Kirchhoff Scattering Inversion:
I. 1-D inversion

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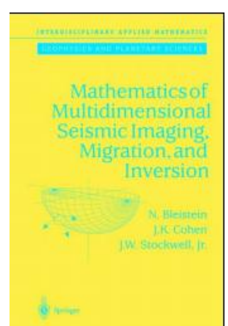
Introduction

- These lectures will introduce the theory of Kirchhoff migration and imaging from an inversion perspective
- They are intended to teach some geophysics to mathematicians and some mathematics to geophysicists
- Recommended reference:
  Bleistein, Cohen & Stockwell, 2001, "*Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion*"



Geophysical Ideas

- Velocity model, background velocity
- Acoustic properties – velocity, density, impedance
- Reflection coefficients
- Impulsive source
- Time trace, sections, gathers, stacks, offset
- Kirchhoff migration, diffraction stacks
- Raytracing, Kirchhoff approximate data
- Migration weights, Beylkin determinant

Mathematical Ideas

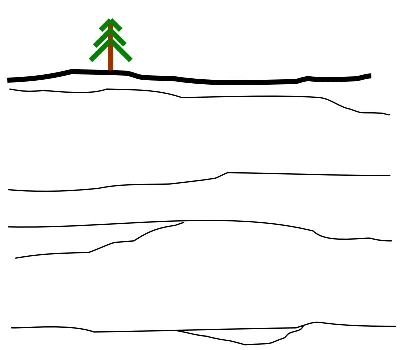
- Wave equation, Helmholtz equation, radiation conditions
- Green's function for homogeneous media
- Green's theorem (a.k.a. Green's 2^{nd} identity)
- Integral solution to Helmholtz equation
- Lippman-Schwinger equation, Born approx'n
- WKBJ approximation (eikonal & transport eqns)
- Stationary phase approximation
- Fourier transform – asymptotics

Overview

- Fundamental concepts ←
- Forward Scattering in 1-D
- Inverse Scattering in 1-D

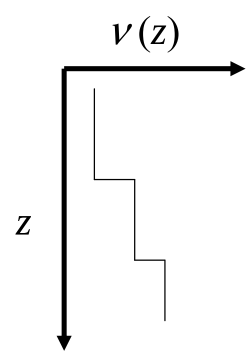
Target image: 1-D

Geophysics



- Earth model
- Stratigraphic layers
- Geological features

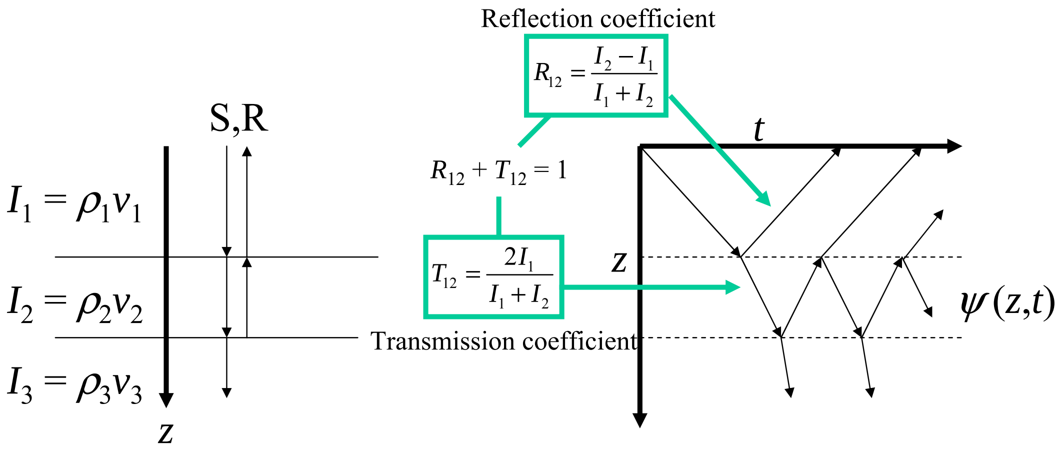
Mathematics



- Velocity profile
- Constant velocities
- Boundary discontinuities

Traveling Disturbance: 1-D

Geophysics | Mathematics

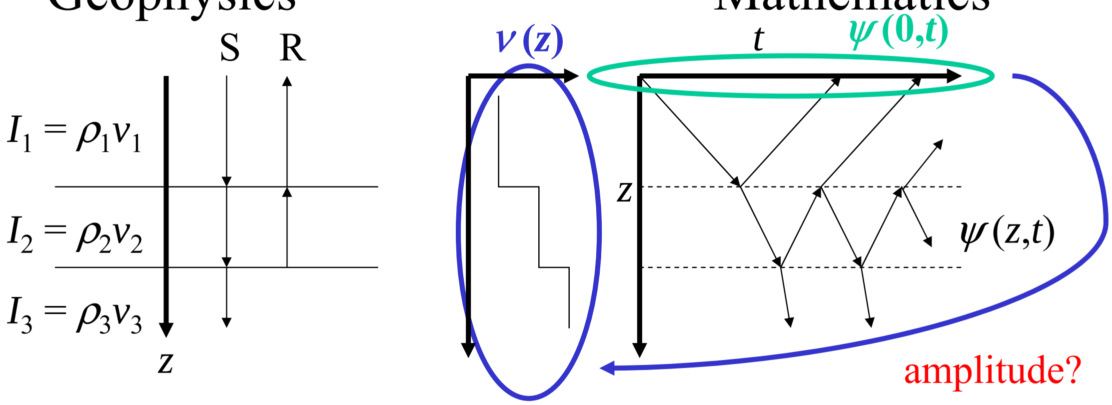


- Dynamite or Vibroseis Source (S)
- Earth filter
- Noise
- Geophone detection (R)

- Impulse source
- Frequency band
- Inherent uncertainty
- Sampled only on boundary

Traveling Disturbance: 1-D

Geophysics | Mathematics

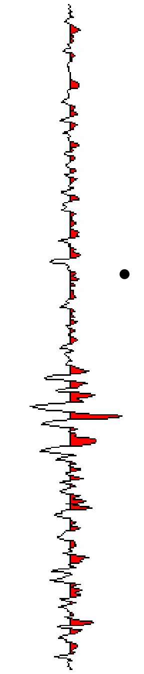


- Dynamite or Vibroseis Source (S)
- Earth filter
- Noise
- Geophone detection (R)

- Impulse source
- Frequency band
- Inherent uncertainty
- Sampled only on boundary

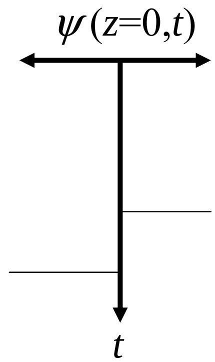
Observed Data: 1-D

Geophysics



- Time trace obtained from receivers (geophones)

Mathematics



- Wave field sampled at boundary
- Provides boundary condition for inversion

Making waves: 1-D

Wave Equation (PDE)

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{v(z)^2}\frac{\partial^2}{\partial t^2}\right]\Psi(z,t) = F(z,t) \qquad \left[\frac{\partial^2}{\partial z^2} - \frac{1}{v(z)^2}\frac{\partial^2}{\partial t^2}\right]G(z,t) = \delta(z)\delta(t)$$

Helmholtz Equation (ODE)

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v(z)^2}\right]\psi(z,\omega) = f(z,\omega) \qquad \left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v(z)^2}\right]g(z,\omega) = \delta(z)$$

Radiation Condition (BC)

$$\left[\frac{\partial}{\partial z} \mp \frac{i\omega}{v(z)}\right]\psi(z,\omega) \to 0 \quad \text{as} \quad z \to \pm\infty$$

Making waves: 3-D

Wave Equation (PDE)

$$\left[\nabla^2 - \frac{1}{v(\mathbf{r})^2}\frac{\partial^2}{\partial t^2}\right] G(\mathbf{r}-\mathbf{r}_0,t) = \delta(\mathbf{r}-\mathbf{r}_0)\delta(t)$$

Helmholtz Equation (lower order PDE)

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] g(\mathbf{r}-\mathbf{r}_0,\omega) = \delta(\mathbf{r}-\mathbf{r}_0)$$

Radiation Condition (BC)

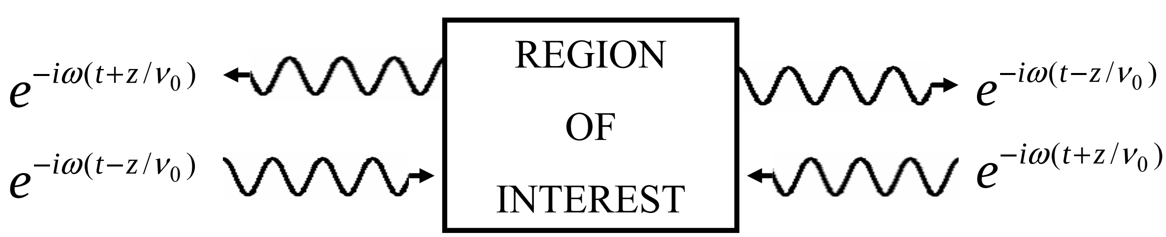
$$r\left[\frac{\partial}{\partial r} - \frac{i\omega}{v(\mathbf{r})}\right]\psi(\mathbf{r},\omega) \to 0 \quad \text{as} \quad r \to \infty$$

Radiation Condition

Boundary condition at infinity

- specifies that there are no sources at infinity

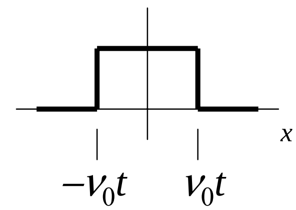
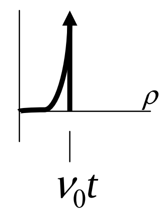
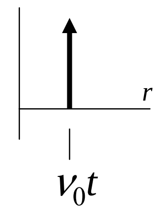
$$\left[\frac{\partial}{\partial z} \mp \frac{i\omega}{v(z)}\right]\psi(z,\omega) \to 0 \quad \text{as} \quad z \to \pm\infty$$



$$\frac{\partial}{\partial z} e^{-i\omega(t\mp z/v_0)} = \pm\frac{i\omega}{v_0} e^{-i\omega(t\mp z/v_0)}$$

…and similarly for 3-D

Green's functions for constant $v(x)$

	Helmholtz equation	Wave equation	
1-D:	$g(x,\omega) = -\frac{1}{2}\frac{\exp(i\omega\lvert x\rvert/v_0)}{i\omega/v_0}$	$g(x,t) = \frac{v_0}{2}H(v_0 t - \lvert x\rvert)$	
2-D:	$g(\boldsymbol{\rho},\omega) = \frac{i}{4}H_0^{(1)}(\omega\rho/v_0)$	$g(\boldsymbol{\rho},t) = \frac{1}{2\pi}\frac{v_0 H(v_0 t - \rho)}{\sqrt{v_0^2 t^2 - \rho^2}}$	
3-D:	$g(\mathbf{r},\omega) = \frac{\exp(i\omega r/v_0)}{4\pi r}$	$g(\mathbf{r},t) = \frac{\delta(t - r/v_0)}{4\pi r}$	

Exercise

Verify that, for x or $r > 0$, the 1-D and 3-D homogeneous ($v(\mathbf{r})$ is constant) Helmholtz equations are satisfied by the expressions on the previous slide. For the 3-D case use the spherically symmetric Helmholtz equation:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rg(r,\omega)] + \frac{\omega^2}{v_0^2}g(r,\omega) = \delta(\mathbf{r})$$

Solution

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[\frac{\exp(i\omega r/v_0)}{4\pi}\right]+\frac{\omega^2}{v_0^{\,2}}\frac{\exp(i\omega r/v_0)}{4\pi r}=0, \qquad r>0$$

$$=\frac{1}{r}\left[-\frac{\omega^2}{v_0^{\,2}}\frac{\exp(i\omega r/v_0)}{4\pi}\right]+\frac{\omega^2}{v_0^{\,2}}\frac{\exp(i\omega r/v_0)}{4\pi r}$$

$$=0$$

Exercise

Apply an inverse Fourier transform to the 3-D homogeneous Green's function for the Helmholtz equation to obtain the corresponding Green's function for the wave equation.

Solution

The inverse Fourier transform is carried out by the following integral:

$$g(r,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(r,\omega)\exp(-i\omega t)\,d\omega \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{4\pi r}\exp\left(\frac{i\omega r}{v_0}\right)\exp(-i\omega t)\,d\omega$$

$$=\frac{1}{2\pi}\frac{1}{4\pi r}\int_{-\infty}^{\infty}\exp\left[i\omega\left(\frac{r}{v_0}-t\right)\right]d\omega$$

We then employ a common definition of the delta function,

$$\delta(x-x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[ik\left(x-x_0\right)\right]dk,$$

to obtain the final result:

$$g(R,t)=\frac{1}{4\pi r}\delta\left(\frac{r}{v_0}-t\right)=\frac{1}{4\pi r}\delta\left(t-\frac{r}{v_0}\right)=\frac{v_0\delta\left(v_0 t-r\right)}{4\pi r}$$

Exercise

Apply a Fourier transform to the 1-D homogeneous Green's function for the wave equation to obtain the corresponding Green's function for the Helmholtz equation.

Solution

To carry out the Fourier transform, the upper limit cannot be evaluated as is. It is necessary to add a small, positive imaginary component to the frequency, which can then be set to zero at the end. This procedure is possible only because $g(|x|,t)$ is causal, i.e., $t > 0$.

$$g(|x|,\omega) = \int_0^\infty \frac{v_0}{2} H\left(v_0 t - |x|\right) \exp(i\omega t)\, dt \qquad = \frac{v_0}{2} \int_{|x|/v_0}^\infty \exp\left(i\omega t\right) dt$$

$$= \frac{v_0}{2} \frac{\lim_{t\to\infty} \exp\left(i\omega t\right) - \exp\left(i\omega |x| / v_0\right)}{i\omega}$$

$$= \frac{v_0}{2} \frac{\lim_{t\to\infty} \exp\left[i\left(\omega_R + i|\omega_I|\right)t\right] - \exp\left(i\omega |x| / v_0\right)}{i\omega}$$

$$= \frac{v_0}{2} \frac{\lim_{t\to\infty} \exp\left(-|\omega_I| t\right) \exp\left(i\omega_R t\right) - \exp\left(i\omega |x| / v_0\right)}{i\omega}$$

$$= -\frac{v_0}{2} \frac{\exp\left(i\omega |x| / v_0\right)}{i\omega}$$

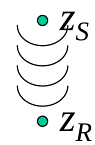
Overview

- Fundamental concepts
- Forward Scattering in 1-D ←
- Inverse Scattering in 1-D

Constructing a forward modeling formula

Helmholtz equations:

$$\left[\frac{d^2}{dz^2}+\frac{\omega^2}{v(z)^2}\right]g(z,\omega;z_R)=-\delta(z-z_R) \qquad \left[\frac{d^2}{dz^2}+\frac{\omega^2}{v(z)^2}\right]\psi(z,\omega;z_S)=-f(z,\omega;z_S)$$



$$\frac{d^2 g}{dz^2}=-\delta(z-z_R)-\frac{\omega^2}{v^2}g \qquad \frac{d^2\psi}{dz^2}=-f(z_S)-\frac{\omega^2}{v^2}\psi$$

Do substitutions on this side

$$\int_{-\infty}^{\infty}\left\{\psi\frac{d^2 g}{dz^2}-g\frac{d^2\psi}{dz^2}\right\}dz=\int_{-\infty}^{\infty}\left\{\psi\frac{d^2 g}{dz^2}-g\frac{d^2\psi}{dz^2}\right\}dz$$

Do integration by parts on this side

<u>Exercise</u>: derive this integral solution to the Helmholtz equation

$$\psi(z_R,\omega;z_S)=\int_{-\infty}^{\infty}g(z,z_R,\omega)f(z,\omega;z_S)dz$$

Solution

Do substitutions on this side

$$\int_{-\infty}^{\infty}\left\{\psi\frac{d^2 g}{dz^2}-g\frac{d^2\psi}{dz^2}\right\}dz=\int_{-\infty}^{\infty}\left\{\psi\frac{d^2 g}{dz^2}-g\frac{d^2\psi}{dz^2}\right\}dz$$

Do integration by parts on this side

$$\int_{-\infty}^{\infty}\left\{\psi\left[-\delta(z-z_R)-\frac{\omega^2}{v^2}g\right]-g\left[-f(z_S)-\frac{\omega^2}{v^2}\psi\right]\right\}dz=\left[\psi\frac{dg}{dz}-g\frac{d\psi}{dz}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left\{\frac{d\psi}{dz}\frac{dg}{dz}-\frac{dg}{dz}\frac{d\psi}{dz}\right\}dz$$

$$\int_{-\infty}^{\infty}\left\{-\psi\delta(z-z_R)+gf(z_S)-\frac{\omega^2}{v^2}g\psi+\frac{\omega^2}{v^2}g\psi\right\}dz=\left[\psi\frac{dg}{dz}-g\frac{d\psi}{dz}\right]_{-\infty}^{\infty}$$

$$-\int_{-\infty}^{\infty}\psi(z,\omega;z_S)\delta(z-z_R)dz+\int_{-\infty}^{\infty}g(z,\omega;z_R)f(z,\omega;z_S)dz=\left[\psi\frac{dg}{dz}-g\frac{d\psi}{dz}\right]_{-\infty}^{\infty}$$

Will vanish at each limit because of radiation condition

$$\psi(z_R,\omega;z_S)=\int_{-\infty}^{\infty}g(z,z_R,\omega)f(z,\omega;z_S)dz$$

$g(z_R,z,\omega)$ contains information on b.c.'s & $v(z)$

Forward Scattering

Let the total wavefield be defined as the sum of incident and scattered waves:

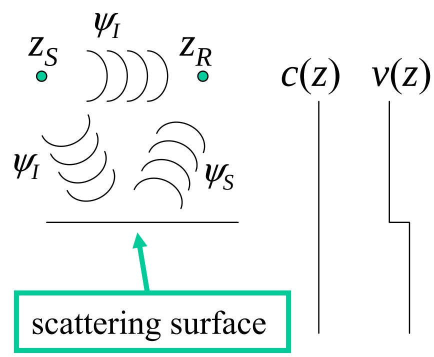
$$\psi_T = \psi_I + \psi_S$$

Let $v(z)$ be the actual velocity model, and define a background velocity model $c(z)$ so that ψ_I satisfies

$$\frac{d^2\psi_I}{dz^2} = -\delta(z - z_S) - \frac{\omega^2}{c^2}\psi_I$$

while ψ_T satisfies

$$\frac{d^2(\psi_I + \psi_S)}{dz^2} = -\delta(z - z_S) - \frac{\omega^2}{c^2}[1 + \alpha(z)](\psi_I + \psi_S)$$



$$\frac{1}{v(z)^2} = \frac{1}{c(z)^2}[1 + \alpha(z)]$$

Exercise: Derive the forward scattering integral for ψ_S

$$\psi_S(z_R, z_S, \omega) = \omega^2 \int_{-\infty}^{\infty} \frac{\alpha(z)}{c(z)^2} g(z, z_R, \omega)[\psi_I(z, z_S, \omega) + \psi_S(z, z_S, \omega)]dz$$

Solution

The Green's function for $c(z)$ satisfies this Helmholtz relation:

$$\frac{d^2 g}{dz^2} = -\delta(z - z_R) - \frac{\omega^2}{c^2} g$$

Subtract ψ_T and ψ_I relations:

$$\frac{d^2(\psi_I + \psi_S)}{dz^2} = -\delta(z - z_S) - \frac{\omega^2}{c^2}[1 + \alpha(z)](\psi_I + \psi_S)$$

$$-\left[\frac{d^2\psi_I}{dz^2} = -\delta(z - z_S) - \frac{\omega^2}{c^2}\psi_I\right]$$

$$\frac{d^2\psi_S}{dz^2} = -\frac{\omega^2}{c^2}\alpha(z)\psi_I - \frac{\omega^2}{c^2}[1 + \alpha(z)]\psi_S$$

effective source

1-D Green's theorem:

$$\int_{-\infty}^{\infty}\left\{\psi_S \frac{d^2 g}{dz^2} - g\frac{d^2\psi_S}{dz^2}\right\}dz = \left[\psi_S \frac{dg}{dz} - g\frac{d\psi_S}{dz}\right]_{-\infty}^{\infty}$$

$$\int_{-\infty}^{\infty}\left\{-\psi_S \delta(z - z_R) + g\frac{\omega^2}{c^2}\alpha(z)[\psi_I + \psi_S]\right\}dz = 0$$

radiation condition

$$\psi_S(z_R, z_S, \omega) = \omega^2 \int_{-\infty}^{\infty} \frac{\alpha(z)}{c(z)^2} g(z, z_R, \omega)[\psi_I(z, z_S, \omega) + \psi_S(z, z_S, \omega)]dz$$

Forward scattering integral

Born Approximation = Linearization

Lippman-Schwinger equation

$$\psi_S(z_R,z_S,\omega)=\omega^2\int_{-\infty}^{\infty}\frac{1}{c(z)^2}\alpha(z)g(z,z_R,\omega)\left[\psi_I(z,z_S,\omega)+\psi_S(z,z_S,\omega)\right]dz$$

Both related to scattering potential: product is "quadratic" in perturbation

→ *weak-scattering approximation*

Linearization

$$\psi_S(z_R,z_S,\omega)=\omega^2\int_{-\infty}^{\infty}\frac{\alpha(z)}{c(z)^2}g(z_R,z,\omega)\psi_I(z,z_S,\omega)dz$$

Note: this may also be considered a *single-scattering approximation*

Overview

- Fundamental concepts
- Forward Scattering in 1-D
- Inverse Scattering in 1-D ←

From forward to inverse scattering

$$\psi_S(z_R, z_S, \omega) = \int_{-\infty}^{\infty} \alpha(z) \frac{\omega^2}{c(z)^2} g(z_R, z, \omega) \psi_I(z, z_S, \omega) dz$$

function of ω — function of z — function of ω and z

Acts as a transform between space and time:

$$\psi_S(\omega) = \int_{-\infty}^{\infty} \alpha(z) T(\omega, z)\, dz$$

If inverse transform exists, then $\alpha(z) = \int_{-\infty}^{\infty} \psi_S(\omega) T^{-1}(\omega, z)\, d\omega$

- obtain velocity profile from wavefield on boundary ($z = z_R$)

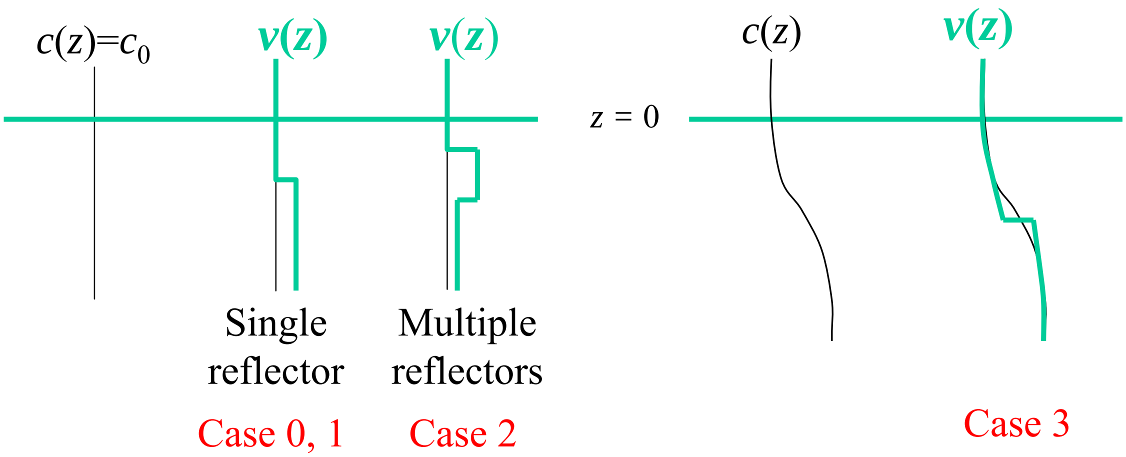
1-D cases to consider

- Case 0:
  - $f(\omega) = 1$ (infinite bandwidth)
  - single reflector
  - $c(z)$ = constant
- Case 1: band-limited
- Case 2: multiple reflectors
- Case 3: $c(z)$ non-constant

Inverse Scattering Cases

Constant background velocity

Non-constant background



Constant $c(z)$ Cases 0, 1, 2

$$\psi_S(z_R, z_S, \omega) = \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} \alpha(z) g(z, z_R, \omega) \psi_I(z, z_S, \omega) dz$$

↓ impulsive source

$$\psi_S(z_R, z_S, \omega) = \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} \alpha(z) g(z, z_R, \omega) g(z, z_S, \omega) dz$$

$$g(z, z_0, \omega) = -\frac{1}{2} \frac{\exp(i\omega |z - z_0| / c_0)}{i\omega / c_0}$$

Let $\alpha(z) = 0$ for $z < z_R, z_S$. Then T becomes the Fourier Transform.

$$\psi_S(z_R, z_S, \omega) = -\frac{\exp[-i\omega(z_R + z_S)/c_0]}{4} \int_0^{\infty} \alpha(z) \exp[i(2\omega/c_0)z] dz$$

"k"

"t"

Exercise: From the modeling expression above obtain the inversion formula below:

$$\alpha(z) = -4 \exp[i\omega(z_R + z_S)/c_0] \frac{1}{2\pi} \frac{2}{c_0} \int_{-\infty}^{\infty} \psi_S(z_R, z_S, \omega) \exp[-i\omega(2z/c_0)] d\omega$$

Solution

$$\psi_S(z_R, z_S, \omega) = -\frac{\exp[-i\omega(z_R + z_S)/c_0]}{4}\int_0^\infty \alpha(z)\exp[i(2\omega/c_0)z]dz$$

Apply $-4\exp[i\omega(z_R + z_S)/c_0](1/2\pi)\int_{-\infty}^{\infty} d(2\omega/c_0)\exp[-i(2\omega/c_0)z]$ to both sides

$$\alpha(z) = -4\exp[i\omega(z_R + z_S)/c_0]\frac{1}{2\pi}\frac{2}{c_0}\int_{-\infty}^{\infty}\psi_S(z_R, z_S, \omega)\exp[-i\omega(2z/c_0)]d\omega$$

Case 0

Exact solutions for $\psi(z)$ exist for piecewise constant $v(z)$.

- In each segment it is a homogeneous Green's function for the given velocity
- It is piecewise continuous at interfaces
- Radiation conditions are satisfied at $\pm\infty$

Let z_R, $z_S = 0$ and $v(z) = c_0 + (c_1 - c_0)H(z-h)$

Exercise: Derive the relation $\alpha(z) = -4RH(z-h) + O(R^2)$. Recall $R = (c_1 - c_0)/(c_1 + c_0)$

Then $\psi_S(z = 0, \omega) = -\frac{c_0 R\exp[i\omega(2h/c_0)]}{2i\omega}$

cf.

$$\alpha^{\text{inv}}(z) = \frac{2R}{\pi}\int_{-\infty}^{\infty}\frac{\exp[i\omega(2[h-z]/c_0)]}{i\omega}d\omega$$
$$= -4RH(z-h)$$

Effect of Linearization

Solution

$$v(z) = c_0 + (c_1 - c_0)H(z-h)$$

$$\frac{1}{v(z)^2} = [1+\alpha(z)]\frac{1}{c_0^2}$$

$$\alpha(z>h) = \frac{c_0^2}{c_1^2} - 1$$

Now rearrange the definition of R to solve for c_1:

$$R = \frac{c_1 - c_0}{c_1 + c_0} \quad \rightarrow \quad c_1 = \frac{1+R}{1-R}c_0$$

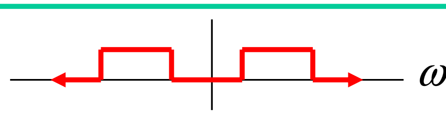
Substitute this above to obtain

$$\alpha(z>h) = \left(\frac{1-R}{1+R}\right)^2 - 1 = \frac{-4R}{(1+R)^2} = -4R + O\left(R^2\right)$$

Case 1: Band Limiting

Replace $\psi_S(z=0,\omega)$ by $f(\omega)\psi_S(z=0,\omega)$

$f(\omega)$:



Can still solve numerically

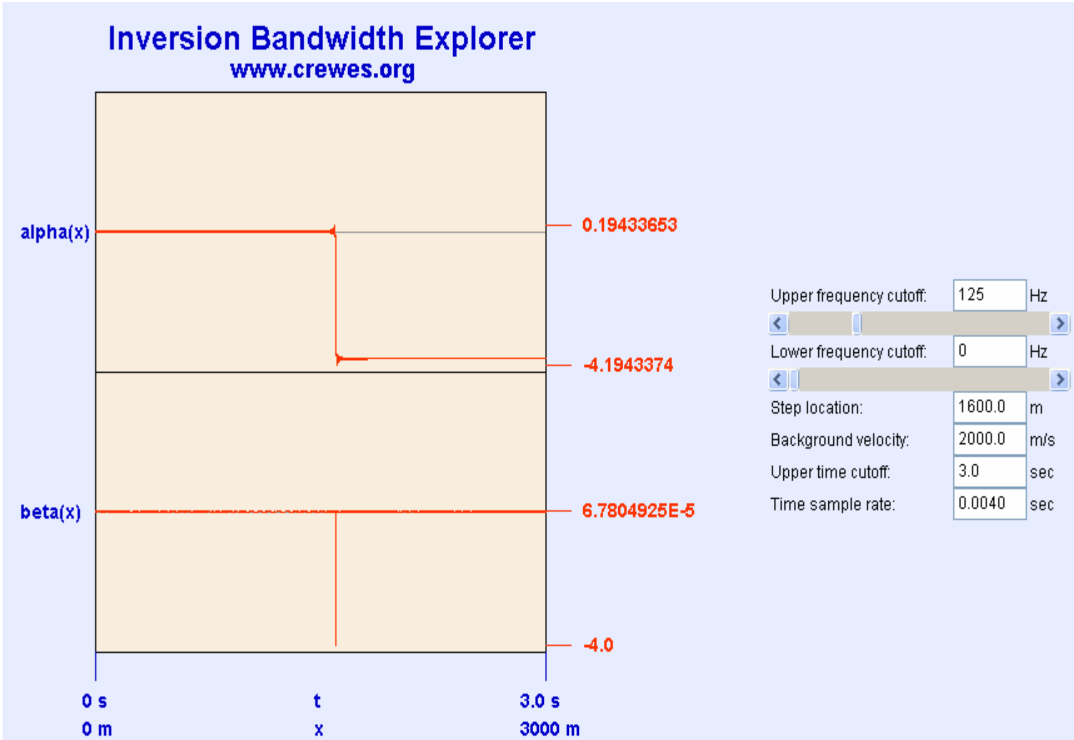
Use interactive software (www.crewes.org/~ursenbac) to explore the questions below:

- What is the effect of changing the upper band limit?
- What is the effect of changing the lower band limit?
- What is the effect of applying a derivative?

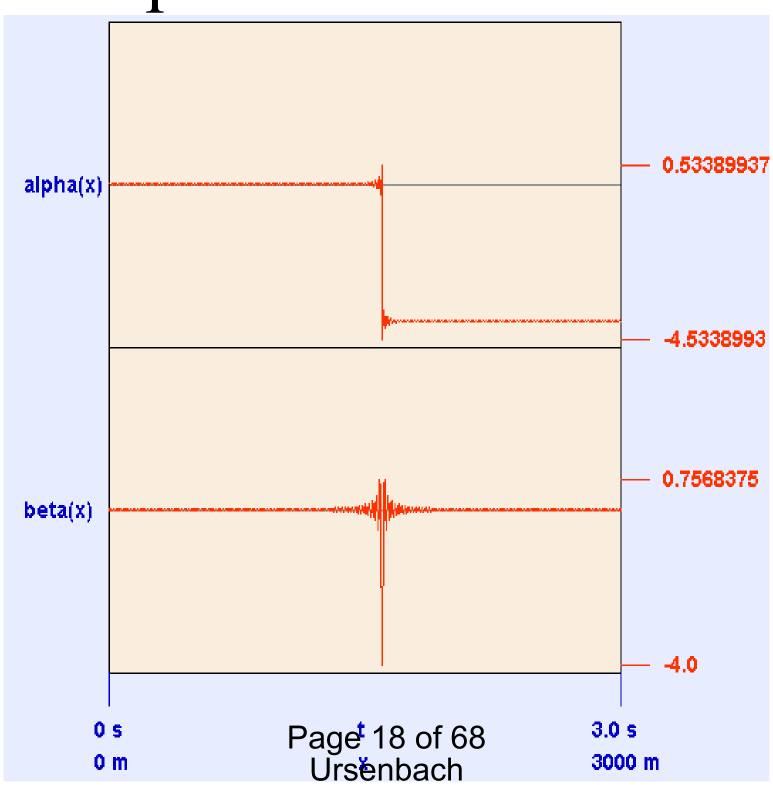
It is helpful to define the quantity $\beta(z) = \frac{\partial}{\partial z}\alpha(z)$

This can also be obtained by inserting $2i\omega/c_0$ into the ω-integral

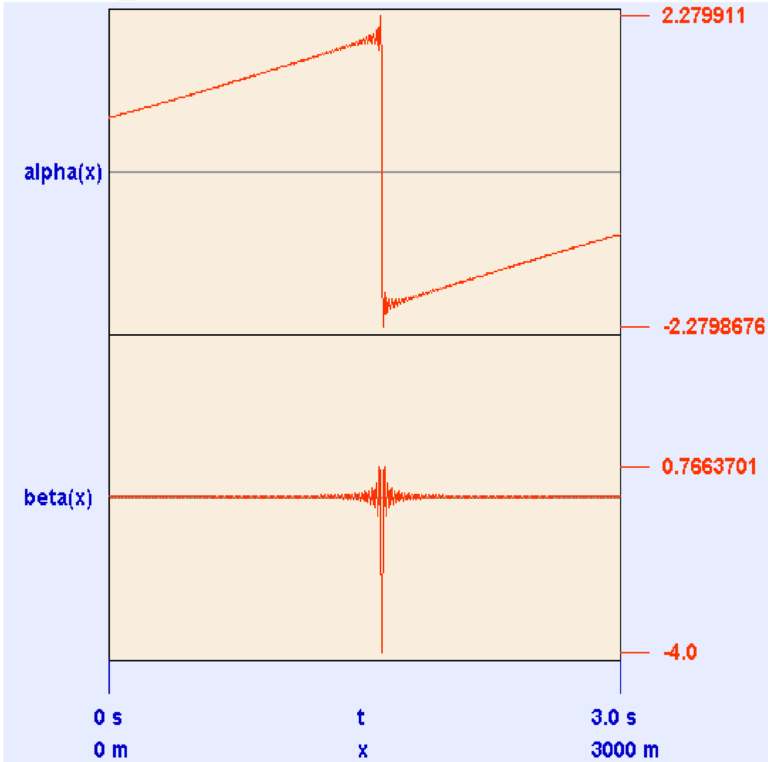
Spectrum: 0-125 Hz



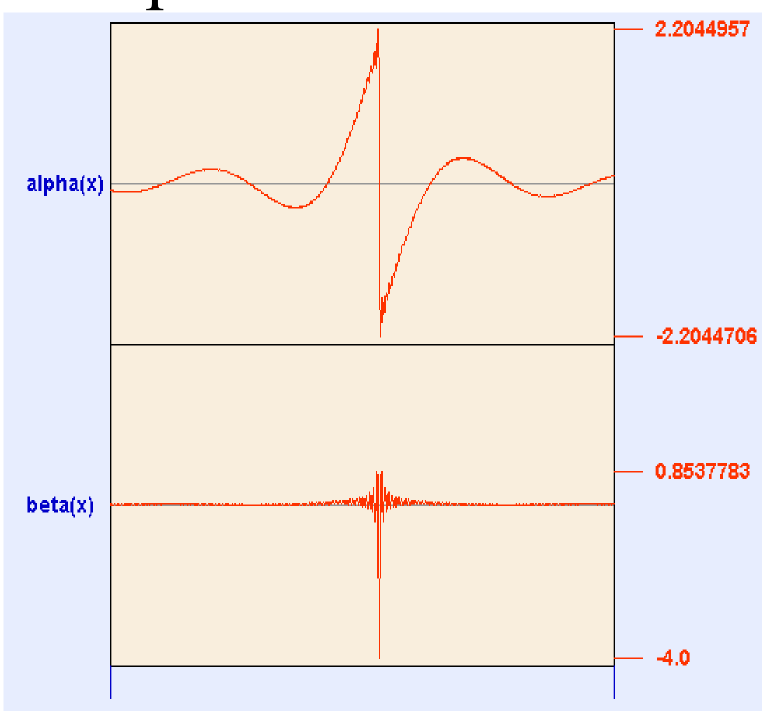
Spectrum: 0-50 Hz



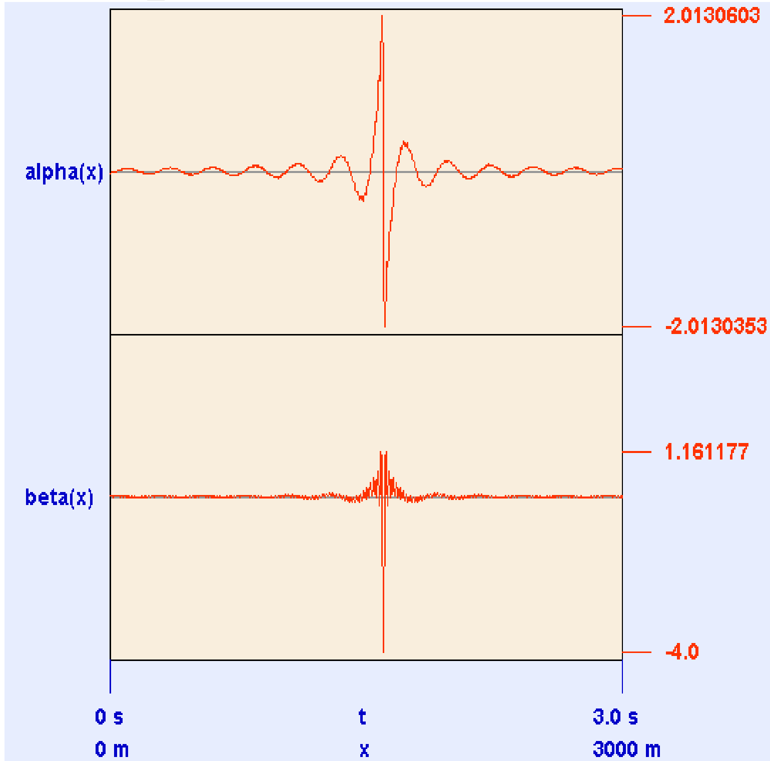
Spectrum: 0.1-50 Hz



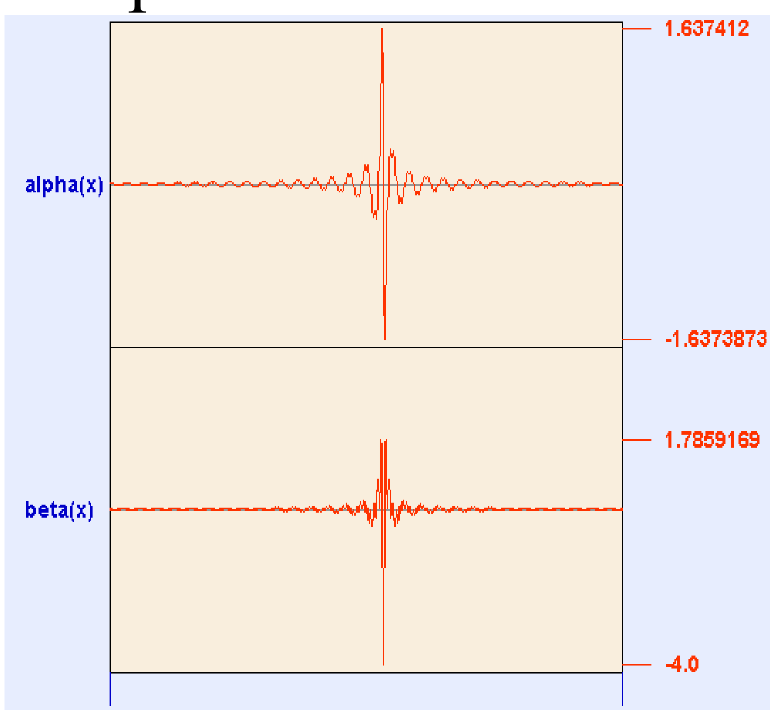
Spectrum: 1-50 Hz



Spectrum: 4-50 Hz



Spectrum: 10-50 Hz



Change of goal

Band limitations suggest seeking $\beta(\mathbf{r})$ rather than $\alpha(\mathbf{r})$

$\alpha(\mathbf{r})$	$\beta(\mathbf{r})$
• Velocity perturbation	• Reflectivity function
• Earth parameters	• Geological structures
• Layer properties	• Interface properties
• Requires low frequencies	• Can be described by high frequencies
• Dynamic	• Kinematic
• Inversion	• Migration
• "True-amplitude migration"	• "Structural inversion"

Time vs. Depth Migration

- True inversion will convert (x,y,t) data to (x,y,z) image – this is depth migration
- Many procedures instead generate an (x,y,t) image – this is time migration
- Time migration
  - quicker, less sensitive to velocity errors, less affected by overburden
  - does not reposition, and is poor for lateral inhomogeneity
- Depth migration
  - more accurate, positions to correct depth, handles complex models
  - time-consuming, requires raytracing & iterations to get model correct

Case 2: Multiple reflectors

Exact solutions for $\psi(z)$ still exist, but more complicated (Bleistein et al., Ex. 2.11)

Let z_R, $z_S = 0$ and $v(z) = c_0 + (c_1 - c_0)H(z-h_1) + (c_2 - c_1)H(z-h_2)$

Exercise: Derive $\alpha(z) = -[4R_1 + O(R_1^2)]H(z-h_1) - [4R_2 + O(R_iR_j)]H(z-h_2)$

$$R_1 = \frac{c_1 - c_0}{c_1 + c_0} \qquad R_2 = \frac{c_2 - c_1}{c_2 + c_1}$$

$$\psi_S(z=0,\omega) = -\frac{c_0}{2i\omega}\left[R_1 \exp[i\omega(2h_1/c_0)] + R_2(1-R_1^2)\sum_{n=1}^{\infty}(-R_1R_2)^{n-1}\exp\{i\omega[2h_1/c_0 + n(h_2-h_1)/c_1]\}\right]$$

Solution

From $v(z) = c_0 + (c_1 - c_0)H(z-h_1) + (c_2 - c_1)H(z-h_1)$ deduce

$$v(z) = \begin{cases} c_0 & , \quad z < h_1 \\ c_1 & , \quad h_1 < z < h_2 \\ c_2 & , \quad z > h_2 \end{cases}, \qquad \alpha(z) = \begin{cases} 0 & , \quad z < h_1 \\ \dfrac{c_0^2}{c_1^2} - 1 & , \quad h_1 < z < h_2 \\ \dfrac{c_0^2}{c_2^2} - 1 & , \quad z > h_2 \end{cases}.$$

Rewrite $\alpha(z > h_2)$ as follows:

$$\frac{c_0^2}{c_2^2} - 1 = \frac{c_0^2}{c_2^2} - \frac{c_0^2}{c_1^2} + \frac{c_0^2}{c_1^2} - 1 = \frac{c_0^2}{c_1^2}\frac{c_1^2}{c_2^2} - \frac{c_0^2}{c_1^2} + \frac{c_0^2}{c_1^2} - 1 = \left(\frac{c_1^2}{c_2^2} - 1\right)\frac{c_0^2}{c_1^2} + \frac{c_0^2}{c_1^2} - 1$$

Using the results of the earlier exercise this can be written as

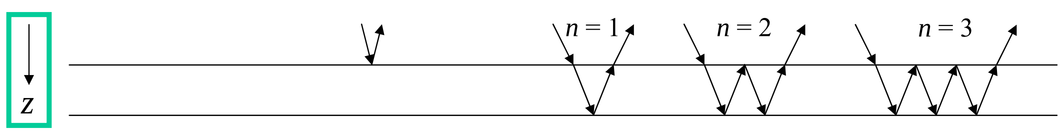
$$\left[-4R_2 + O\left(R_2^2\right)\right]\left(\frac{1-R_1}{1+R_1}\right)^2 + \left[-4R_1 + O\left(R_1^2\right)\right] = \left[-4R_2 + O\left(R_2^2\right)\right]\left[1 - 2R_1 + O\left(R_1^2\right)\right] + \left[-4R_1 + O\left(R_1^2\right)\right]$$

$$= -4R_1 - 4R_2 + O\left(R_1^2\right) + 8R_1R_2 + O\left(R_2^2\right) + \ldots$$

$$= -4R_1 - 4R_2 + O\left(R_iR_j\right)$$

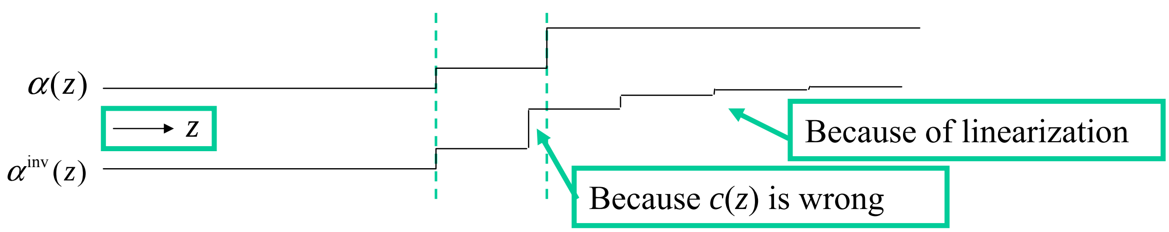
Case 2: Multiple reflectors

$$\psi_S(z=0,\omega) = -\frac{c_0}{2i\omega}\left[R_1 \exp[i\omega(2h_1/c_0)] + R_2(1-R_1^2)\sum_{n=1}^{\infty}(-R_1R_2)^{n-1}\exp\{i\omega[2h_1/c_0 + n(h_2-h_1)/c_1]\}\right]$$

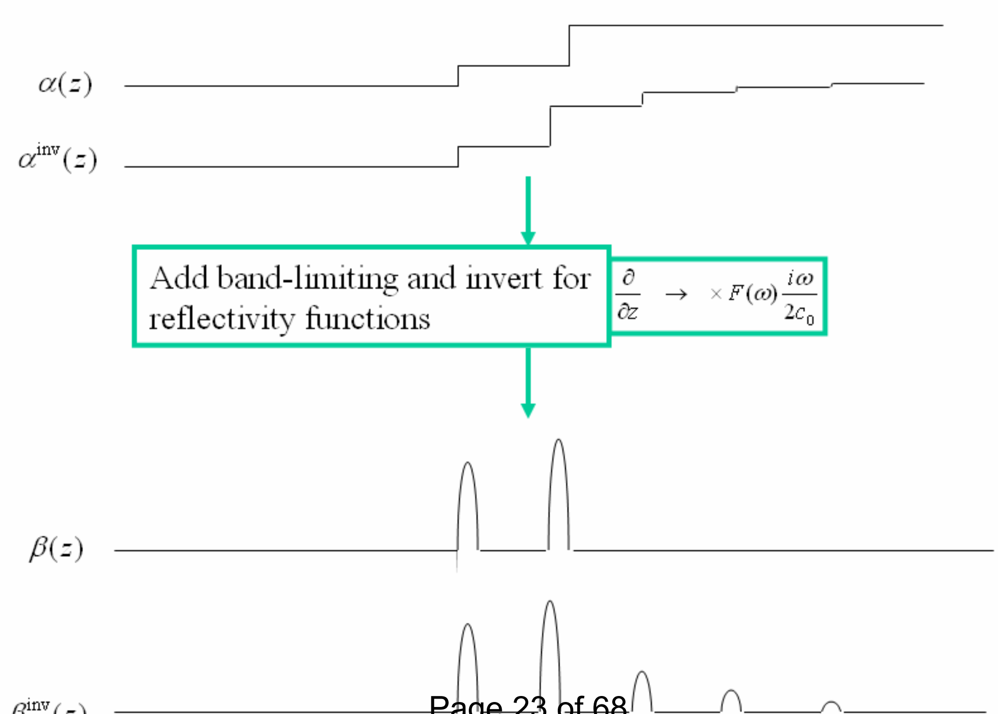


Inversion yields

$$\alpha^{\text{inv}}(x) = -4\left[R_1 H(x-h_1) + R_2(1-R_1^2)H((h_2-h_1)c_0/c_1 + h_1 - x) + (n = 2,3,4\dots\ \text{terms})\right]$$



Case 2



Case 3: WKBJ approximation

A high-frequency approximation for hyperbolic PDEs. Assume an oscillatory solution written as

$$\psi(z,\omega) = A(z)\exp[i\omega\tau(z)]$$

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v(z)^2}\right]\psi(z,\omega) = 0$$

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v(z)^2}\right]\left\{A(z)\exp[i\omega\tau(z)]\right\} = 0$$

real

$$0 = A_{zz} - A\omega^2\left(\tau_z^2 - \frac{1}{v(z)^2}\right)$$

$$0 \approx \tau_z^2 - \frac{1}{v(z)^2} \quad \text{(for high frequency)}$$

$$\Rightarrow \quad \tau_z = \pm\frac{1}{v(z)} \quad \text{(1-D eikonal equation)}$$

$$\Rightarrow \quad \tau(z) = \pm\int_0^z \frac{dz'}{v(z')} \quad \text{(traveltime } \tau\text{)}$$

imaginary

$$i\omega\left(2A_z\tau_z + A\tau_{zz}\right) = 0$$

$$2A_z\tau_z + A\tau_{zz} = 0 \quad \text{(1-D transport equation)}$$

$$\left(A^2\tau_z\right)_z = 0 \quad \text{(conservation of energy)}$$

$$\Rightarrow \quad A \propto \sqrt{v(z)}$$

$$\psi^{\text{WKBJ}}(z,\omega) \propto \sqrt{v(z)}\exp\left[i\omega\int_0^z \frac{dz'}{v(z')}\right]$$

Case 3

Use the WKBJ approximation to the Green's function (Bleistein et al., Ex. 2.13)

- valid for high frequency and slowly-varying $c(z)$

$$g^{\text{WKBJ}}(z,z_R,\omega) = -\frac{\sqrt{c(z)c(z_R)}}{2i\omega}\exp[i\omega\tau(z,z_R)] \qquad \tau(z,z_R) = \left|\int_{z_R}^{z}\frac{dz'}{c(z')}\right|$$

$$g^{\text{homog.}}(z,z_R,\omega) = -\frac{c_0}{2i\omega}\exp(i\omega|z-z_R|/c_0)$$

~ FT

FT

$$\psi_S(0,z_S,\omega) = -f(\omega)\int_0^\infty \alpha(z)\frac{c(z_S)}{4c(z)}\exp[2i\omega\tau(z,z_S)]dz$$

$$\psi_S(z_R,z_S,\omega) = -\frac{\exp[-i\omega(z_R+z_S)/c_0]}{4}\int_0^\infty \alpha(z)\exp[2i\omega(z/c_0)]dz$$

Strategy: assume inversion relation with unknown inverse transform

$$\alpha(z) = -\int_0^\infty f(\omega)T^{-1}(z,\omega)\psi_S(0,z_S,\omega)\exp[-2i\omega\tau(z,z_S)]d\omega$$

Substitute ψ_S into this expression to obtain

Delta function

$$\alpha(z) = \int_0^\infty t(z',z)\alpha(z')dz'$$

From this ascertain T^{-1}

Case 3

FT

$$\alpha^{\text{WKBJ}}(z) = \frac{-4}{\pi c(z_s)} \int_{-\infty}^{\infty} \psi_S(z_S,\omega)\exp[-2i\omega\tau(z,z_S)]d\omega$$

$$\frac{\partial}{\partial z} \to \times \frac{i\omega}{2c(z)}$$

$$\beta^{\text{WKBJ}}(z) = \frac{-2}{\pi c(z)c(z_s)} \int_{-\infty}^{\infty} i\omega\psi_S(z_S,\omega)\exp[-2i\omega\tau(z,z_S)]d\omega$$

homogeneous

$$\beta(z) = \frac{-2}{\pi c_0^2} \int_{-\infty}^{\infty} i\omega\psi_S(z_S,\omega)\exp[-2i\omega|z-z_S|/c_0]d\omega$$

Summary

- Helmholtz equation + Green’s theorem =
  Forward Scattering Theory (Lippman-Schwinger)
- Born Approximation =
  Linearization and/or single-scattering
- Constant background velocity
  → homogeneous Green’s function
  → inversion of modeling formula by FT^{-1}
  → t is replaced by $2x/c_0$
- WKBJ traveltimes and amplitudes
  → approximate Green’s function
  → inversion by substitution yielding an FT^{-1}
  → t is given by solution of eikonal equation

Kirchhoff Scattering Inversion:
II. 3-D inversion

Chuck Ursenbach, CREWES

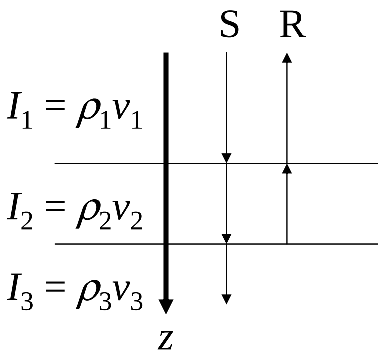
Seismic Imaging Summer School
University of Calgary
August 8, 2006

Overview

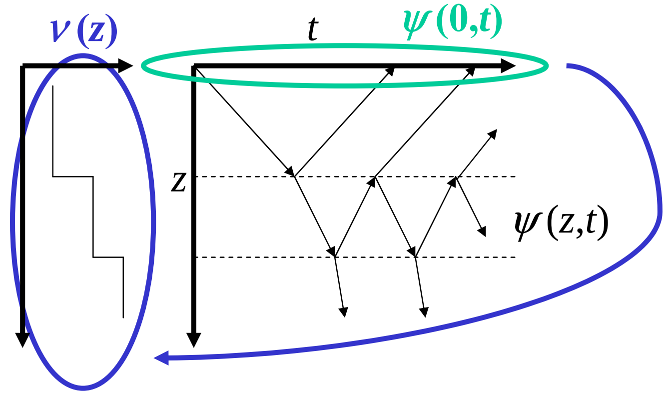
- Fundamental concepts ←
- Normal-incidence Forward Scattering in 3-D
- Normal-incidence Inverse Scattering in 3-D
- Kirchhoff Migration

Traveling Disturbance: 1-D

Geophysics



Mathematics

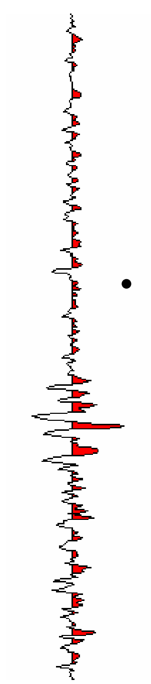


- Dynamite or Vibroseis Source (S)
- Earth filter
- Noise
- Geophone detection (R)

- Impulse source
- Frequency band
- Inherent uncertainty
- Sampled only on boundary

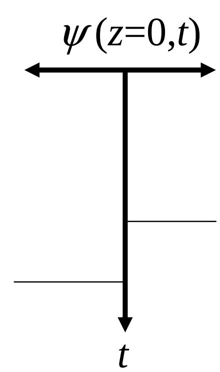
Observed Data: 1-D

Geophysics



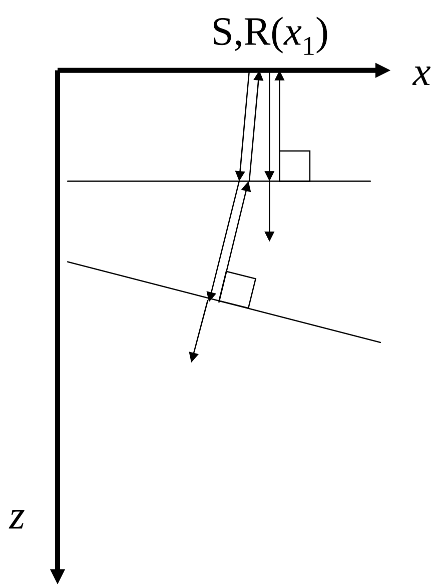
- Time trace obtained from receivers (geophones)

Mathematics

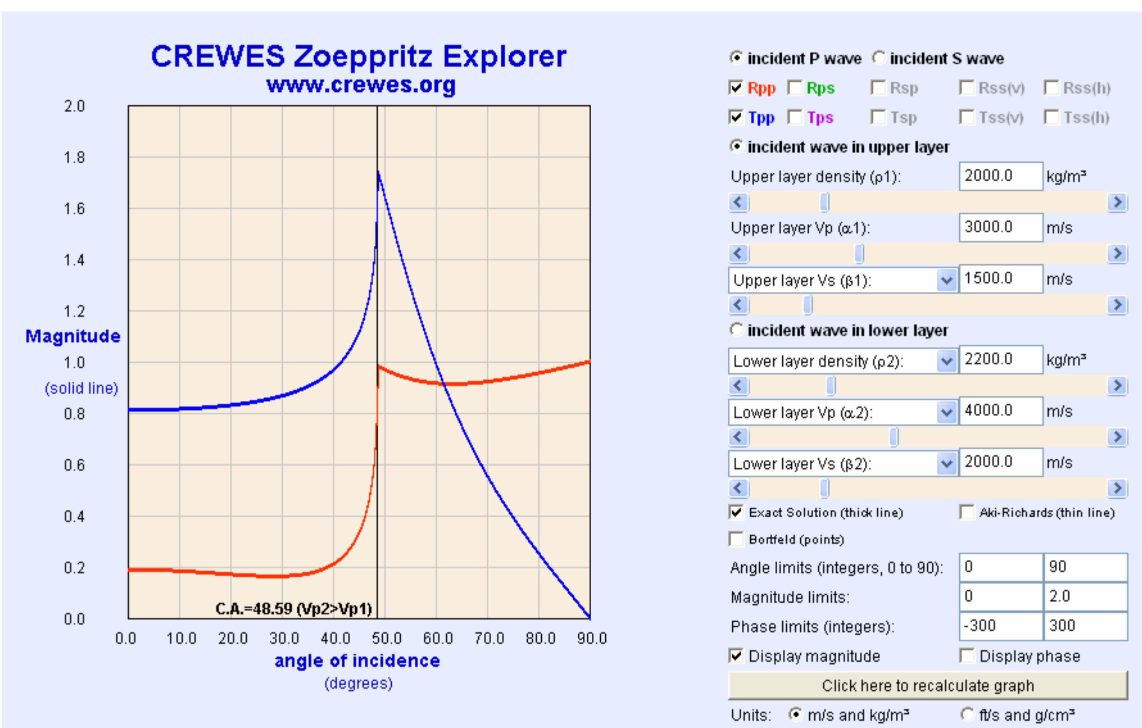


- Wave field sampled at boundary
- Provides boundary condition for inversion

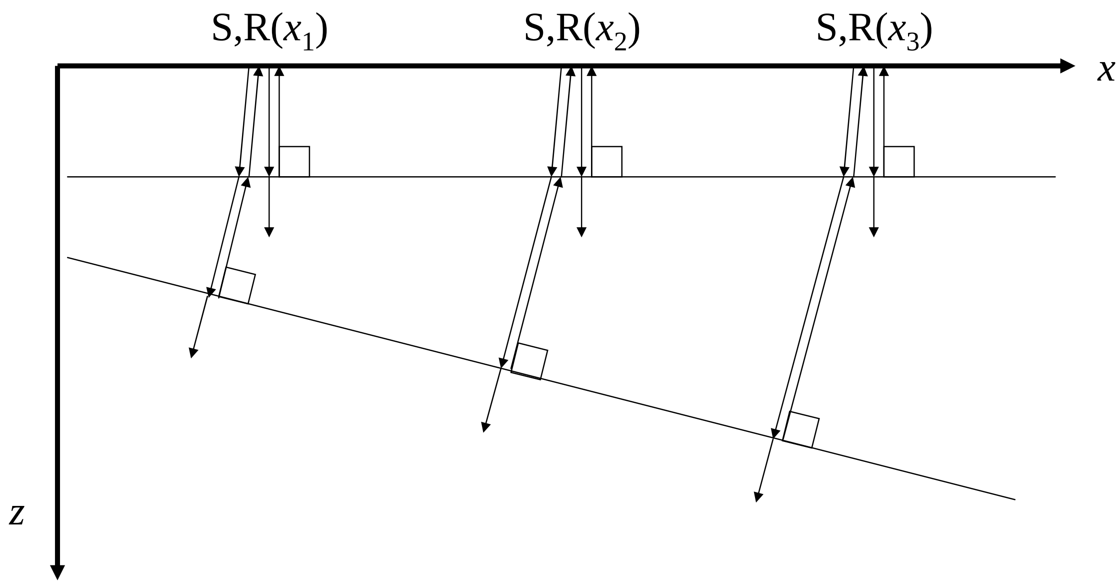
2-D Traveling Disturbance: zero-offset or normal incidence at reflector



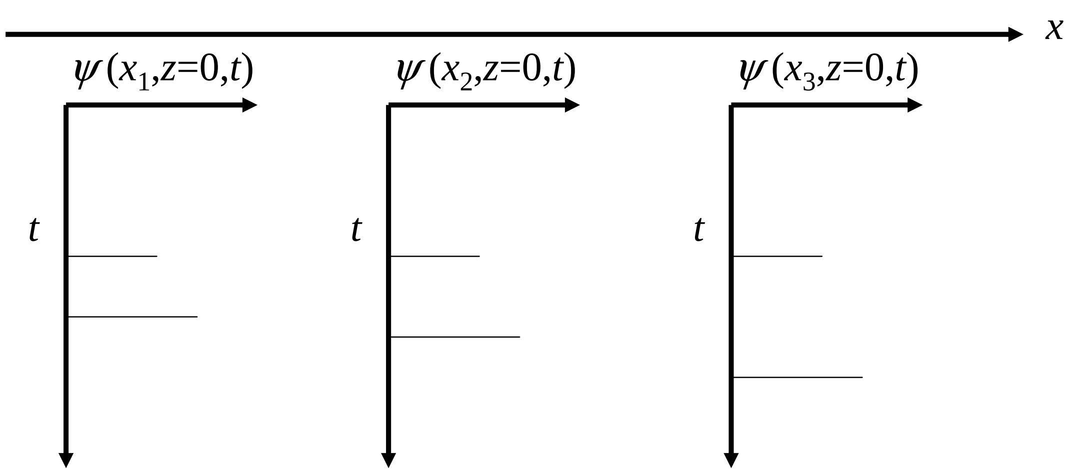
- Acoustic waves
  - Normal-incidence reflectivity:
    - $R_{12} = \frac{I_2 - I_1}{I_1 + I_2}$ $T_{12} = \frac{2I_1}{I_1 + I_2}$
  - Oblique-incidence reflectivity:
    - $R_{12} = \frac{I_2 \cos\theta_1 - I_1 \cos\theta_2}{I_1 \cos\theta_2 + I_2 \cos\theta_1}$ $T_{12} = \frac{2I_1 \cos\theta_2}{I_1 \cos\theta_2 + I_2 \cos\theta_1}$
- Elastic wave
  - Zoeppritz reflection coefficients
  - www.crewes.org/Explorers



2-D Traveling Disturbance: zero-offset or normal incidence

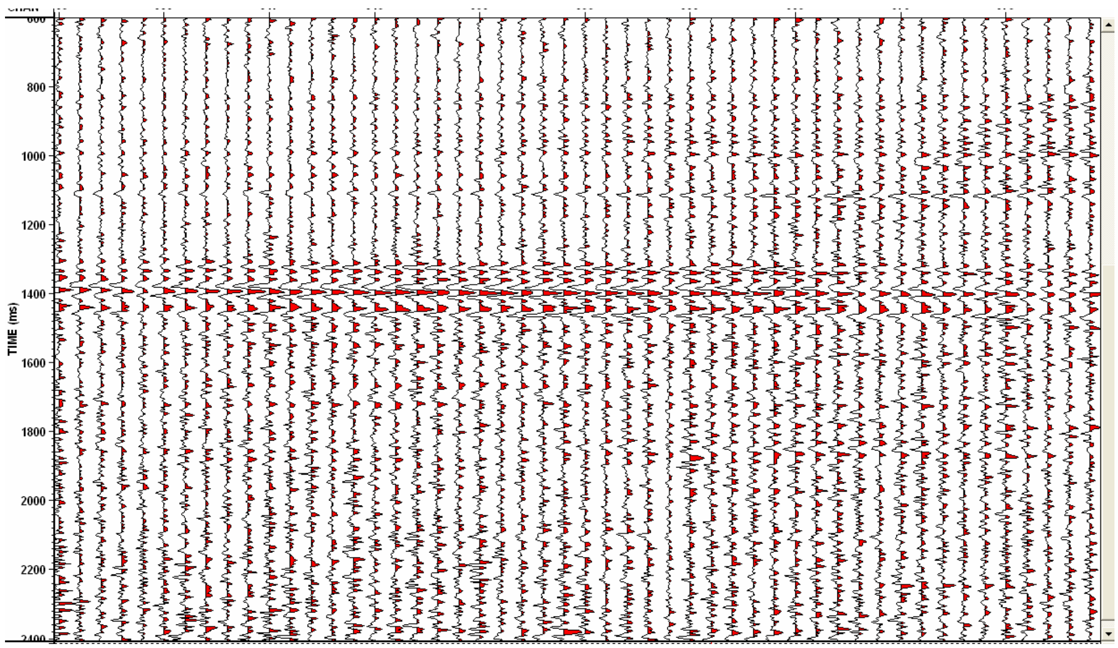


Observed Data: 2-D zero offset

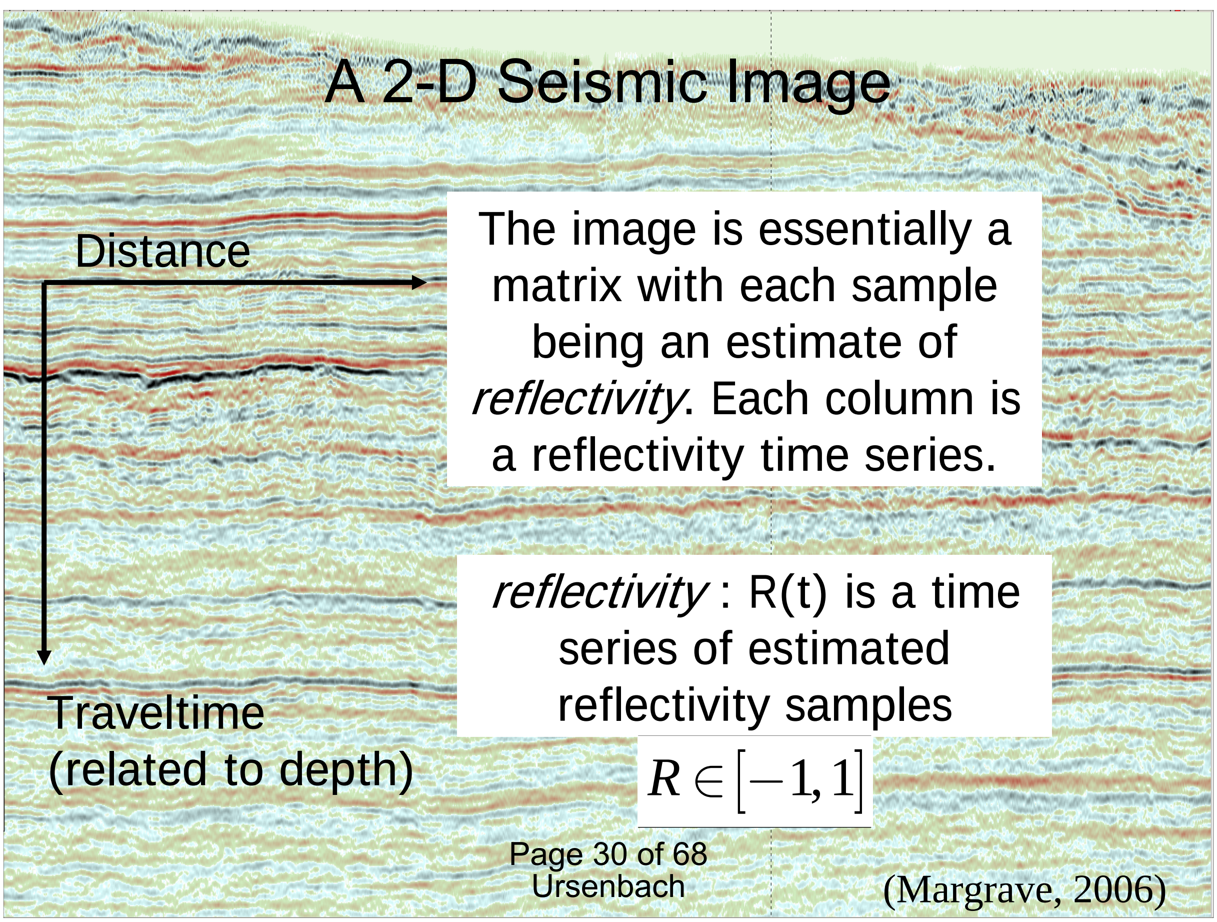


- Zero-offset seismic section
- Forward modeling goal: $v(x,z),\ \psi(x,z,t=0) \rightarrow \psi(x,z,t)$
- Inversion goal: $\psi(x,z=0,t) \rightarrow v(x, z)$
- 3-D adds horizontal y-axis

Zero-offset seismic section



A 2-D Seismic Image



The image is essentially a matrix with each sample being an estimate of *reflectivity*. Each column is a reflectivity time series.

reflectivity : R(t) is a time series of estimated reflectivity samples

$$R \in [-1, 1]$$

(Margrave, 2006)

Overview

- Fundamental concepts
- Normal-Incidence Forward Scattering in 3-D ⟵
- Normal-Incidence Inverse Scattering in 3-D
- Kirchhoff Migration

Exercise

From the following equations, derive the integral solution to Helmholtz equation for an unbounded medium

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right]\psi(\mathbf{r},\mathbf{r}_S,\omega) = f(\mathbf{r},\mathbf{r}_S)$$ Helmholtz equation for source at $\mathbf{r}_S$

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right]g(\mathbf{r}-\mathbf{r}_R,\omega) = \delta(\mathbf{r}-\mathbf{r}_R)$$ Helmholtz equation for Green's function

$$r\left[\frac{\partial}{\partial r} - \frac{i\omega}{v(\mathbf{r})}\right]\psi(\mathbf{r},\omega) \to 0 \quad \text{as} \quad r \to \infty$$ Radiation Condition (BC)

$$\int_V \left\{\psi\nabla^2 g - g\nabla^2\psi\right\} dV = \int_S \left\{\psi\frac{dg}{dn} - g\frac{d\psi}{dn}\right\} dS$$ Green's theorem (also called Green's 2nd identity)

$\frac{dg}{dn} \equiv \mathbf{n}\cdot\nabla g$

Solution

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] g(\mathbf{r}-\mathbf{r}_R,\omega) = \delta(\mathbf{r}-\mathbf{r}_R)$$

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] \psi(\mathbf{r},\mathbf{r}_S,\omega) = f(\mathbf{r},\mathbf{r}_S)$$

$$\int_V \left\{\psi\nabla^2 g - g\nabla^2\psi\right\} dV = \int_S \left\{\psi\frac{dg}{dn} - g\frac{d\psi}{dn}\right\} dS$$

Radiation condition for unbounded problem

$$\int_V \left\{\psi\left[-\delta(\mathbf{r}-\mathbf{r}_G) - \frac{\omega^2}{v^2} g\right] - g\left[-f(\mathbf{r}_S) - \frac{\omega^2}{v^2}\psi\right]\right\} dV = 0$$

$$\psi(\mathbf{r}_S,\mathbf{r}_R) = -\int_V g(\mathbf{r},\mathbf{r}_R)\, f(\mathbf{r},\mathbf{r}_S)\, dV$$

Integral solution to Helmholtz equation

3-D Forward Scattering

$$\frac{1}{v(\mathbf{r})^2} = \frac{1}{c(\mathbf{r})^2}\left[1+\alpha(\mathbf{r})\right]$$

True velocity

Background velocity

Velocity Perturbation

$$\psi = \psi_I + \psi_S$$

$$\left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2}\right] \psi_I(\mathbf{r},\mathbf{r}_S,\omega) = f(\mathbf{r},\mathbf{r}_S)$$

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] \psi(\mathbf{r},\mathbf{r}_S,\omega) = f(\mathbf{r},\mathbf{r}_S)$$

Exercise: Derive the following Helmholtz-like equation for ψ_S

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] \psi_S(\mathbf{r},\mathbf{r}_S,\omega) = \left[\frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right] \psi_I(\mathbf{r},\mathbf{r}_S,\omega)$$

Solution

Using the two Helmholtz equation on the previous slide, substitute for $v(\mathbf{r})$ and ψ and then subtract

$$\left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2} + \frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right]\left[\psi_I(\mathbf{r},\mathbf{r}_S,\omega) + \psi_S(\mathbf{r},\mathbf{r}_S,\omega)\right] = f(\mathbf{r},\mathbf{r}_S)$$

$$-\left\{\left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2}\right]\psi_I(\mathbf{r},\mathbf{r}_S,\omega) = f(\mathbf{r},\mathbf{r}_S)\right\}$$

$$\rightarrow \quad \left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2} + \frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right]\psi_S(\mathbf{r},\mathbf{r}_S,\omega) = \underbrace{\left[\frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right]\psi_I(\mathbf{r},\mathbf{r}_S,\omega)}_{\text{Effective source}}$$

3-D Forward Scattering

$$\left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2} + \frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right]\psi_S(\mathbf{r},\mathbf{r}_S,\omega) = \left[\frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right]\psi_I(\mathbf{r},\mathbf{r}_S,\omega)$$

Exercise: Using 1) the result above, 2) the Green's function for the background velocity model and 3) Green's theorem, derive the following modeling formula for ψ_S

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega) = \omega^2 \int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2} g(\mathbf{r},\mathbf{r}_R,\omega)\psi_I(\mathbf{r},\mathbf{r}_S,\omega)d\mathbf{r}$$

Solution

$$\left[\nabla^2 + \frac{\omega^2}{c(\mathbf{r})^2}\right] g(\mathbf{r}-\mathbf{r}_R,\omega) = \delta(\mathbf{r}-\mathbf{r}_R)$$

$$\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] \psi_S(\mathbf{r},\mathbf{r}_S,\omega) = \left[\frac{\omega^2}{c(\mathbf{r})^2}\alpha(\mathbf{r})\right] \psi_I(\mathbf{r},\mathbf{r}_S,\omega)$$

$$\int_V \left\{\psi_S \nabla^2 g - g\nabla^2 \psi_S\right\} dV = \int_{S_V} \left\{\psi_S \frac{dg}{dn} - g\frac{d\psi_S}{dn}\right\} dS_V$$

vanishes by radiation condition

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega) = \omega^2 \int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2} g(\mathbf{r},\mathbf{r}_R,\omega)\left[\psi_I(\mathbf{r},\mathbf{r}_S,\omega) + \psi_S(\mathbf{r},\mathbf{r}_S,\omega)\right] d\mathbf{r}$$

Born approximation

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega) = \omega^2 \int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2} g(\mathbf{r},\mathbf{r}_R,\omega)\psi_I(\mathbf{r},\mathbf{r}_S,\omega) d\mathbf{r}$$

Kirchhoff Modeling Formula

- The Born approximation is used to derive a modeling formula from which the inversion formula is derived
- We may wish to test the inversion formula with modeled data, but must use a different model
- The Kirchhoff modeling formula can be used for this purpose

Kirchhoff Modeling Formula

- Employing Green's theorem and the Kirchhoff approximation

$$\psi_S = R\psi_I, \quad \frac{\partial \psi_S}{\partial n} = -R\frac{\partial \psi_I}{\partial n}, \quad \text{on } S_V,$$

one can derive the Kirchhoff modeling formula

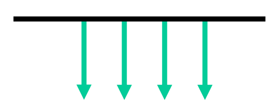
$$\psi_S(\mathbf{r}_R, \mathbf{r}_S, \omega) = \int_{S_V} R\frac{\partial(\psi_I g)}{\partial n} dS_V$$

cf. Margrave notes, p. 27, 28

3-D nomenclature

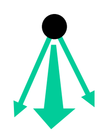
<u>1-D: $v(\boldsymbol{r}) = v(z)$</u>

planar source $\longrightarrow$ no spreading



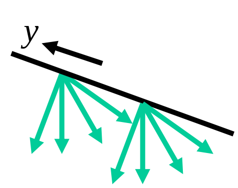
<u>"1.5"-D: $v(\boldsymbol{r}) = v(z)$</u>

point source $\longrightarrow$ spherical spreading



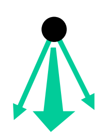
<u>2-D: $v(\boldsymbol{r}) = v(x,z)$</u>

linear source $\longrightarrow$ cylindrical spreading



<u>"2.5"-D: $v(\boldsymbol{r}) = v(x,z)$</u>

point source $\longrightarrow$ spherical spreading



Overview

- Fundamental concepts
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- Normal-incidence Inverse Scattering in 3-D ⟵
- Kirchhoff Migration

Inverse Scattering Cases

In 1-D, only one lateral source / receiver position is possible

In 3-D, there are surface coverage issues

How much of the surface is covered with receivers? (geophysics)

What is the support of $\psi_S(x,y,0)$? (mathematics)

Case 4: 1.5-D, $c(\mathbf{r}) = c_0$; $\alpha(\mathbf{r}) = \alpha(z)$; $\mathbf{r}_S = \mathbf{r}_R = (0, 0, 0)$

- every zero-offset trace is identical (except for noise)
- replication → infinite coverage
- differs from 1-D in spherical spreading

Case 5: 3-D, $c(\mathbf{r}) = c_0$; $\alpha(\mathbf{r})$; $\mathbf{r}_S = \mathbf{r}_R = (x_{SR}, y_{SR}, 0)$

- coverage is now an inescapable issue
- first assume infinite coverage, and $f(\omega) = 1$

Case 4 modeling formula

$c(\mathbf{r}) = c_0$

$$g(\mathbf{r},\omega) = \frac{\exp(i\omega r / c_0)}{4\pi r}$$

$$\psi_I(\mathbf{r},\omega) = f(\omega)\frac{\exp(i\omega r / c_0)}{4\pi r}$$

$$\psi_S(\mathbf{r}_R = \mathbf{r}_{RS}, \mathbf{r}_S = \mathbf{r}_{RS}, \omega) = \omega^2 \int_V \frac{\alpha(z)}{c_0^2} g(\mathbf{r},\mathbf{r}_R,\omega)\, \psi_I(\mathbf{r},\mathbf{r}_S,\omega)\, d\mathbf{r}$$

$\mathbf{r}_{RS} = (x_{RS}, y_{RS}, 0)$

$r = |\mathbf{r} - \mathbf{r}_{RS}|$

$$\psi_S(x_{RS}, y_{RS}, \omega) = f(\omega)\omega^2 \int_V \frac{\alpha(z)}{c_0^2} \frac{\exp(2i\omega r / c_0)}{(4\pi r)^2} d\mathbf{r}$$

Exercise: 1) Convert **r** to cylindrical coordinates. 2) Integrate over θ. 3) Integrate by parts (Note: $\partial r / \partial \rho = \rho / r$) and retain leading term in high frequency approximation.

$$\psi_S(x_{RS}, y_{RS}, \omega) \approx f(\omega)\frac{i\omega}{16\pi c_0}\int_0^\infty \frac{\alpha(z)}{z}\exp(2i\omega z / c_0) dz$$

Spherical spreading

Solution

$$\psi_S(x_{RS}, y_{RS}, \omega) = f(\omega)\omega^2 \int_V \frac{\alpha(z)}{c_0^2} \frac{\exp(2i\omega r / c_0)}{(4\pi r)^2} d\mathbf{r}$$

$$= f(\omega)\frac{\omega^2}{(4\pi c_0)^2}\int_0^\infty dz\, \alpha(z)\int_0^{2\pi} d\theta \int_0^\infty d\rho \frac{\exp(2i\omega r / c_0)}{r^2}, \quad r = \sqrt{\rho^2 + z^2}$$

$$= f(\omega)\frac{\omega^2}{8\pi c_0^2}\int_0^\infty dz\, \alpha(z)\int_0^\infty d\rho \frac{\exp(2i\omega r / c_0)}{r^2},$$

$$= f(\omega)\frac{\omega^2}{8\pi c_0^2}\int_0^\infty dz\, \alpha(z)\left[-\exp(2i\omega z / c_0)\right]\sum_{n=0}^{\infty} n!\left(\frac{c_0}{2i\omega z}\right)^{n+1}, \quad (\mathrm{Im}(\omega) \geq 0)$$

$$\approx f(\omega)\frac{i\omega}{16\pi c_0}\int_0^\infty \frac{\alpha(z)}{z}\exp(2i\omega z / c_0)\, dz \qquad \text{(high-frequency approximation)}$$

Case 4 inversion formula

Forward modeling

$$\psi_S(x_{RS}, y_{RS}, \omega) \approx f(\omega)\frac{i\omega}{16\pi c_0}\int_0^{\infty}\frac{\alpha(z)}{z}\exp(2i\omega z/c_0)dz$$

↓ **Exercise**: Derive the inversion formula

Inversion formula

$$\alpha(z) = 16z\int_{-\infty}^{\infty}\frac{f(\omega)\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}\exp(-2i\omega z/c_0)d\omega$$

↓ $\times\frac{i\omega}{2c_0}$

Reflectivity formula

$$\beta(z) = \frac{8z}{c_0}\int_{-\infty}^{\infty} f(\omega)\psi_S(x_{RS}, y_{RS}, \omega)\exp(-2i\omega z/c_0)d\omega$$

Solution

$$\psi_S(x_{RS}, y_{RS}, \omega) \approx f(\omega)\frac{i\omega}{16\pi c_0}\int_0^{\infty}\frac{\alpha(z)}{z}\exp[i\underbrace{(2\omega/c_0)}_{k}z]\,dz$$

$$\frac{f(\omega)}{16\pi c_0}\int_0^{\infty}dz\frac{\alpha(z)}{z}\exp[i(2\omega/c_0)z] = \frac{\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\exp(-ikz)f(\omega)\frac{1}{16\pi c_0}\int_0^{\infty}dz'\frac{\alpha(z')}{z'}\exp[i(2\omega/c_0)z'] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}\exp(-ikz)dk$$

$$\int_0^{\infty}dz'\frac{\alpha(z')}{z'}\frac{1}{16\pi c_0}\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,f(\omega)\exp[-ik(z-z')] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}\exp[-i(2\omega/c_0)z]\,d(2\omega/c_0)$$

$$\int_0^{\infty}dz'\frac{\alpha(z')}{z'}\frac{1}{16\pi c_0}\delta_B(z-z') = \frac{1}{2\pi}\frac{2}{c_0}\int_{-\infty}^{\infty}\frac{\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}\exp(-2i\omega z/c_0)d\omega$$

$$\frac{\alpha_B(z)}{z}\frac{1}{16} = \int_{-\infty}^{\infty}\frac{\psi_S(x_{RS}, y_{RS}, \omega)}{i\omega}\exp(-2i\omega z/c_0)d\omega$$

The subscript *B* denotes band-limited versions of δ(z-z') and α (z).

Case 5 inversion formula

Forward modeling

$$\psi_S(x_0, y_0, \omega) = \omega^2 \int_{z>0} \frac{\alpha(\mathbf{r})}{c_0^2} \frac{\exp(2i\omega r / c_0)}{(4\pi r)^2} d\mathbf{r}$$

$\rho_0 \equiv (x_0, y_0)$

$\kappa \equiv (k_x, k_y)$

Inversion formula

$$\alpha(\mathbf{r}) = \frac{8c_0^3}{i\pi^2} \int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} dk_z \frac{k_z}{\omega^2} \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\} \int_0^{\infty} dt\, t \Psi_S(\boldsymbol{\rho}_0, t) \exp(i\omega t) \left[1 + \frac{2i}{\omega t}\right]$$

delta function?

$$k_z = \sqrt{\frac{\omega_0^2}{c_0^2} - k_x^2 - k_y^2}$$

high frequency approx.

$$\times \frac{i\omega}{2c_0}$$

$$\beta(\mathbf{r}) = \frac{4c_0^2}{\pi^2} \int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} dk_z \frac{k_z}{\omega} \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\} \int_0^{\infty} dt\, t \Psi_S(\boldsymbol{\rho}_0, t) \exp(i\omega t)$$

Case 5 and Stolt's 3-D migration

$$\beta(\mathbf{r}) = \frac{4c_0^2}{\pi^2} \int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} dk_z \frac{k_z}{\omega} \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\} \int_0^{\infty} dt\, t \Psi_S(x_0, y_0, t) \exp(i\omega t)$$

$dk_z \to d\omega$

Integrate by parts in ω

$$\beta(\mathbf{r}) = \frac{8z}{\pi^2 c_0^2} \int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} d\omega \frac{\omega}{k_z} \left[\int_0^{\infty} dt\, \Psi_S(x_0, y_0, t) \exp(i\omega t)\right] \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\}$$

Stolt's 3-D migration formula:

$$\psi(X, Y, Z, 0) = (2\pi)^{-3/2} \int dP \int dQ \int d\omega\, B(P, Q, \omega) \exp[i(PX + QY - 2\omega Z / c)]$$

Stolt, R. H., 1978, Geophysics **43**, 23-48, equations 62-67

Case 5 with Stationary phase

$$\beta(\mathbf{r}) = \frac{4c_0^2}{\pi^2}\int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} dk_z \frac{k_z}{\omega} \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\}\int_0^{\infty} dt\, t\Psi_S(x_0, y_0, t)\exp(i\omega t)$$

$dk_z \rightarrow d\omega$

$$\beta(\mathbf{r}) = \frac{4}{\pi^2}\int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa} \int_{-\infty}^{\infty} d\omega \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\}\int_0^{\infty} dt\, t\Psi_S(x_0, y_0, t)\exp(i\omega t)$$

Stationary phase approx. to $\int d\boldsymbol{\kappa}$

$$\beta(\mathbf{r}) = \frac{4z}{\pi c_0}\int_{-\infty}^{\infty} \frac{d\boldsymbol{\rho}_0}{r^2} \int_{-\infty}^{\infty} d\omega\, i\omega \exp(-2i\omega r / c_0)\int_0^{\infty} dt\, t\Psi_S(\boldsymbol{\rho}_0, t)\exp(i\omega t)$$

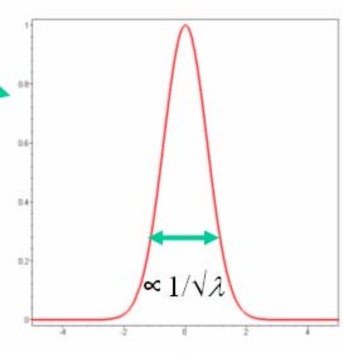
$$r = \sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2 + z^2}$$

Asymptotic Approximations ($\lambda >> 1$)

Laplace's Method

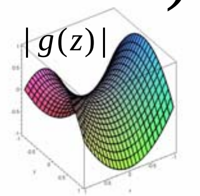
For real $f(x)$, with max at x_0, then $f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$

$$\begin{aligned} I(\lambda) &= \int_A^B h(x)\exp[\lambda f(x)]dx \\ &\approx \exp[\lambda f(x_0)]\int_{-\infty}^{\infty} h(x)\exp[\lambda f''(x_0)(x-x_0)^2/2]dx \\ &\approx h(x_0)\exp[\lambda f(x_0)]\int_{-\infty}^{\infty} \exp[\lambda f''(x_0)(x-x_0)^2/2]dx \\ &= h(x_0)\exp[\lambda f(x_0)]\sqrt{\frac{2\pi}{\lambda|f''(x_0)|}} \end{aligned}$$



Asymptotic Approximations ($\lambda >> 1$)

Steepest Descent *or* Saddle-point method



$$I(\lambda) = \int_A^B h(z)\exp[\lambda g(z)]dz$$

$g(z) = f(x)$ → Laplace's Method

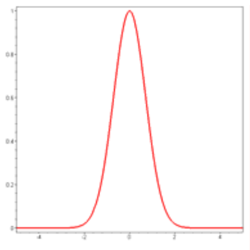
$g(z) = if(x)$ → Stationary-phase approximation

Laplace's Method

$$I(\lambda) = \int_A^B h(x)\exp[\lambda f(x)]dx$$

$$\approx h(x_0)\exp[\lambda f(x_0)]\sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}$$

$\exp[\lambda f(x)]$

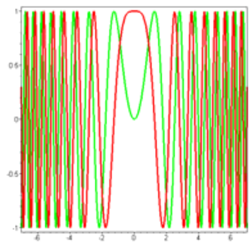


Stationary-phase approximation

$$I(\lambda) = \int_A^B h(x)\exp[i\lambda f(x)]dx$$

$$\approx h(x_0)\exp[i\lambda f(x_0)]\sqrt{\frac{2\pi}{i\lambda f''(x_0)}}$$

$\exp[i\,\lambda f(x)]$



Stationary-Phase

One-dimensional

$$I(\lambda) = \int_A^B h(x)\exp[i\,\lambda f(x)]dx$$

$$\approx h(x_0)\exp[i\,\lambda f(x_0)]\sqrt{\frac{2\pi}{i\,\lambda f''(x_0)}}$$

Multi-dimensional

$$I(\lambda) = \int h(\mathbf{x})\exp[i\,\lambda f(\mathbf{x})]d^n x$$

$$\approx h(\mathbf{x}_0)\exp[i\,\lambda f(\mathbf{x}_0)]\left(\frac{2\pi}{\lambda}\right)^{n/2}\sqrt{\frac{\pm 1}{i^m \det f_{ij}}}$$

$$I = \int_{-\infty}^{\infty} d\boldsymbol{\kappa}\exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\}$$

$$= \int_{-\infty}^{\infty} d(k\eta_1)\int_{-\infty}^{\infty} d(k\eta_2)\exp\{2ik[\boldsymbol{\eta}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - \eta_z z]\}$$

$$= k^2\int_{-\infty}^{\infty} d\eta_1\int_{-\infty}^{\infty} d\eta_2\exp\{2ik[\boldsymbol{\eta}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - \eta_z z]\}$$

$$= k^2\int_{-\infty}^{\infty} d\eta_1\int_{-\infty}^{\infty} d\eta_2\exp\{2ikr[\boldsymbol{\eta}\cdot\frac{\boldsymbol{\rho}-\boldsymbol{\rho}_0}{r} - \eta_z\frac{z}{r}]\}$$

$\int_{-\infty}^{\infty} d\omega$

$r = \sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2 + z^2}$

Case 5 and Stationary phase

$$\beta(\mathbf{r}) = \frac{4}{\pi^2}\int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \int_{-\infty}^{\infty} d\boldsymbol{\kappa}\int_{-\infty}^{\infty} d\omega \exp\{2i[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - k_z z]\}\int_0^{\infty} dt\, t\Psi_S(x_0, y_0, t)\exp(i\omega t)$$

Stationary phase approx. to $\int d\boldsymbol{\kappa}$

3-D: $$\beta(\mathbf{r}) = \frac{4z}{\pi c_0}\int_{-\infty}^{\infty} \frac{d\boldsymbol{\rho}_0}{r^2}\int_{-\infty}^{\infty} d\omega\, i\omega \exp(-2i\omega r / c_0)\int_0^{\infty} dt\, t\Psi_S(\boldsymbol{\rho}_0, t)\exp(i\omega t)$$

2.5-D: $$\beta(\mathbf{r}) = \frac{4z}{\sqrt{\pi c_0}}\int_{-\infty}^{\infty} \frac{dx_0}{r^{3/2}}\int_{-\infty}^{\infty} d\omega \sqrt{i\omega} \exp(-2i\omega r / c_0)\int_0^{\infty} dt\, t\Psi_S(\boldsymbol{\rho}_0, t)\exp(i\omega t)$$

Case 5 and Kirchhoff Migration

Stationary phase result:

$$\beta(\mathbf{r}) = \frac{4z}{\pi c_0}\int_{-\infty}^{\infty} \frac{d\boldsymbol{\rho}_0}{r^2}\int_{-\infty}^{\infty} d\omega\, i\omega \exp(-2i\omega r / c_0)\int_0^{\infty} dt\, t\Psi_S(\boldsymbol{\rho}_0, t)\exp(i\omega t)$$

Int. by parts (ω); high freq. apprx.; FT

$$\beta(\mathbf{r}) = \frac{8z}{\pi c_0^2}\int_{-\infty}^{\infty} \frac{d\boldsymbol{\rho}_0}{r}\int_{-\infty}^{\infty} d\omega\, i\omega \exp(-2i\omega r / c_0)\psi_S(\boldsymbol{\rho}_0, \omega)$$

Schneider’s Kirchhoff migration formula:

cf. Margrave notes, p. 21

$$U(x, y, z, 0) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\iint dx\, dy \frac{U(x, y, 0, R/C)}{R}$$

Schneider, W. A., 1978, Geophysics **43**, 49-76, equations 62-67

Overview

- Fundamental concepts
- Normal-incidence Forward Scattering in 3-D
- Normal-incidence Inverse Scattering in 3-D
- Kirchhoff Migration ←

Kirchhoff Migration

$$\beta(\mathbf{r}) = \frac{8z}{\pi c_0^2} \int_{-\infty}^{\infty} \frac{d\rho_0}{r} \int_{-\infty}^{\infty} d\omega \, i\omega \exp(-2i\omega r / c_0) \psi_S(\rho_0, \omega)$$

rearrange

$$\beta(\mathbf{r}) = \frac{8}{\pi c_0^2} \int_{-\infty}^{\infty} d\rho_0 \underbrace{\frac{z}{r}}_{\cos\theta} \int_{-\infty}^{\infty} d\omega \, \underbrace{\exp(-2i\omega r / c_0)}_{\text{kernal of } FT^{-1} \text{ for } t = 2r/c_0} \underbrace{\left[i\omega \psi_S(\rho_0, \omega) \right]}_{\text{FT of time derivative of } \Psi}$$

$$\beta(\mathbf{r}) = \frac{8}{\pi c_0^2} \int_{-\infty}^{\infty} d\rho_0 \cos\theta \left[\frac{\partial}{\partial t} \Psi_S(\rho_0, t) \right]_{t = \frac{2r}{c_0}}$$

cf. Yilmaz, 2001, equation 4.5

$$\left(\sqrt{v_{rms} r} \to v_{rms} r\right)$$

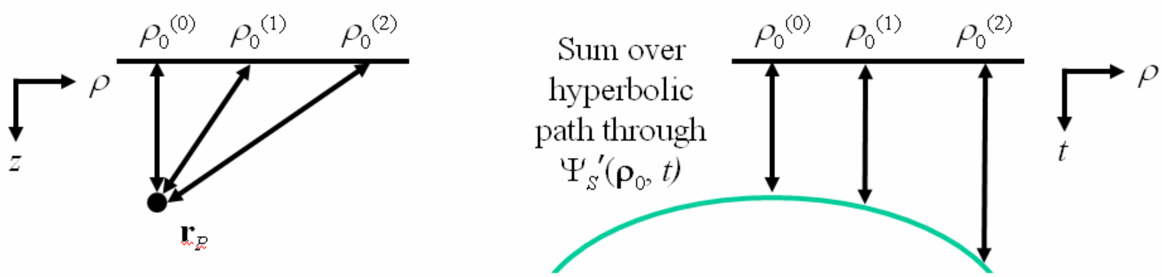
$$P_{out} = \frac{\Delta x}{2\pi} \sum_x \left[\frac{\cos\theta}{v_{rms} r} \rho(t) * P_{in} \right]$$

Kirchhoff Migration

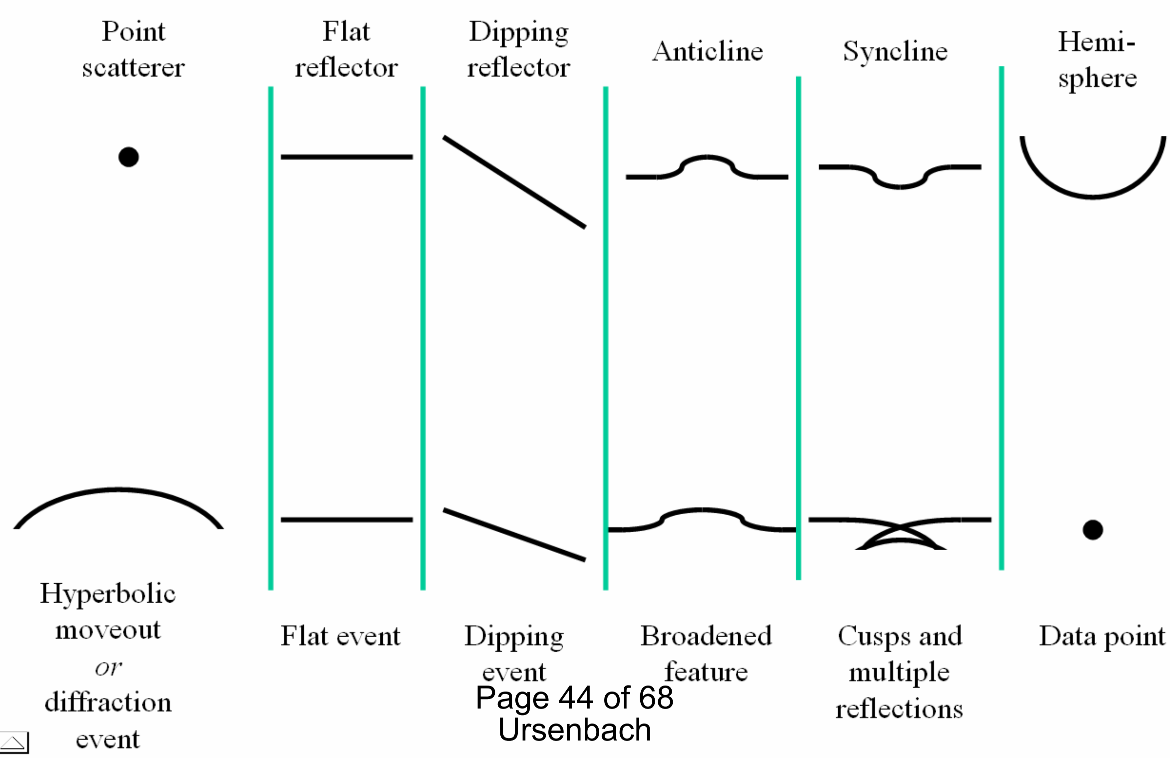
$$\beta(\mathbf{r}) = \frac{8}{\pi c_0^2} \int_{-\infty}^{\infty} d\boldsymbol{\rho}_0 \cos\theta \left[\frac{\partial}{\partial t} \Psi'_S(\boldsymbol{\rho}_0, t) \right]_{t=\frac{2r}{c_0}}$$

- Suppose that reflector is a single point
- Assume it is at $\mathbf{r} = \mathbf{r}_P$, and sum $\Psi_S'(\boldsymbol{\rho}_0, t = 2r / c_0)$ over all receivers

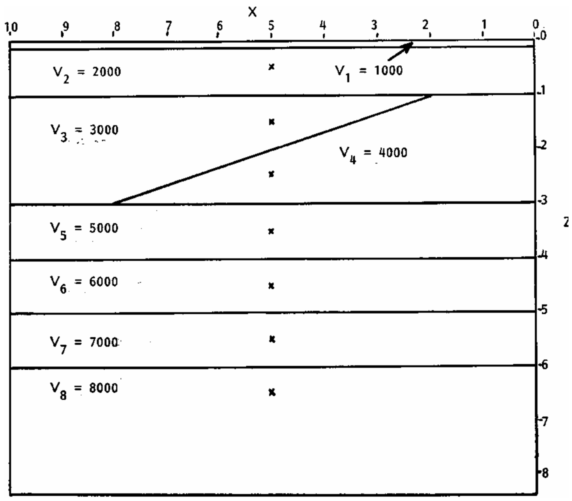
$$r = \sqrt{(x_0 - x_P)^2 + (y_0 - y_P)^2 + z_P^2}$$

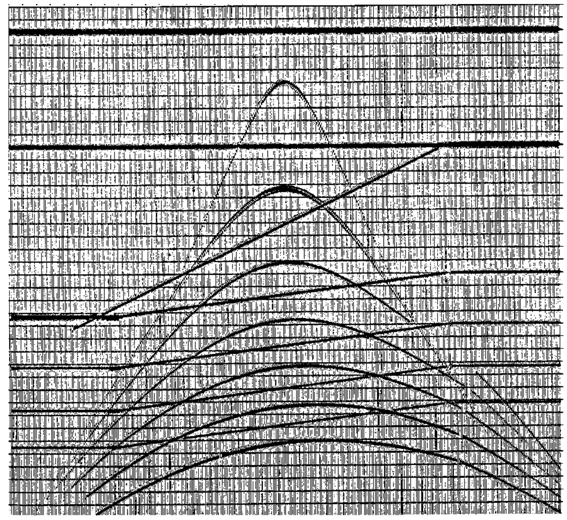


Modeled and Migrated Structures



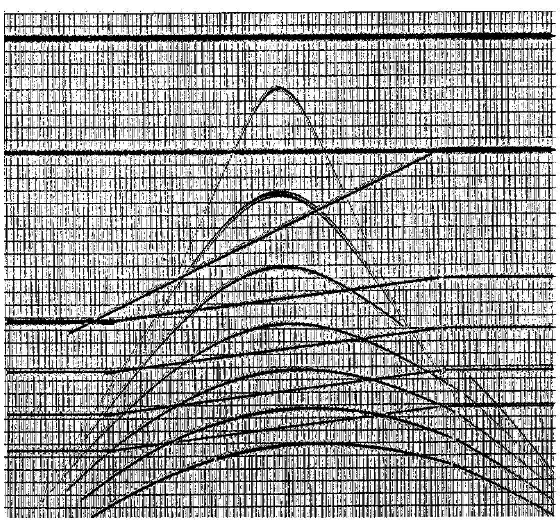
Synthetic Seismic Data
Forward Modeling by Raytracing

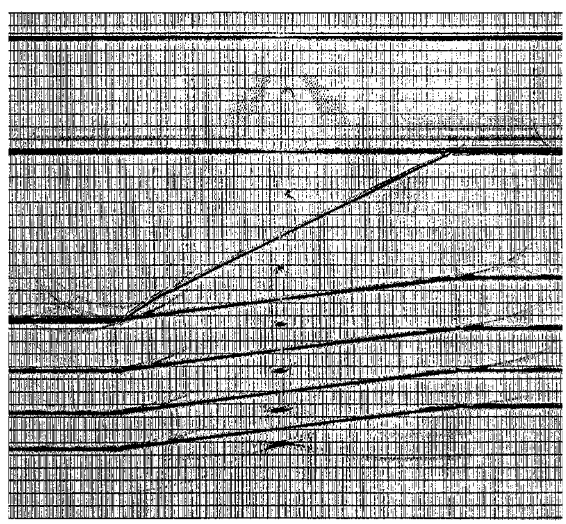




J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

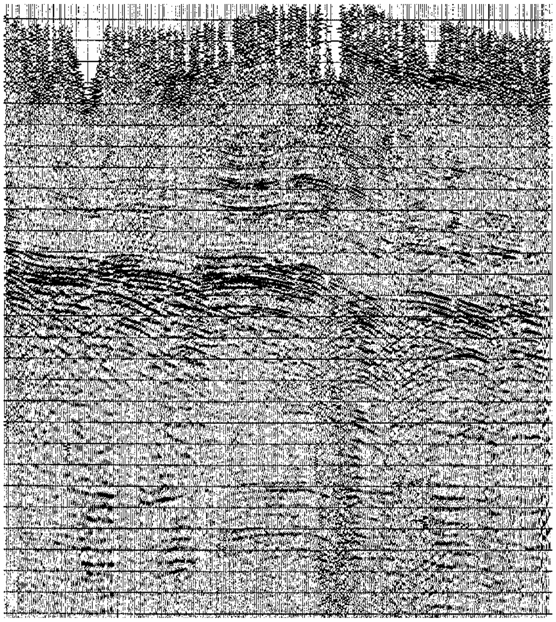
Synthetic Seismic Data
Inversion by Kirchhoff Migration

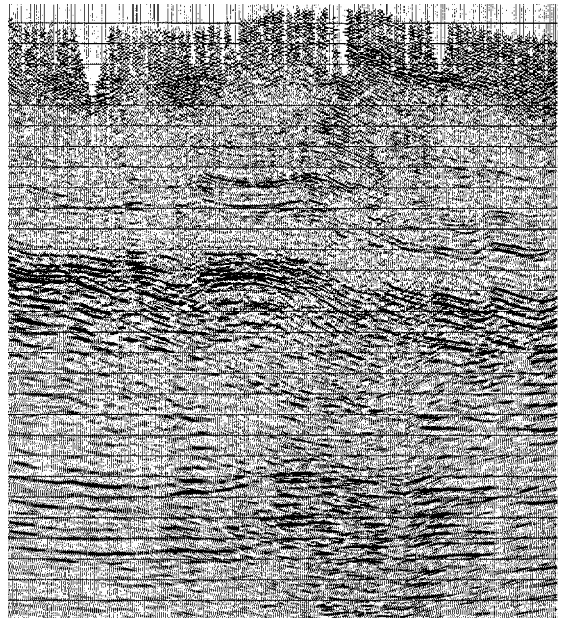




J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

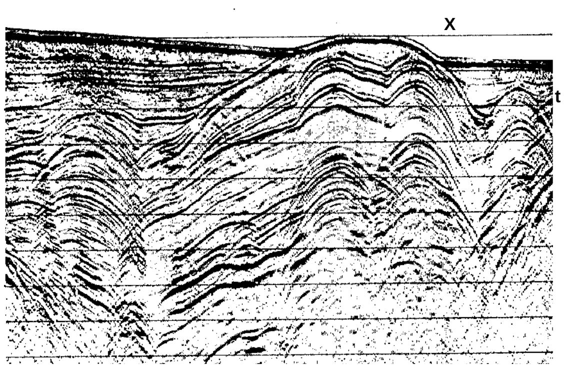
Real Seismic Data
Inversion by Kirchhoff Migration

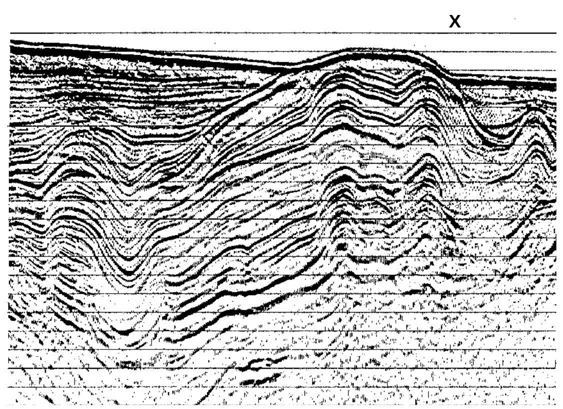




J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

Real Seismic Data
Inversion by Depth Migration





J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

Summary

- General 3-D modeling formulas developed in similar fashion to 1-D (Helmholtz equations, Green's theorem, Born approximation)
- Kirchhoff modeling formula introduced
- 2.5 D based on 3-D wave equation
- Homogeneous background velocity suggests obtaining inversion formulas by FT^{-1}
- Trickier because of spreading factor $1/r$
- Results comparable to Stolt's 3-D migration
- Stationary phase apprx. → Kirchhoff migration

Kirchhoff Scattering Inversion:
III. General Recording Geometries

Chuck Ursenbach, CREWES
Seismic Imaging Summer School
University of Calgary
August 9, 2006

The Laws of Seismic Inversion

1. You can only invert as well as you can model
2. You can't ever invert quite as well as you can model
3. The ideal inversion would involve complete coverage, full bandwidth, infinite SNR, and infinite computing time

(sung to the tune of "The Laws of Thermodynamics)

Overview

- Inversion Formulas for General Geometries ⟵
- Beylkin Determinant
- Ray Theory and its Uses
- Comments on Kirchhoff methods

3-D perturbation and linearization

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega)=\omega^2\int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2}\, g(\mathbf{r},\mathbf{r}_R,\omega)\left[\psi_I(\mathbf{r},\mathbf{r}_S,\omega)+\psi_S(\mathbf{r},\mathbf{r}_S,\omega)\right]d\mathbf{r}$$

Born approximation

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega)=\omega^2\int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2}\, g(\mathbf{r},\mathbf{r}_R,\omega)\psi_I(\mathbf{r},\mathbf{r}_S,\omega)d\mathbf{r}$$

$$\psi_I(\mathbf{r},\mathbf{r}_S,\omega)=F(\omega)g(\mathbf{r},\mathbf{r}_S,\omega)$$

WKBJ approximation to Green's function

- <u>high frequency</u> , <u>slowly-varying $c(z)$</u>

$$g^{\mathrm{WKBJ}}(\mathbf{r},\mathbf{r}_R,\omega)\approx A(\mathbf{r},\mathbf{r}_R)\exp[i\omega\tau(\mathbf{r},\mathbf{r}_R)]$$

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega)=\omega^2 F(\omega)\int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2}\, A(\mathbf{r},\mathbf{r}_R,\omega)A(\mathbf{r}_S,\mathbf{r},\omega)\exp[i\omega\left(\tau(\mathbf{r},\mathbf{r}_R)+\tau(\mathbf{r}_S,\mathbf{r})\right)]d\mathbf{r}$$

Inversion Formula

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega)=\omega^2F(\omega)\int_V\frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2}\underbrace{A(\mathbf{r},\mathbf{r}_R)A(\mathbf{r}_S,\mathbf{r})}_{\equiv\, a(\mathbf{r},\mathbf{r}_S,\mathbf{r}_R)}\exp[i\omega\underbrace{(\tau(\mathbf{r},\mathbf{r}_R)+\tau(\mathbf{r}_S,\mathbf{r}))}_{\equiv\,\phi(\mathbf{r},\mathbf{r}_S,\mathbf{r}_R)}]d\mathbf{r}$$

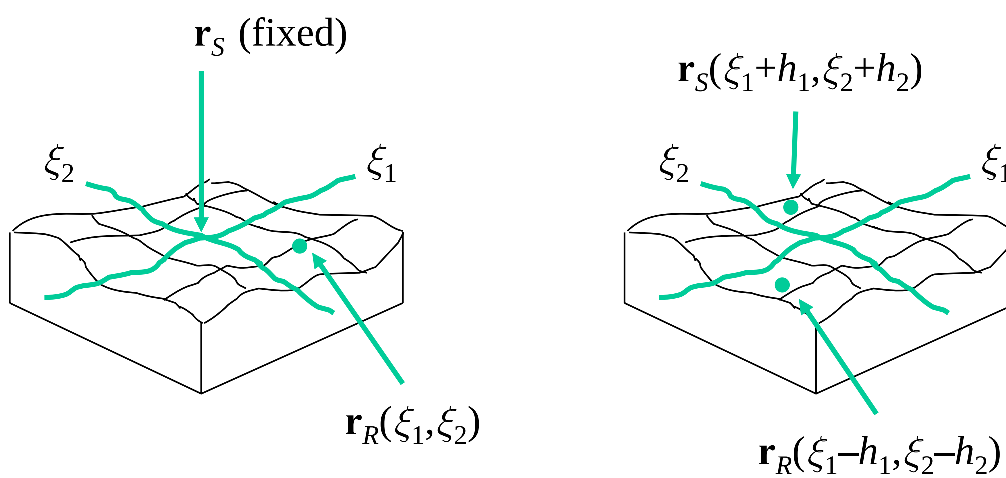
Strategy: assume inversion relation with unknown inverse transform

$$\alpha(\mathbf{r})=\int_{-\infty}^{\infty}d\omega\int d\underbrace{\{\mathbf{r}_R,\mathbf{r}_S\}}_{\text{6 variables?}}\,T^{-1}(\mathbf{r},\mathbf{r}_R,\mathbf{r}_S)\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega)\exp[-i\omega\phi(\mathbf{r},\mathbf{r}_R,\mathbf{r}_S)]$$

Source and Receiver Variables

Common shot gather

Common offset gather



From Modeling to Inversion

$$\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \underbrace{\int d^2\xi}_{\text{2 variables!}} T^{-1}(\mathbf{r},\xi)\,\psi_S(\mathbf{r}_R(\xi),\mathbf{r}_S(\xi),\omega)\exp[-i\omega\phi(\mathbf{r},\mathbf{r}_R(\xi),\mathbf{r}_S(\xi))]$$

Or, more compactly,

$$\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \int d^2\xi\, T^{-1}(\mathbf{r},\xi)\,\psi_S(\xi,\omega)\exp[-i\omega\phi(\mathbf{r},\xi)]$$

Substitute modeling formula for ψ_S in the expression above to obtain

$$\alpha(\mathbf{r}) = \int t(\mathbf{r}',\mathbf{r})\,\alpha(\mathbf{r}')\,d\mathbf{r}'$$

Bandlimited delta function (→ $t(\mathbf{r}',\mathbf{r})$)

From this ascertain T^{-1}

Exercise: Determine expression for $t(\mathbf{r}',\mathbf{r})$.

Solution

Substitute

$$\psi_S(\mathbf{r}_R,\mathbf{r}_S,\omega) = \omega^2 F(\omega)\int_V \frac{\alpha(\mathbf{r})}{c(\mathbf{r})^2}\,a(\mathbf{r},\mathbf{r}_R,\mathbf{r}_S)\exp[i\omega\phi(\mathbf{r},\mathbf{r}_R,\mathbf{r}_S)]d\mathbf{r}$$

into

$$\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \int d^2\xi\, T^{-1}(\mathbf{r},\xi)\,\psi_S(\xi,\omega)\exp[-i\omega\phi(\mathbf{r},\xi)]$$

to obtain

$$\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \int d^2\xi\, T^{-1}(\mathbf{r},\xi)\left\{\omega^2 F(\omega)\int_V \frac{\alpha(\mathbf{r}')}{c(\mathbf{r}')^2}\,a(\mathbf{r}',\mathbf{r}_R,\mathbf{r}_S)\exp[i\omega\phi(\mathbf{r}',\mathbf{r}_R,\mathbf{r}_S)]d\mathbf{r}'\right\}\exp[-i\omega\phi(\mathbf{r},\xi)]$$

$$= \int_V \alpha(\mathbf{r}')\left\{\int_{-\infty}^{\infty} d\omega\,\omega^2 F(\omega)\int d^2\xi\, T^{-1}(\mathbf{r},\xi)\frac{1}{c(\mathbf{r}')^2}\,a(\mathbf{r}',\mathbf{r}_R,\mathbf{r}_S)\exp[i\omega\phi(\mathbf{r}',\mathbf{r}_R,\mathbf{r}_S)]\exp[-i\omega\phi(\mathbf{r},\xi)]\right\}d\mathbf{r}'$$

from which we write

$$t(\mathbf{r},\mathbf{r}') = \delta_B(\mathbf{r}-\mathbf{r}') = \int_{-\infty}^{\infty} d\omega \int d^2\xi\;\; T^{-1}(\mathbf{r},\xi)\,\omega^2 F(\omega)\frac{a(\mathbf{r}',\xi)}{c(\mathbf{r}')^2}\exp[i\omega(\phi(\mathbf{r}',\xi)-\phi(\mathbf{r},\xi))]$$

Zero-phase → Main Contribution

Formal definition: $\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3k \, \exp[i\mathbf{k}\bullet(\mathbf{r}'-\mathbf{r})]$

From previous slide (without band-limiting $F(\omega)$):

$$\delta(\mathbf{r}-\mathbf{r}') = \int_{-\infty}^{\infty} d\omega \int d^2\xi \;\; T^{-1}(\mathbf{r},\xi)\,\omega^2 \frac{a(\mathbf{r}',\xi)}{c(\mathbf{r}')^2}\exp[i\omega(\phi(\mathbf{r}',\xi)-\phi(\mathbf{r},\xi))]$$

Maximum contribution for phase = 0

$\mathbf{r}' \sim \mathbf{r}$

$$\frac{a(\mathbf{r}',\xi)}{c(\mathbf{r}')^2} \approx \frac{a(\mathbf{r},\xi)}{c(\mathbf{r})^2}$$

$$i\omega(\phi(\mathbf{r}',\xi)-\phi(\mathbf{r},\xi)) = i\omega\nabla_{\mathbf{r}'}\phi(\mathbf{r}',\xi)\,|_{\mathbf{r}'=\mathbf{r}}\bullet(\mathbf{r}'-\mathbf{r})$$

Has wavenumber units → $\equiv \mathbf{k}$

$$\int d^3k \quad \to \quad \int_{-\infty}^{\infty} d\omega \int d^2\xi \left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right|$$

Exercise: Derive expression for $T^{-1}(\mathbf{r},\xi)$ and thus obtain an inversion relation.

Solution

$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3k \, \exp[i\mathbf{k}\bullet(\mathbf{r}'-\mathbf{r})]$ may be written as

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \int d^2\xi \left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right| \exp[i\omega\nabla_{\mathbf{r}'}\phi(\mathbf{r}',\xi)\,|_{\mathbf{r}'=\mathbf{r}}\bullet(\mathbf{r}'-\mathbf{r})]$$

Compare this to

$$\delta(\mathbf{r}-\mathbf{r}') = \int_{-\infty}^{\infty} d\omega \int d^2\xi \;\; T^{-1}(\mathbf{r},\xi)\,\omega^2 \frac{a(\mathbf{r}',\xi)}{c(\mathbf{r}')^2}\exp[i\omega(\phi(\mathbf{r}',\xi)-\phi(\mathbf{r},\xi))]$$

to obtain

$$T^{-1}(\mathbf{r},\xi)\,\omega^2\frac{a(\mathbf{r}',\xi)}{c(\mathbf{r}')^2} \approx \frac{1}{(2\pi)^3}\left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right| \qquad T^{-1}(\mathbf{r},\xi) \approx \frac{1}{\omega^2}\frac{c(\mathbf{r}')^2}{a(\mathbf{r}',\xi)}\frac{1}{(2\pi)^3}\left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right|$$

Substitute:

$$\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} d\omega \int d^2\xi \, T^{-1}(\mathbf{r},\xi)\,\psi_S(\xi,\omega)\exp[-i\omega\phi(\mathbf{r},\xi)]$$

$$\alpha(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2}\int d^2\xi \frac{c(\mathbf{r})^2}{a(\mathbf{r},\xi)}\left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right|\psi_S(\xi,\omega)\exp[-i\omega\phi(\mathbf{r},\xi)]$$

Overview

- Inversion Formulas for General Geometries
- Beylkin Determinant ←
- Ray Theory and its Uses
- Comments on Kirchhoff methods

Beylkin Determinant

$$\alpha(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2}\int d^2\xi \frac{c(\mathbf{r})^2}{a(\mathbf{r},\boldsymbol{\xi})}\left|\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)}\right| \psi_S(\boldsymbol{\xi},\omega)\exp[-i\omega\phi(\mathbf{r},\boldsymbol{\xi})]$$

$$\mathbf{k} = \omega\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)$$

$$\frac{\partial(k_1,k_2,k_3)}{\partial(\omega,\xi_1,\xi_2)} = \begin{vmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ \omega\frac{\partial^2\phi}{\partial x\partial\xi_1} & \omega\frac{\partial^2\phi}{\partial y\partial\xi_1} & \omega\frac{\partial^2\phi}{\partial z\partial\xi_1} \\ \omega\frac{\partial^2\phi}{\partial x\partial\xi_2} & \omega\frac{\partial^2\phi}{\partial y\partial\xi_2} & \omega\frac{\partial^2\phi}{\partial z\partial\xi_2} \end{vmatrix} = \omega^2 \begin{vmatrix} \nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \\ \frac{\partial}{\partial\xi_1}\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \\ \frac{\partial}{\partial\xi_2}\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \end{vmatrix} \equiv \omega^2 h(\mathbf{r},\xi)$$

The Beylkin determinant is defined as $1/\omega^2$ times the Jacobian of the transformation from wavenumber space to (ω, ξ_1, ξ_2)

Inversion Formula

The determinant h must be non-zero for some range of (ξ_1, ξ_2) for each desired value of $\mathbf{r}$ to be imaged.

$$\alpha(\mathbf{r}) = \frac{1}{8\pi^3}\int d^2\xi \frac{|h(\mathbf{r},\xi)|\,c(\mathbf{r})^2}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\ \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

$$\times i\omega\left|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)\right|$$

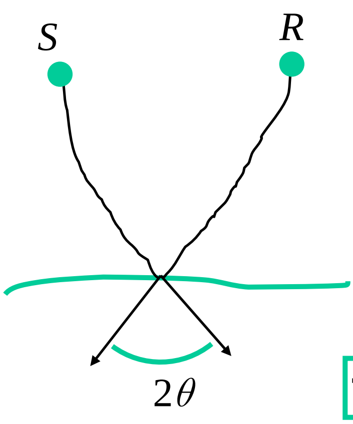
$$\text{cf. } (\pm ik)$$

$$\text{cf. } \frac{2i\omega}{c_0}$$

$$\times\frac{1}{c(\mathbf{r})^2\left|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)\right|^2} \quad \left(=\frac{1}{4} \text{ for 1D}\right)$$

$$\beta(\mathbf{r}) = \frac{1}{8\pi^3}\int d^2\xi \frac{|h(\mathbf{r},\xi)|}{a(\mathbf{r},\xi)\,|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)|}\int_{-\infty}^{\infty} d\omega\ \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

Defining a new reflectivity function



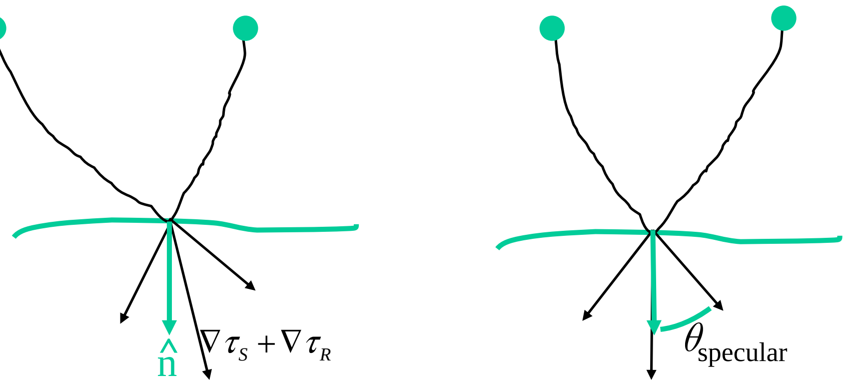
$$\left|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)\right|^2$$
$$= \left|\nabla_{\mathbf{r}}\tau(\mathbf{r},\mathbf{r}_S)\right|^2 + \left|\nabla_{\mathbf{r}}\tau(\mathbf{r},\mathbf{r}_R)\right|^2 + 2\nabla_{\mathbf{r}}\tau(\mathbf{r},\mathbf{r}_S)\bullet\nabla_{\mathbf{r}}\tau(\mathbf{r},\mathbf{r}_R)$$
$$= \frac{1}{c(\mathbf{r})^2} + \frac{1}{c(\mathbf{r})^2} + \frac{2\cos 2\theta}{c(\mathbf{r})^2} = \left(\frac{2\cos\theta}{c(\mathbf{r})}\right)^2$$

Thus $c(\mathbf{r})^2\left|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)\right|^2 = 4\cos^2\theta \quad (= 4 \text{ for normal incidence})$

$$\beta(\mathbf{r}) = \frac{1}{8\pi^3}\int d^2\xi \frac{|h(\mathbf{r},\xi)|}{a(\mathbf{r},\xi)\,|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)|}\int_{-\infty}^{\infty} d\omega\ \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_G,\omega)$$

$$\beta_1(\mathbf{r}) = \frac{1}{8\pi^3}\int d^2\xi \frac{|h(\mathbf{r},\xi)|}{a(\mathbf{r},\xi)\,|\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi)|^2}\int_{-\infty}^{\infty} d\omega\ \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_G,\omega)$$

Determining $R_B(\theta_{\text{specular}})$, $\cos\theta_{\text{specular}}$



Using Kirchhoff-approximate data:

$$\beta^{\text{PEAK}}(\mathbf{r}) \sim R(\mathbf{r},\theta_{\text{specular}})\,\frac{\cos\theta_{\text{specular}}}{\pi c(\mathbf{r})}\int_{-\infty}^{\infty} d\omega\, F(\omega)$$

$$\beta_1^{\text{PEAK}}(\mathbf{r}) \sim R(\mathbf{r},\theta_{\text{specular}})\,\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega)$$

$$\rightarrow \quad R_B(\mathbf{r},\theta_{\text{specular}}), \quad 2\frac{\cos\theta_{\text{specular}}}{c(\mathbf{r})}$$

Notation for Beylkin Determinant

$$h(\mathbf{r},\xi) = \begin{vmatrix} \nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \\ \dfrac{\partial}{\partial\xi_1}\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \\ \dfrac{\partial}{\partial\xi_2}\nabla_{\mathbf{r}}\phi(\mathbf{r},\xi) \end{vmatrix} \equiv \begin{vmatrix} \mathbf{p}_S + \mathbf{p}_R \\ \dfrac{\partial(\mathbf{p}_S + \mathbf{p}_R)}{\partial\xi_1} \\ \dfrac{\partial(\mathbf{p}_S + \mathbf{p}_R)}{\partial\xi_2} \end{vmatrix} \equiv \begin{vmatrix} \mathbf{p}_S + \mathbf{p}_R \\ \mathbf{v}_S + \mathbf{v}_R \\ \mathbf{w}_S + \mathbf{w}_R \end{vmatrix} = (\mathbf{p}_S + \mathbf{p}_R)\bullet\left[(\mathbf{v}_S + \mathbf{v}_R)\times(\mathbf{w}_S + \mathbf{w}_R)\right]$$

Note: $\mathbf{p}_S \parallel (\mathbf{v}_S \times \mathbf{w}_S)$

Proof: $\mathbf{p}_S\bullet\mathbf{v}_S = \mathbf{p}_S\bullet\dfrac{\partial\mathbf{p}_S}{\partial\xi_1} = \dfrac{1}{2}\dfrac{\partial(\mathbf{p}_S\bullet\mathbf{p}_S)}{\partial\xi_1} = \dfrac{1}{2}\dfrac{\partial\left(1/c(\mathbf{r})^2\right)}{\partial\xi_1} = 0$

So $\mathbf{p}_S \perp \mathbf{v}_S$ and $\mathbf{p}_S \perp \mathbf{w}_S$

Hence: $\mathbf{p}_S\bullet(\mathbf{v}_S\times\mathbf{w}_S) = \pm\dfrac{|\mathbf{v}_S\times\mathbf{w}_S|}{c(\mathbf{r})}$

$\mathbf{p}_R\bullet(\mathbf{v}_S\times\mathbf{w}_S) = \pm\dfrac{|\mathbf{v}_S\times\mathbf{w}_S|}{c(\mathbf{r})}\cos 2\theta$

Beylkin Determinant for common-shot

$$\left|\begin{matrix}\mathbf{p}_S+\mathbf{p}_R\\ \dfrac{\partial(\mathbf{p}_S+\mathbf{p}_R)}{\partial\xi_1}\\ \dfrac{\partial(\mathbf{p}_S+\mathbf{p}_R)}{\partial\xi_2}\end{matrix}\right| \equiv \left|\begin{matrix}\mathbf{p}_S+\mathbf{p}_R\\ \mathbf{v}_R\\ \mathbf{w}_R\end{matrix}\right| = \mathbf{p}_S\bullet(\mathbf{v}_R\times\mathbf{w}_R)+\mathbf{p}_R\bullet(\mathbf{v}_R\times\mathbf{w}_R)$$

$$= \pm\frac{1+\cos 2\theta}{c(\mathbf{r})}\,|\,\mathbf{v}_R\times\mathbf{w}_R\,|$$

$$= \pm 2\frac{\cos^2\theta}{c(\mathbf{r})}\,|\,\mathbf{v}_R\times\mathbf{w}_R\,|$$

$$= \pm 2\cos^2\theta\,\mathbf{p}_R\bullet(\mathbf{v}_R\times\mathbf{w}_R)$$

$$= \pm 2\cos^2\theta\,h_R(\mathbf{r},\xi)$$

Inversion formula for common-shot

$$\beta(\mathbf{r}) = \frac{c(\mathbf{r})}{8\pi^3}\int d^2\xi\,\frac{\cos\theta\,|\,h_R(\mathbf{r},\xi)\,|}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

$$\beta_1(\mathbf{r}) = \frac{c(\mathbf{r})^2}{16\pi^3}\int d^2\xi\,\frac{|\,h_R(\mathbf{r},\xi)\,|}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

Beylkin Determinant for zero-offset

$$\begin{vmatrix} \mathbf{p}_S + \mathbf{p}_R \\ \dfrac{\partial(\mathbf{p}_S + \mathbf{p}_R)}{\partial \xi_1} \\ \dfrac{\partial(\mathbf{p}_S + \mathbf{p}_R)}{\partial \xi_2} \end{vmatrix} = \begin{vmatrix} 2\mathbf{p}_S \\ 2\dfrac{\partial \mathbf{p}_S}{\partial \xi_1} \\ 2\dfrac{\partial \mathbf{p}_S}{\partial \xi_2} \end{vmatrix} \equiv 8\begin{vmatrix} \mathbf{p}_S \\ \mathbf{v}_S \\ \mathbf{w}_S \end{vmatrix} = 8h_S(\mathbf{r},\xi) = 8h_R(\mathbf{r},\xi)$$

Inversion formula for zero-offset

$$\beta(\mathbf{r}) = \frac{c(\mathbf{r})}{2\pi^3}\int d^2\xi \frac{\cos\theta \,|\, h_R(\mathbf{r},\xi)\,|}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

$$\beta_1(\mathbf{r}) = \frac{c(\mathbf{r})^2}{4\pi^3}\int d^2\xi \frac{|\, h_R(\mathbf{r},\xi)\,|}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

No new information

Inversion formula for zero-offset

$$\beta(\mathbf{r}) = \frac{c(\mathbf{r})}{2\pi^3}\int d^2\xi \frac{|h_R(\mathbf{r},\xi)|}{a(\mathbf{r},\xi)}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-i\omega\phi(\mathbf{r},\xi)]\psi_S(\mathbf{r}_S,\mathbf{r}_R,\omega)$$

$$c(\mathbf{r}) \to c_0$$

$$(\xi_1,\xi_2) = (x_{SR}, y_{SR}, 0)$$

$$\beta(\mathbf{r}) = \frac{8z}{\pi c_0}\int d^2\xi \frac{1}{r}\int_{-\infty}^{\infty} d\omega\, i\omega \exp[-2i\omega r / c_0]\psi_S(\xi,\omega)$$

Compare to result in lecture 2

Beylkin Determinant

- Jacobian of transformation between surface measurements and subsurface image
- Replaces surface coordinates with dip angles
- Maps area on surface to area on unit sphere around image point
- May be specialized to various shot/receiver configurations
- Corrects for irregularity of illumination and/or acquisition
- True-amplitude migration weights (Jaramillo et al., 2000)
- Product of kernals for asymptotically inverse operators (Beylkin, 1985)
- Must be invertible for inverse of Kirchhoff modeling operator to exist
- Plays a key role in Kirchhoff data mapping

Overview

- Inversion Formulas for General Geometries
- Beylkin Determinant
- Ray Theory and its Uses ⟵
- Comments on Kirchhoff methods

Ray Theory

Helmholtz Equation ⟶ $\left[\nabla^2 + \frac{\omega^2}{v(\mathbf{r})^2}\right] g(\mathbf{r}-\mathbf{r}_0,\omega) = \delta(\mathbf{r}-\mathbf{r}_0)$

+ WKBJ Approximation ⟶ $\psi(\mathbf{r},\omega) = A(\mathbf{r},\omega)\exp[i\omega\tau(\mathbf{r})]$

= Eikonal Equation ⟶ $|\nabla\tau(\mathbf{r})|^2 = \frac{1}{c(\mathbf{r})^2}$

+ Transport Equation ⟶ $2\nabla A \bullet \nabla\tau + A\nabla^2\tau = 0$

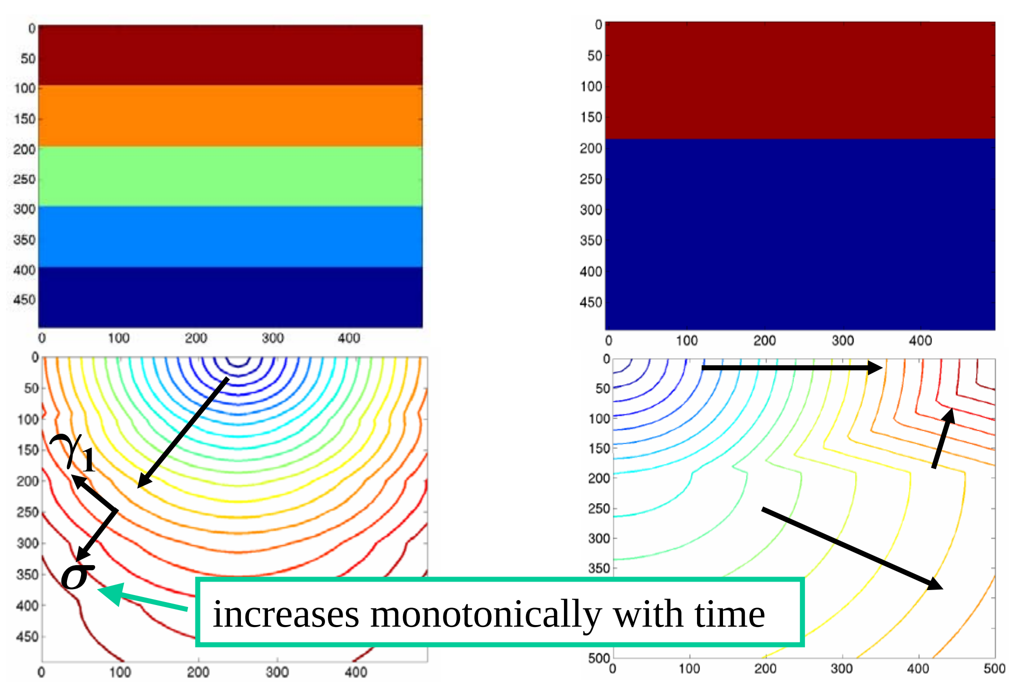
+ higher-order equations

Traveltime calculations

Eikonal Equation $$|\nabla \tau(\mathbf{r})|^2 = \frac{1}{c(\mathbf{r})^2}$$

- Can use the method of characteristics to obtain equal-traveltime surfaces, given a traveltime of zero at the source.
- Gradients normal to these surfaces give rise to paths through the system which are solutions of ODEs. These purely mathematical entities are known as raypaths.
- Three variables (**r**) become two parameters to label the raypath (e.g., θ,ϕ or p_1,p_2 at the source) and one variable to track progress along the ray (variable of the ODE).

Traveltime Calculations



Perez & Bancroft, CREWES Research Report, 2000, vol. 12, ch. 13

Amplitude Calculations

Transport Equation $2\nabla A \cdot \nabla\tau + A\nabla^2\tau = 0$

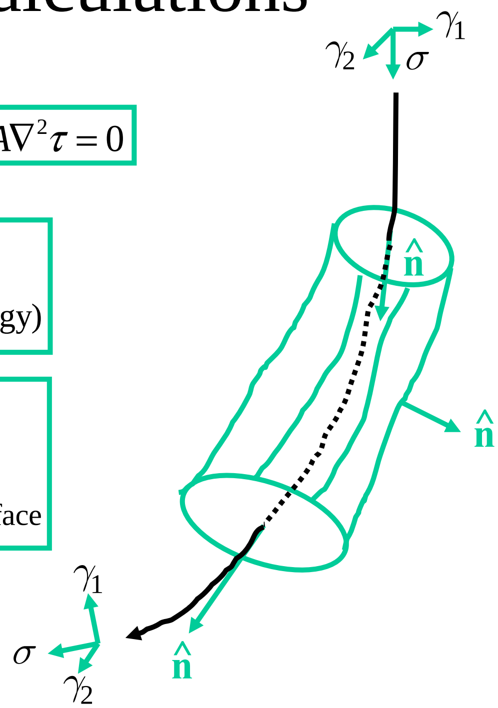
$$A \times \left(2\nabla A \cdot \nabla\tau + A\nabla^2\tau = 0\right)$$

$$\rightarrow \quad \nabla\cdot\left(A^2\nabla\tau\right) = 0 \qquad \text{(conservation of energy)}$$

$$\int_D dV\ \nabla\cdot\left(A^2\nabla\tau\right) = \int_{\partial D} dS\ \left(A^2\nabla\tau\right)\cdot\hat{\mathbf{n}} = 0$$

$$= \int \text{Sides} - \int \text{Entry surface} + \int \text{Exit surface}$$

0



Amplitude Calculations

$$\int_{\partial D} dS\ \left(A^2\nabla\tau\right)\cdot\hat{\mathbf{n}} = 0$$

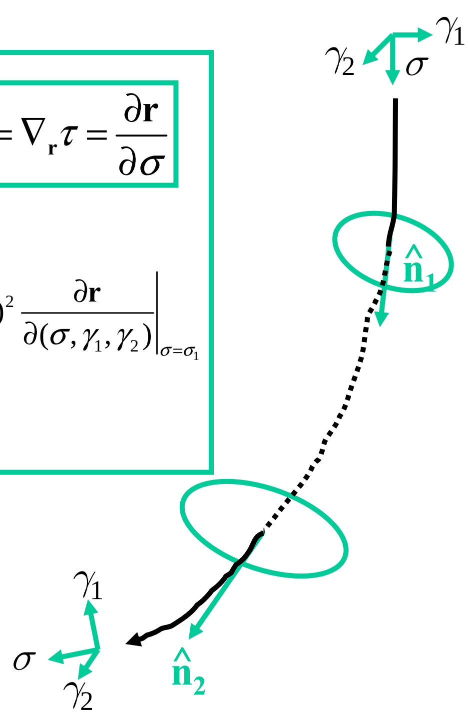
$$\mathbf{p} = \nabla_{\mathbf{r}}\tau = \frac{\partial\mathbf{r}}{\partial\sigma}$$

$$= \int dS_2\ \left(A^2\mathbf{p}\right)\cdot\hat{\mathbf{n}}_2 \ - \ \int dS_1\ \left(A^2\mathbf{p}\right)\cdot\hat{\mathbf{n}}_1$$

$$= \int d\gamma_1 d\gamma_2\ A(\sigma_2)^2 \left.\frac{\partial\mathbf{r}}{\partial(\sigma,\gamma_1,\gamma_2)}\right|_{\sigma=\sigma_2} - \int d\gamma_1 d\gamma_2\ A(\sigma_1)^2 \left.\frac{\partial\mathbf{r}}{\partial(\sigma,\gamma_1,\gamma_2)}\right|_{\sigma=\sigma_1}$$

$$\equiv \int d\gamma_1 d\gamma_2\ \left[A(\sigma_2)^2 J(\sigma_2) - A(\sigma_1)^2 J(\sigma_1)\right] = 0$$

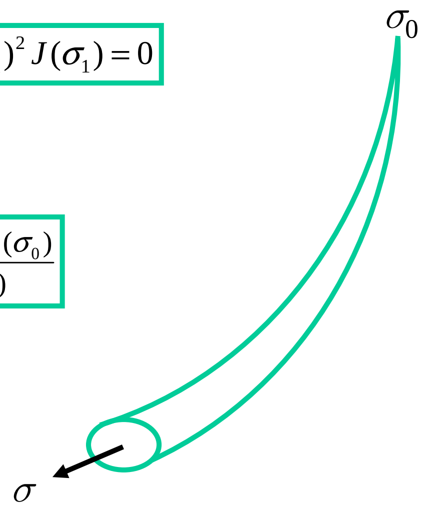
Ray Jacobian



Amplitude Calculations

$$A(\sigma_2)^2 J(\sigma_2) - A(\sigma_1)^2 J(\sigma_1) = 0$$

$$A(\sigma)^2 = \frac{A(\sigma_0)^2 J(\sigma_0)}{J(\sigma)}$$



$A(\sigma_0)$ is singular – obtain $A(\sigma_0)^2\, J(\sigma_0)$ using limits from homogeneous Green's function

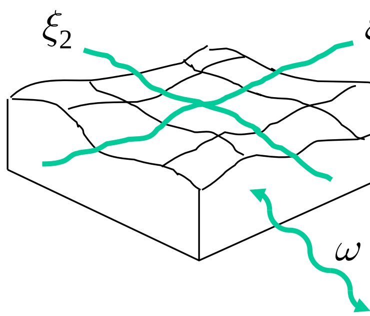
Uses of Ray Theory

- Given $\tau(\mathbf{r})$ and $A(\mathbf{r})$ we can construct the WKBJ approximation to the Green's function for use in the inversion formula
- $J(\mathbf{r})$ used to calculate $A(\mathbf{r})$ can also be used to approximate the Beylkin determinant

Relating Beylkin and Ray Jacobians

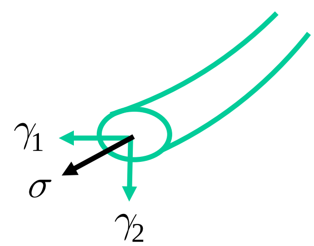
Beylkin Determinant

$$h(\mathbf{r},\xi) = \frac{1}{\omega^2}\frac{\partial \mathbf{k}}{\partial(\omega,\xi_1,\xi_2)}$$



Ray Jacobian

$$J(\mathbf{r},\xi) = \frac{\partial \mathbf{r}}{\partial(\sigma,\gamma_1,\gamma_2)}$$



$h_R(\mathbf{r},\xi)J_R(\mathbf{r},\xi) =$ function of ray and surface parametrizations

Overview

- Inversion Formulas for General Geometries
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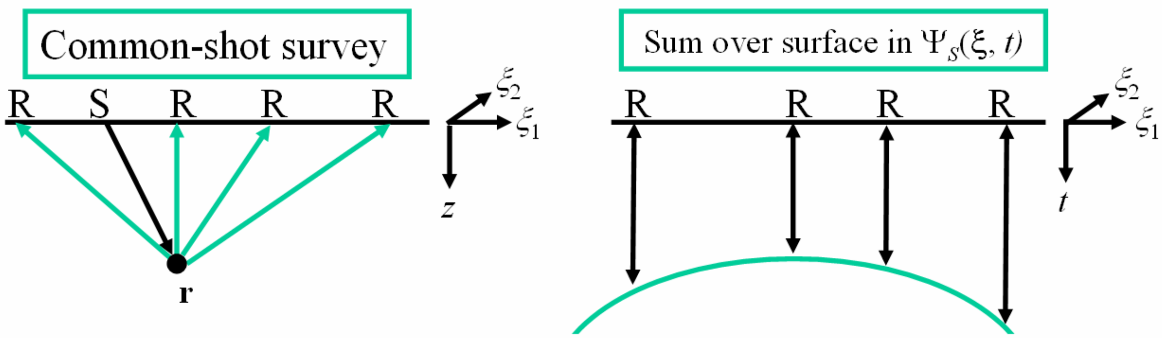
Kirchhoff Inversion as a weighted sum

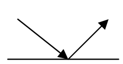
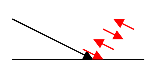
$$\alpha(\mathbf{r}) = \frac{1}{8\pi^3}\int d^2\xi \, \frac{|h(\mathbf{r},\xi)|\, c(\mathbf{r})^2}{a(\mathbf{r},\xi)} \int_0^\infty d\omega \;\; \exp[i\omega\phi(\mathbf{r},\xi)]\psi_S(\xi,\omega)$$

$$= \frac{1}{8\pi^3}\int d^2\xi \, \frac{|h(\mathbf{r},\xi)|\, c(\mathbf{r})^2}{a(\mathbf{r},\xi)} \Psi_S\left(\xi, \tau(\mathbf{r},\mathbf{r}_S) + \tau(\mathbf{r}_R,\mathbf{r})\right)$$

$$\approx \frac{1}{8\pi^3}\sum d^2\xi \, \frac{|h(\mathbf{r},\xi)|\, c(\mathbf{r})^2}{a(\mathbf{r},\xi)} \Psi_S\left(\xi, \tau(\mathbf{r},\mathbf{r}_S) + \tau(\mathbf{r}_R,\mathbf{r})\right)$$



Extensions of Kirchhoff Migration

- Elastic wave
  - Both PP () and PS ()
- Anisotropy (VTI, TTI, others)
  - Traveltimes in homogeneous background no longer hyperbolic
- Multi-arrival
  - More accurate Green's functions
- Variable density
  - Not just velocity model

Alternatives to Kirchhoff Migration

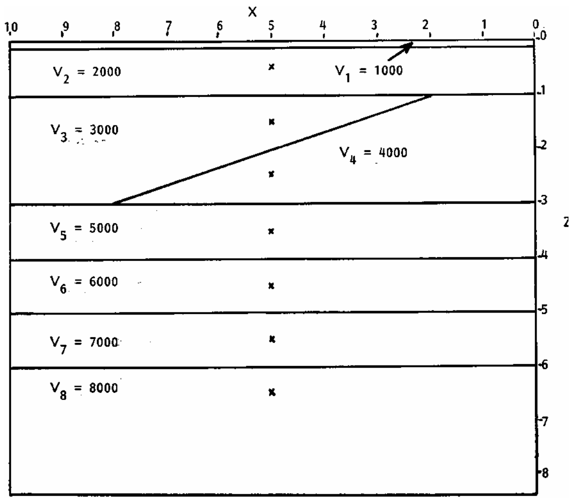
- Beam migration
  - Studies wave behavior along ray trajectory
- Wave-equation (or finite-difference) migration
  - Propagates waves back down to reflector using one-way wave equation
- Reverse-time migration
  - Uses full two-way wave equation

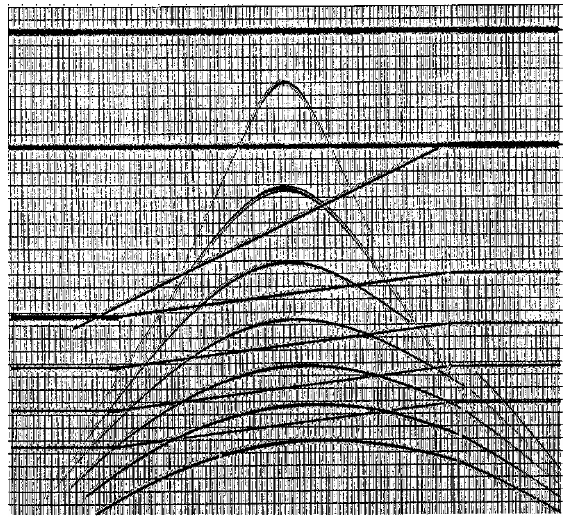
Judging Kirchhoff

- Flexibility: good at imaging irregularly sampled data; any output grid
- Computationally efficient – only common choice for 3D prestack migration

- Only useful when high frequency assumption is valid
- Suffers in presence of strong lateral velocity variations
- Migration smiles
- Limited by quality of Green's function
- Velocity smoothing trade-offs

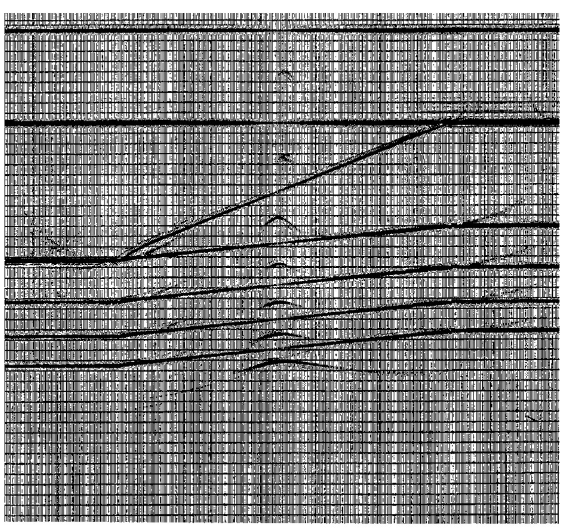
Synthetic Seismic Data
Forward Modeling by Raytracing

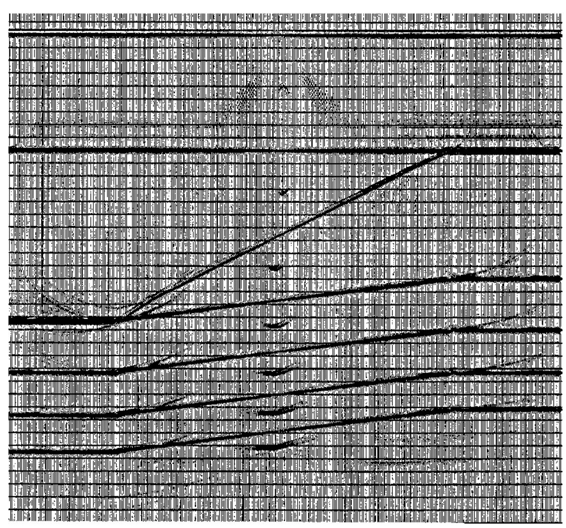




J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

Smiles and Frowns in Kirchhoff Migration



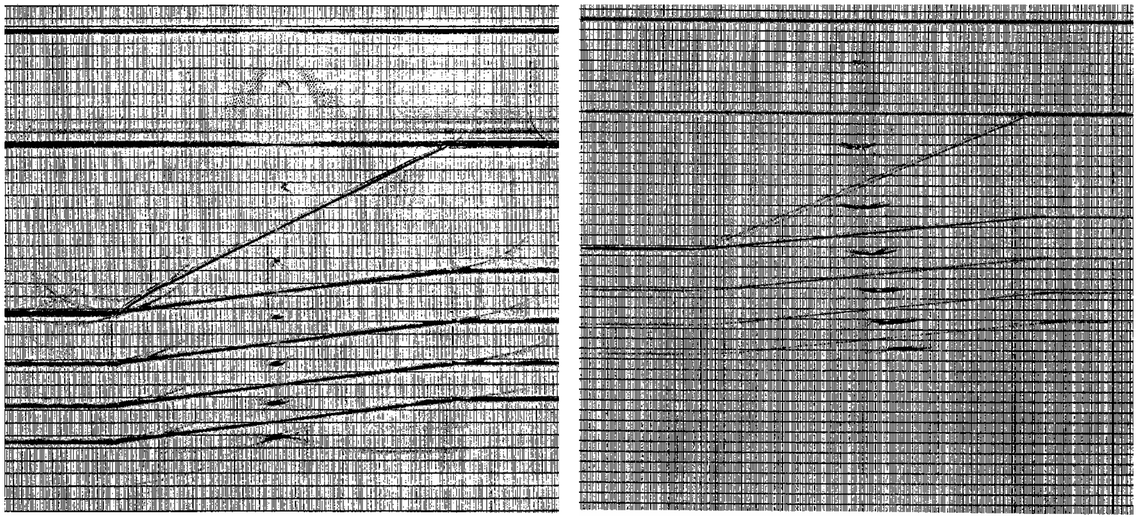


Undermigrated – background velocity too small

Overmigrated – background velocity too large

J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

Comparing Migration Methods

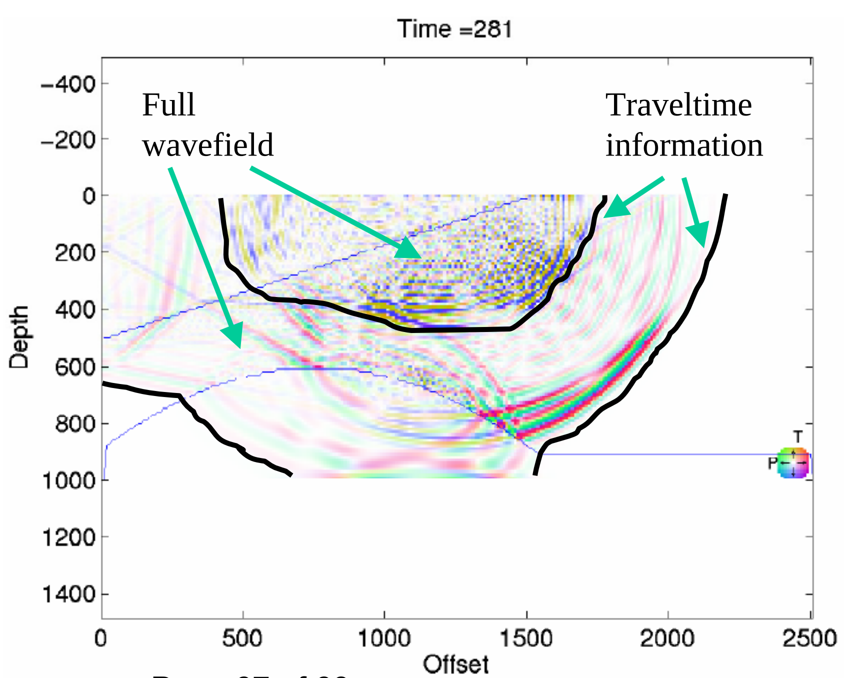


Kirchhoff migration | Wave equation migration

J. Bancroft, 2006, A Practical Understanding of Pre- and Poststack Migrations

Approximate Green's functions

- The Green's functions obtained from raytracing are much less detailed and accurate than those used in wave-equation migration
- e.g., Figure 3, Bevc & Biondi, June 2005, The Leading Edge



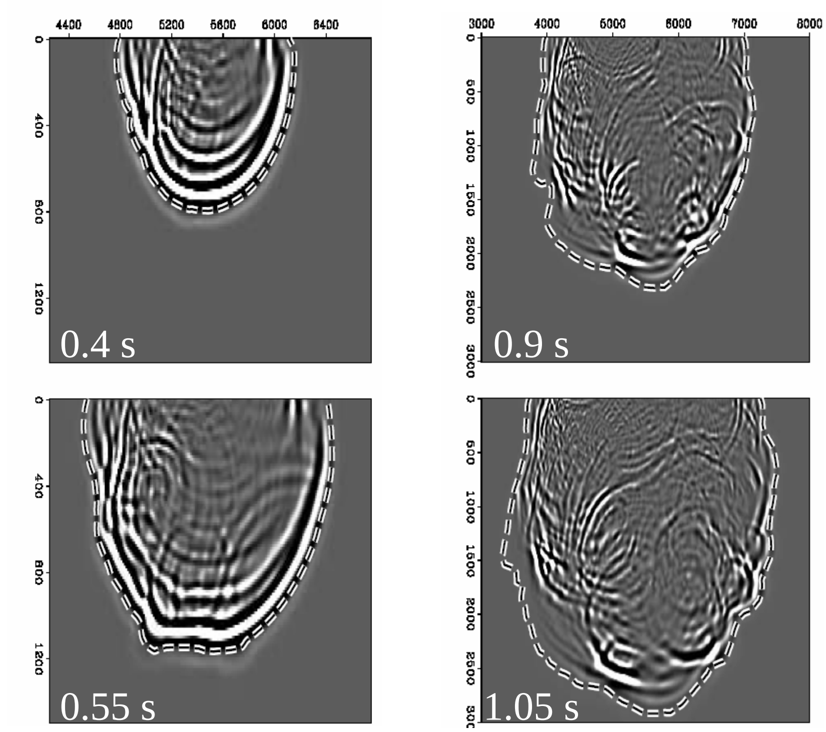
Manning & Margrave, 2004, CREWES Research Report, vol. 16, ch. 14

Approximate Green's function

Snapshots of wavefield propagation through Marmousi model.

D. Bevc, Ph.D. Thesis, Stanford, 1995

Used with permission



Summary

- One can derive a 3-D inversion formula for arbitrary recording geometries in which the influence of the geometry is contained in a Jacobian factor
- This factor (the Beylkin determinant) has a simplified form for certain geometries, e.g., common-source and zero-offset.
- Defining an additional reflectivity function allows one in principle to extract angle dependent reflectivity information.
- Dynamic raytracing (i.e., traveltimes and amplitudes) can be used to create approximate Green's functions and Beylkin determinants for arbitrary velocity models.
- The strength of the Kirchhoff method is in its speed and flexibility. Greater accuracy would require improved Green's functions.