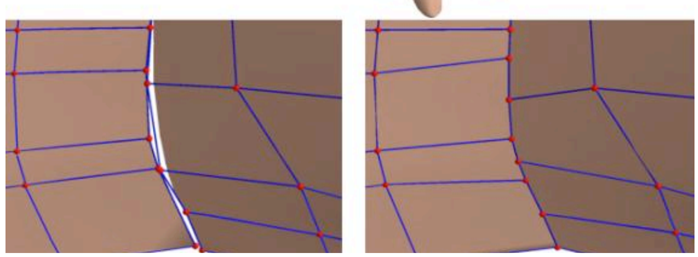
Reduced Number of Control Points







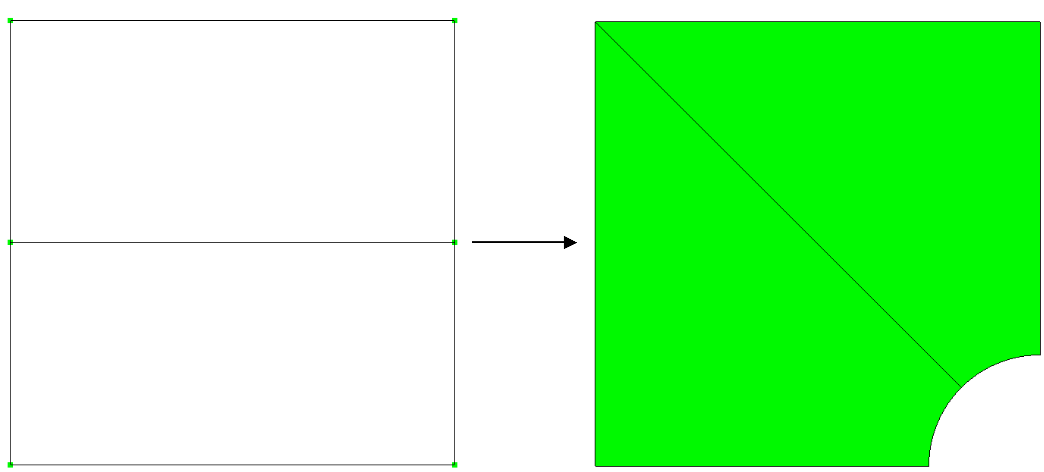
Water tight merging of patches



Elastic Plate with a Hole

- Infinite plate with circular hole under constant stress in *x*-direction
- Uniform and local *h*-refinement for *p* = 2 and 3

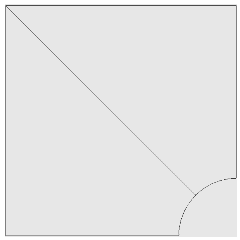
Elastic Plate with a Hole



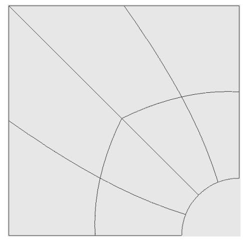
Mesh in parametric space

Mesh in real space

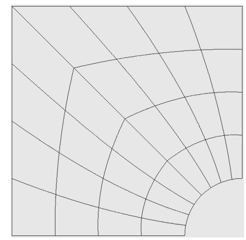
Uniformly Refined T-Meshes
(Standard NURBS)



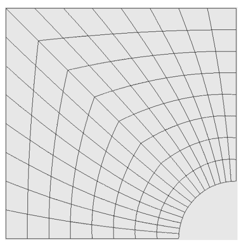
Mesh 1



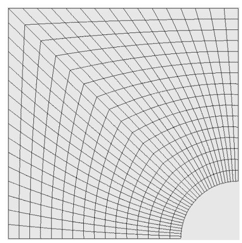
Mesh 2



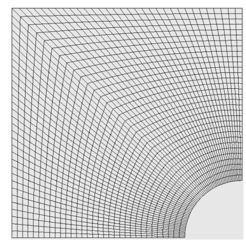
Mesh 3



Mesh 4

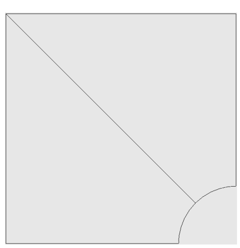


Mesh 5

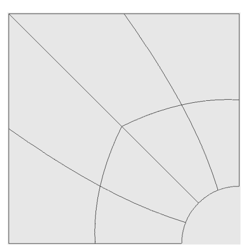


Mesh 6

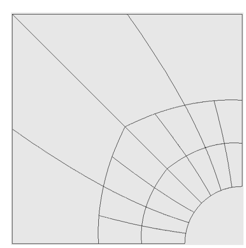
Locally Refined T-Meshes
(T-Splines)



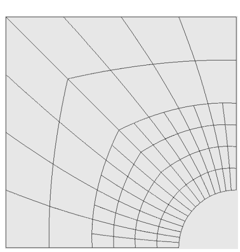
Mesh 1



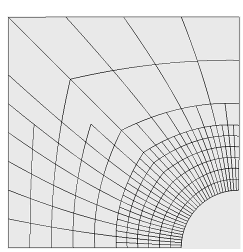
Mesh 2



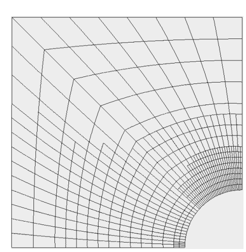
Mesh 3



Mesh 4

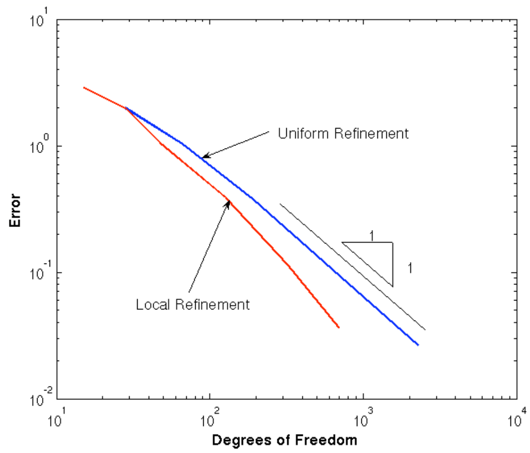


Mesh 5

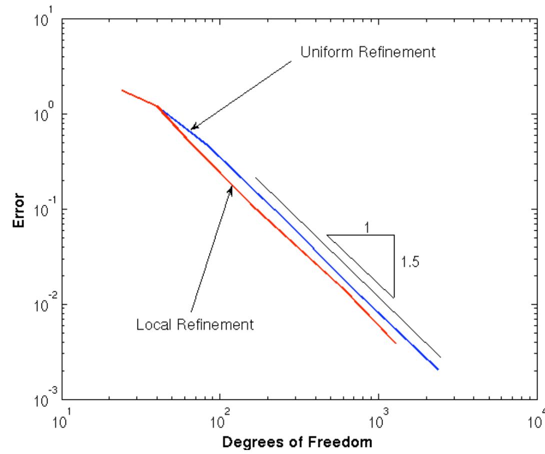


Mesh 6

L^2 Norm of the Error in the Stress (equivalent to H^1 semi-norm)

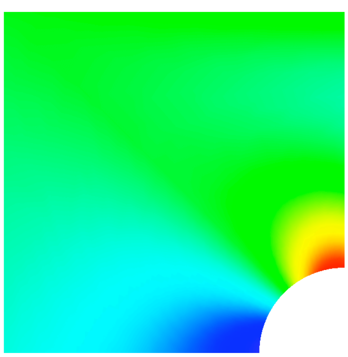


Quadratic T-Splines

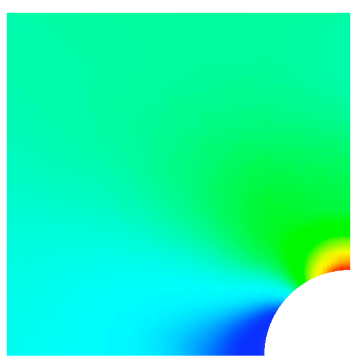


Cubic T-Splines

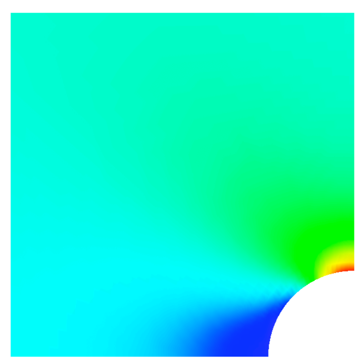
Contours of σ_{xx} for Locally Refined Meshes



Mesh 1

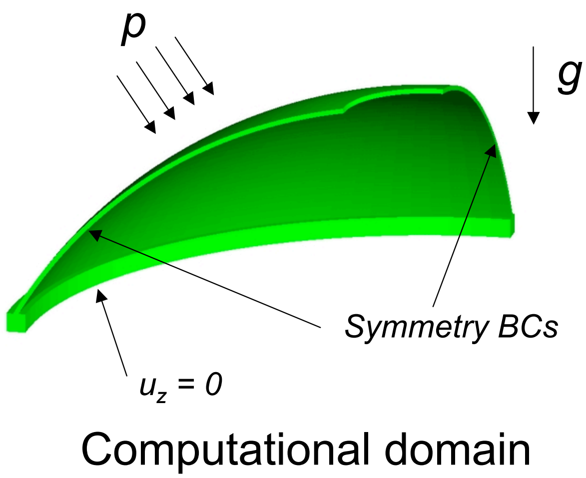


Mesh 3

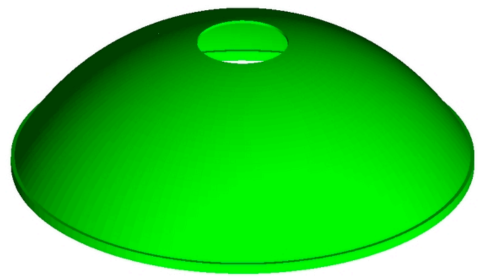


Mesh 5

Hemispherical Shell with Stiffener

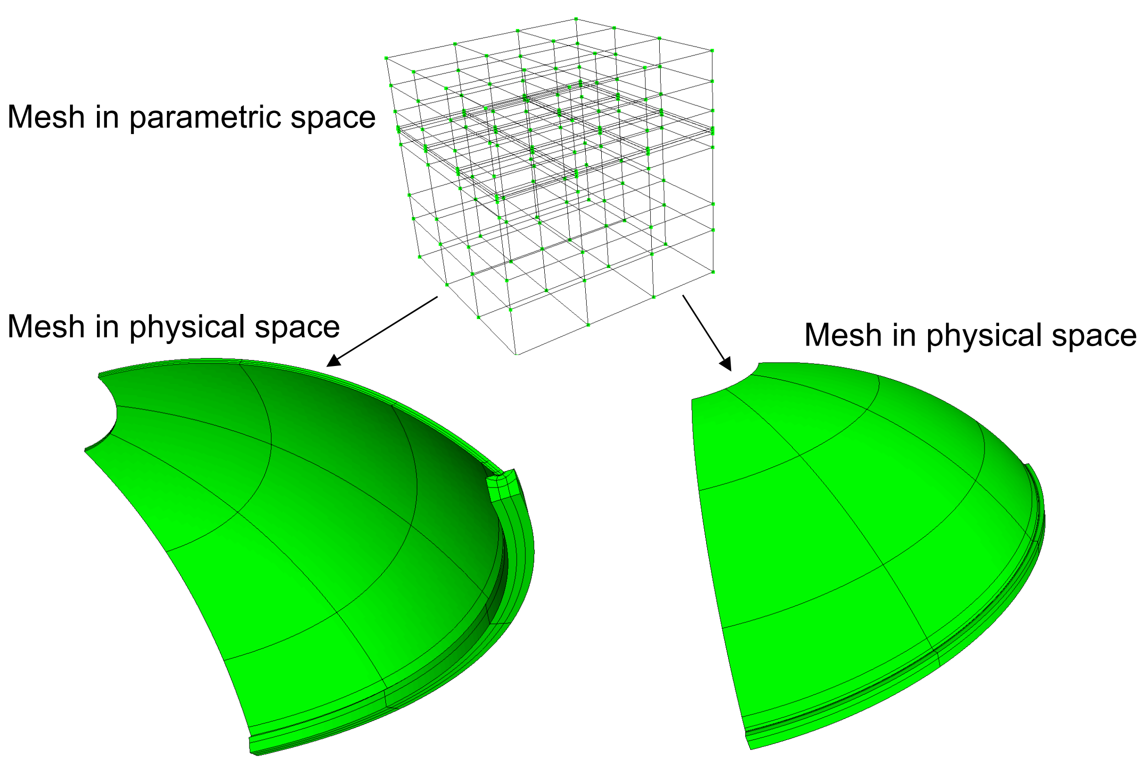


Loading:
- Gravity
- External pressure

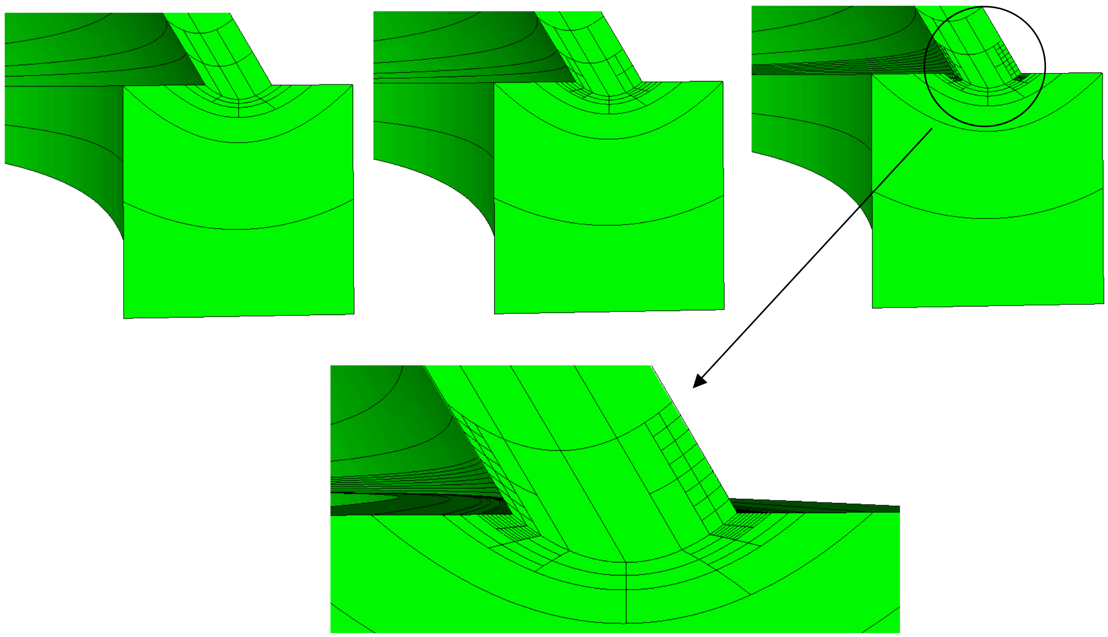


(from E. Rank *et al.*)

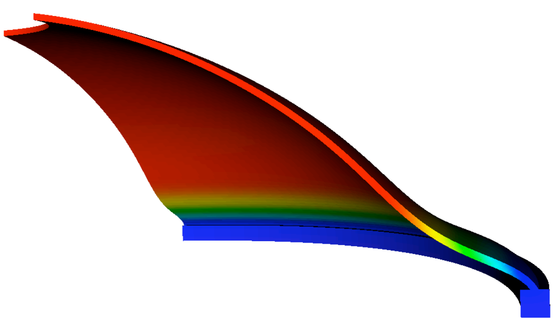
Hemispherical Shell with Stiffener

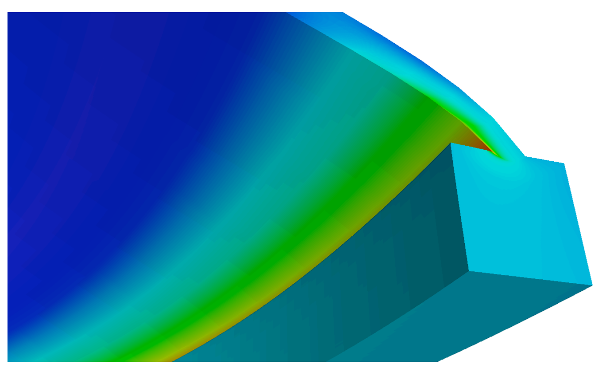


Locally Refined Meshes



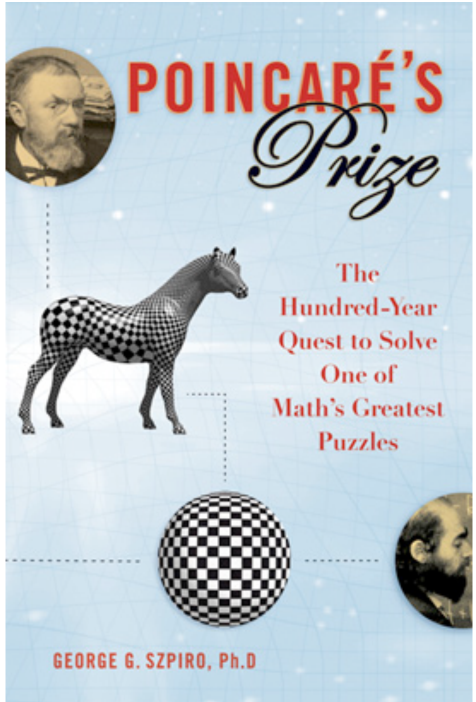
Hemispherical Shell with Stiffener





Vertical displacement (smooth)

Von Mises stress (singular)



Conclusions

- Isogeometric Analysis is a powerful generalization of FEA
  - Mesh refinement is vastly simplified
  - Numerical calculations are encouraging
  - Higher-order accuracy *and* robustness
  - It may play an important role in *unifying* design and analysis

Papers on Isogeometric Analysis

T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Computer Methods in Applied Mechanics and Engineering,* Vol. 194, Nos. 39-41, pp. 4135-4195 (2005).

J.A. Cottrell, A. Reali, Y. Bazilevs, T.J.R. Hughes. Isogeometric analysis of structural vibrations, *Computer Methods in Applied Mechanics and Engineering,* Vol. 195, Nos. 41-43, pp. 5257-5296 (2006).

Y. Bazilevs, L. Beirão da Veiga, J.A. Cottrell, T.J.R. Hughes, G. Sangalli. Isogeometric analysis: approximation, stability and error estimates for *h*-refined meshes, *Mathematical Models and Methods in Applied Science,* Vol. 16, No. 7, pp. 1031-1090, July 2006.

Bazilevs, Y; Calo, VM; Zhang, Y; Hughes, TJR. Isogeometric fluid-structure interaction analysis with applications to arterial blood flow. *Computational Mechanics,* 38 (4-5): 310-322 (2006).

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Geometry is the foundation of analysis

Computational geometry is the future of computational analysis