Overview of Seismic Imaging

Presented at
Seismic Imaging Summer School
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Sponsors web pages

www.pims.math.ca

www.crewes.org

www.cseg.ca

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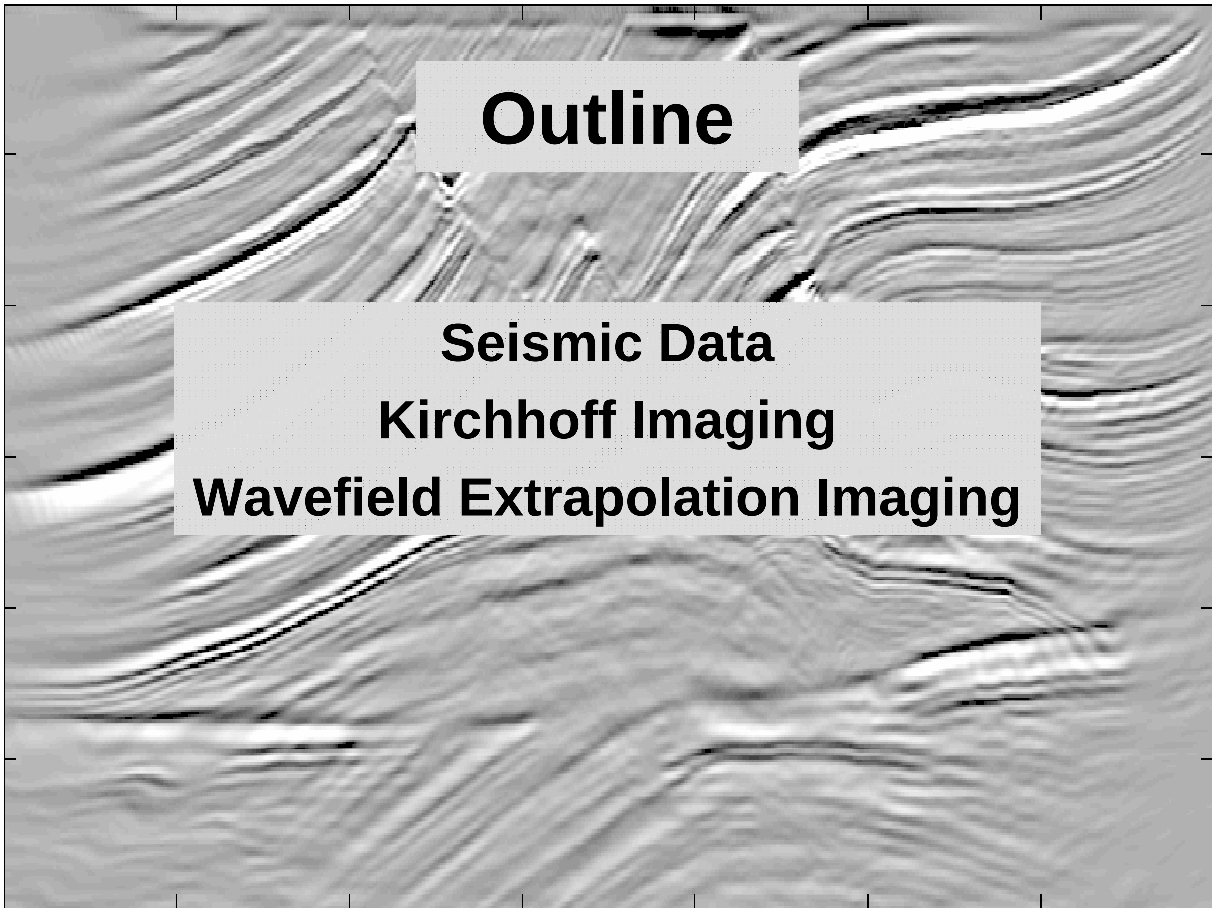
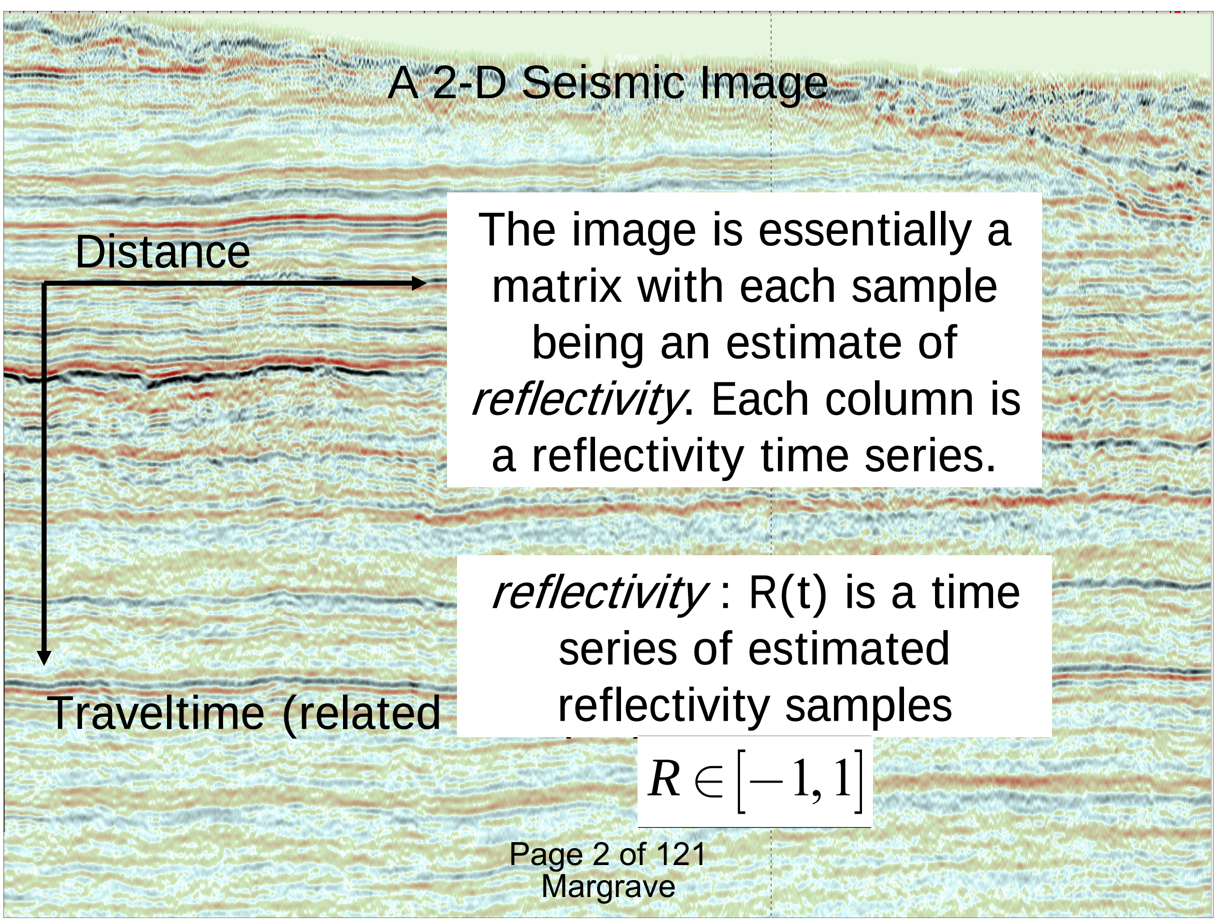
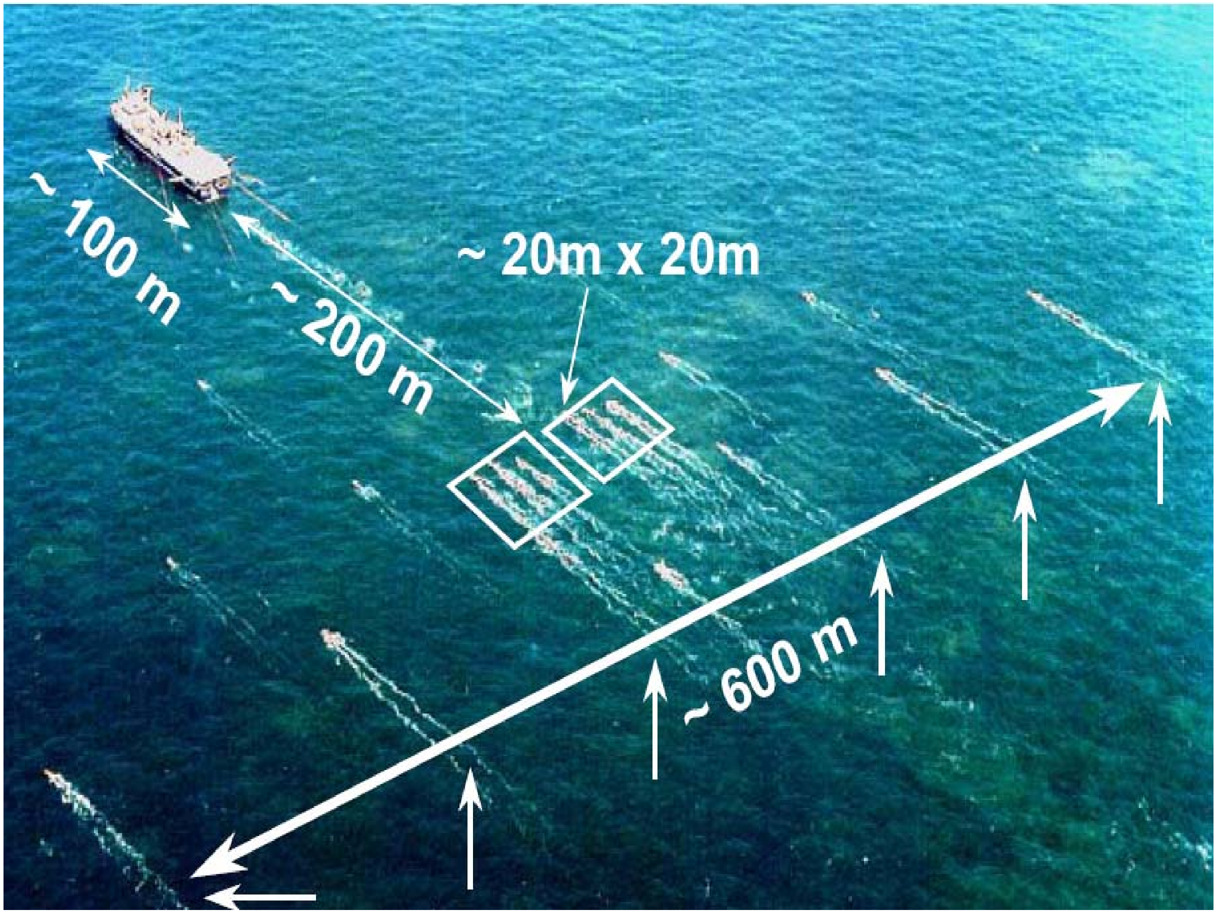
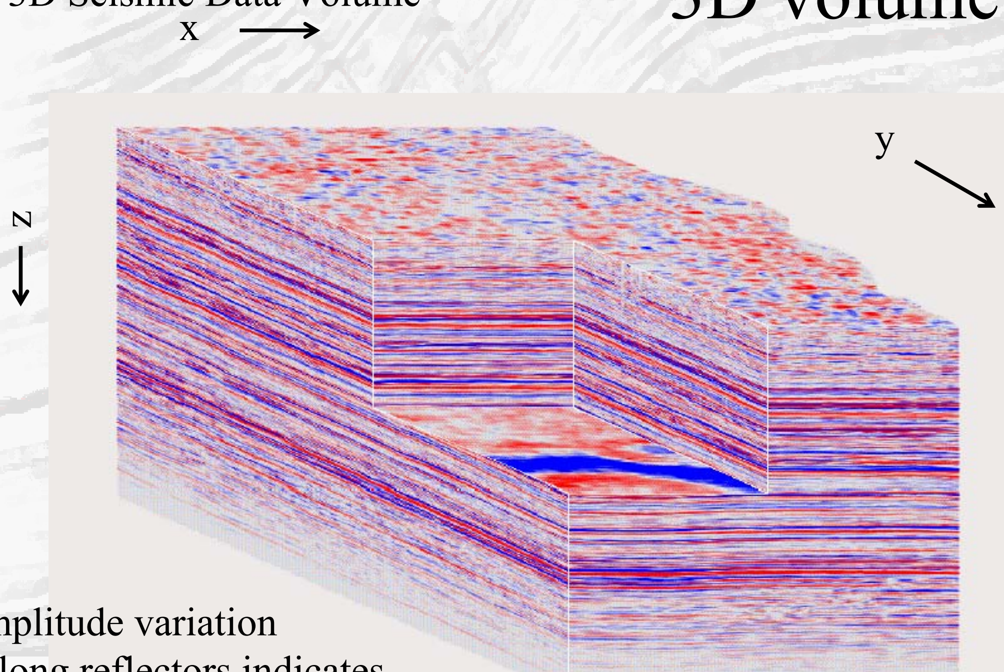

Outline
Seismic Data
Kirchhoff Imaging
Wavefield Extrapolation Imaging


A 2-D Seismic Image
Distance
Traveltime (related
The image is essentially a matrix with each sample being an estimate of *reflectivity*. Each column is a reflectivity time series.
reflectivity : R(t) is a time series of estimated reflectivity samples
$R \in [-1, 1]$



3D volume

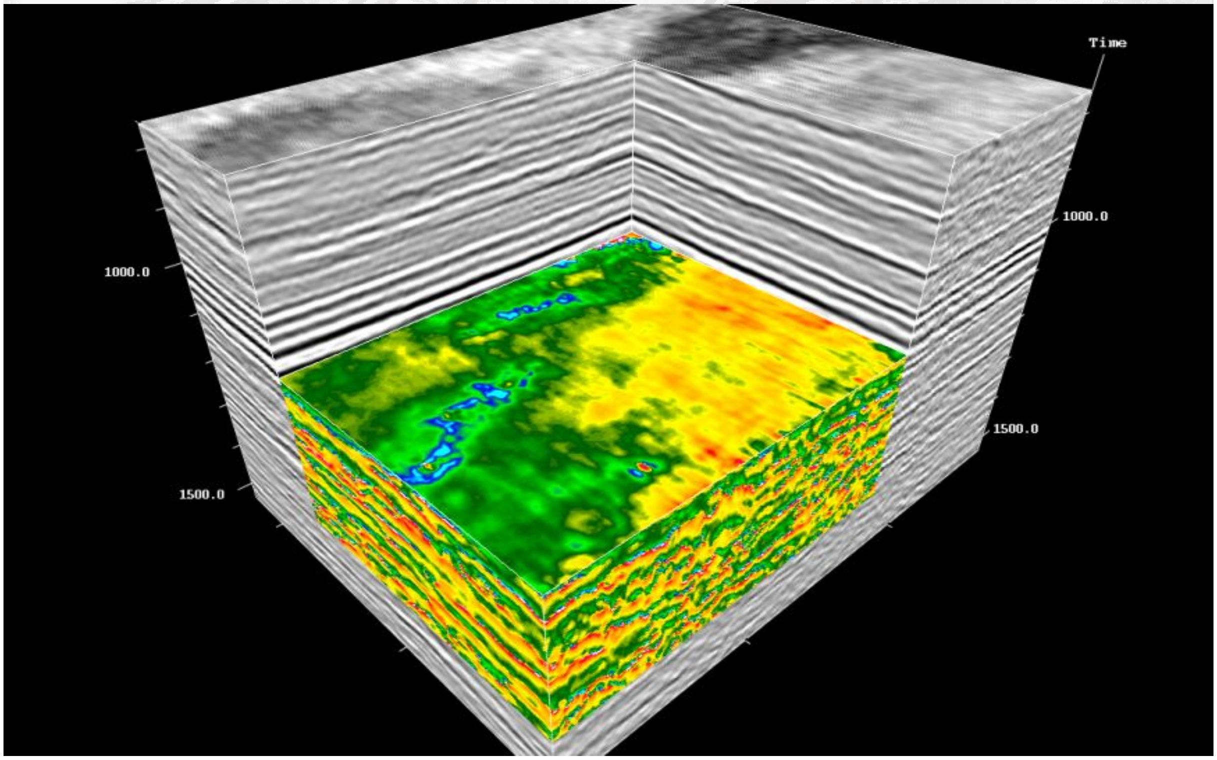
3D Seismic Data Volume



-amplitude variation along reflectors indicates
=> change in impedance contrast
=> rock and fluid properties

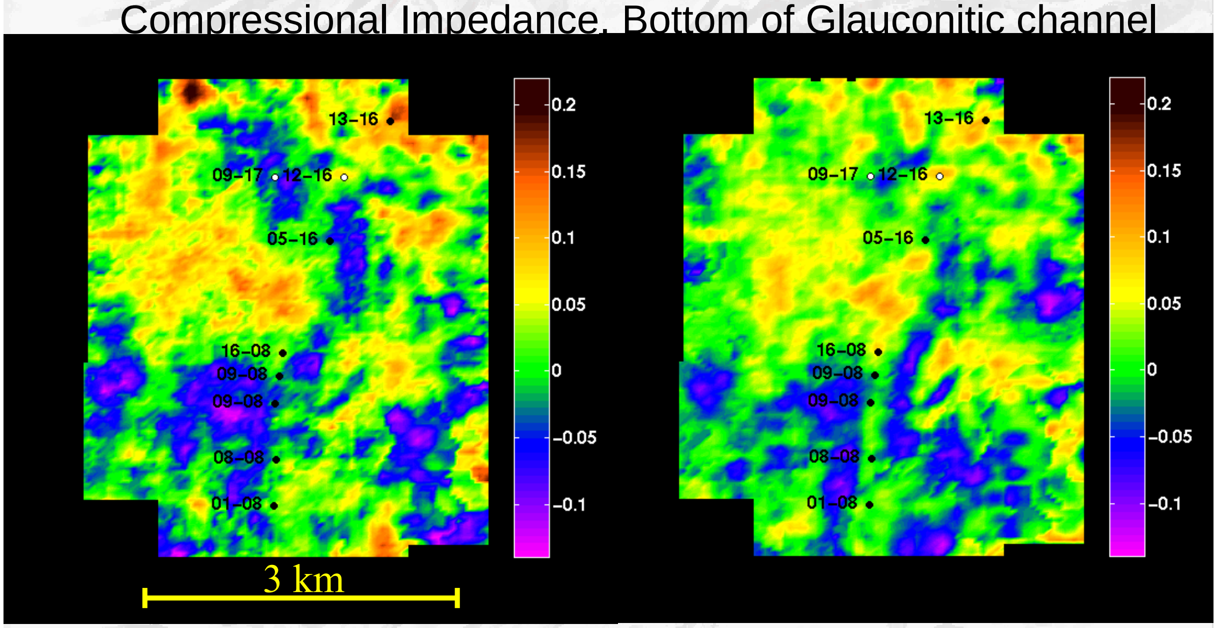
(**WesternGeco**)

3D Seismic Volume



CREWES Blackfoot Survey

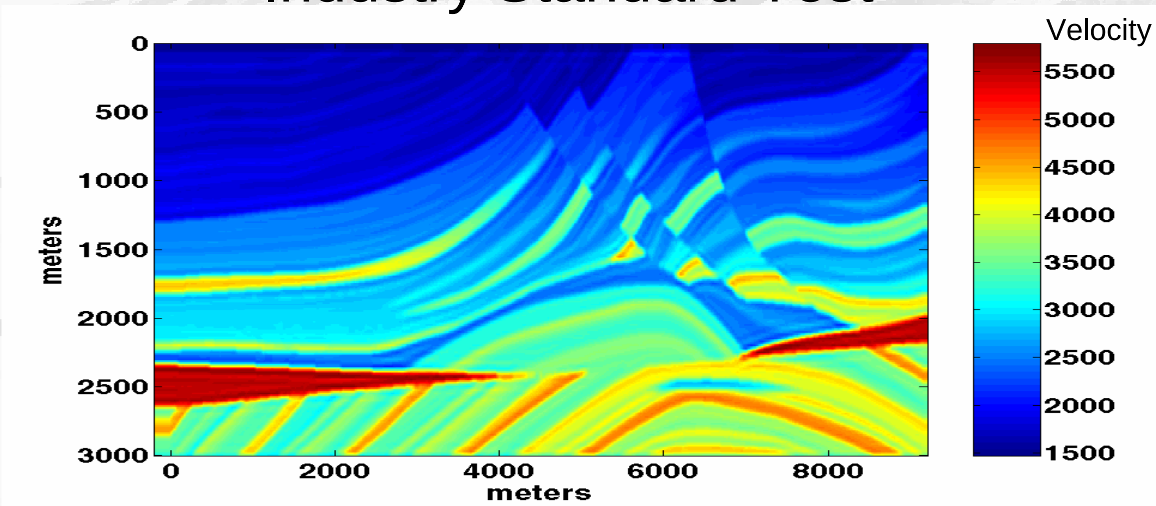
Compressional Impedance, Bottom of Glauconitic channel



P-P (1C)

Simultaneous P-P and P-S (3C)

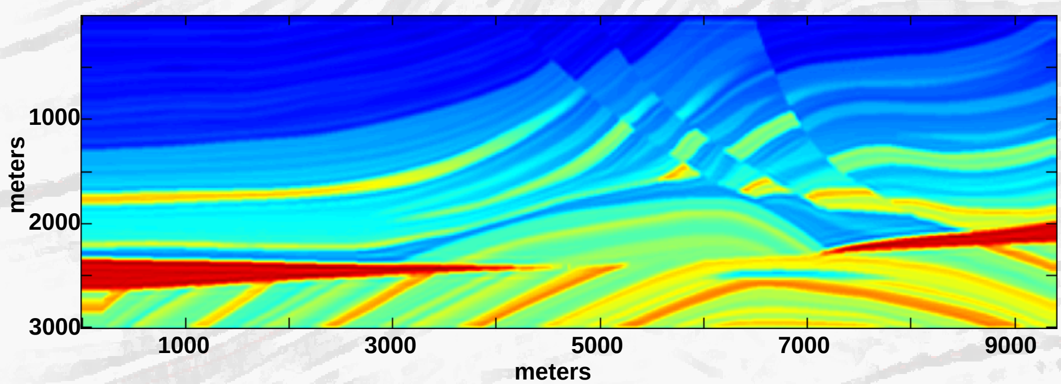
Marmousi Model
Industry Standard Test



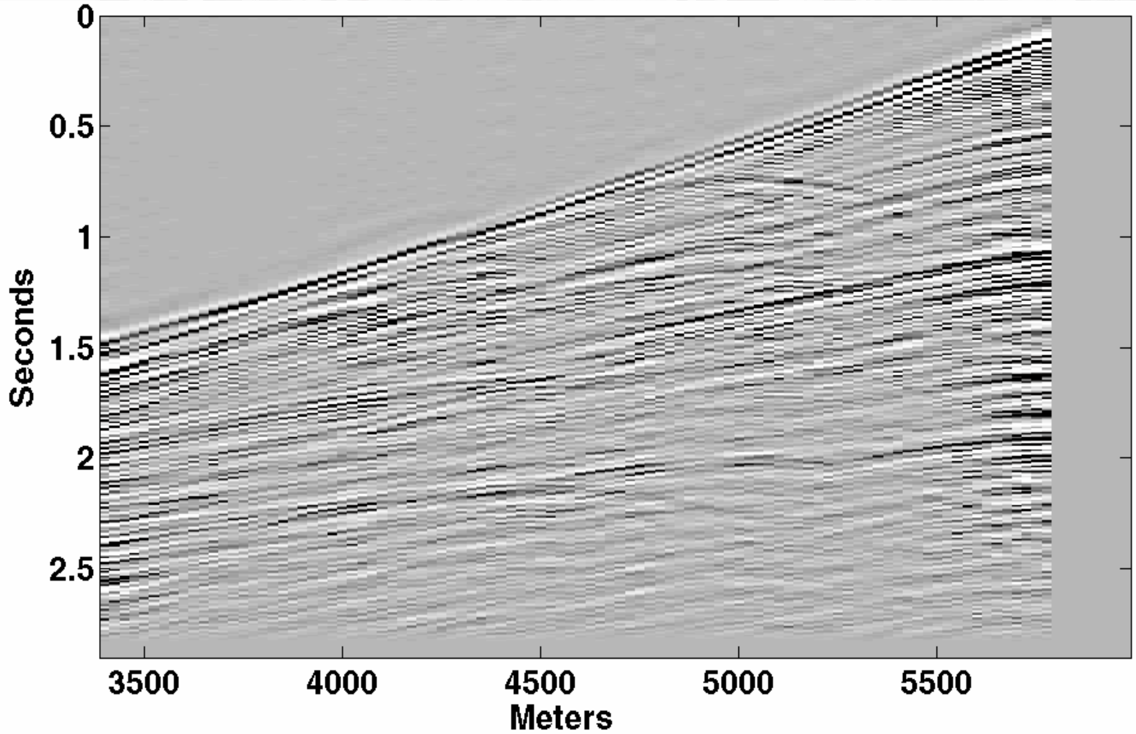
Environmental Difficulties

1) Complex, layered environments
2) Multidimensional environments
3) Inhomogeneous background
4) Large scale (many wavelengths)
5) Strongly inhomogeneous environments
6) Focusing and defocusing regimes

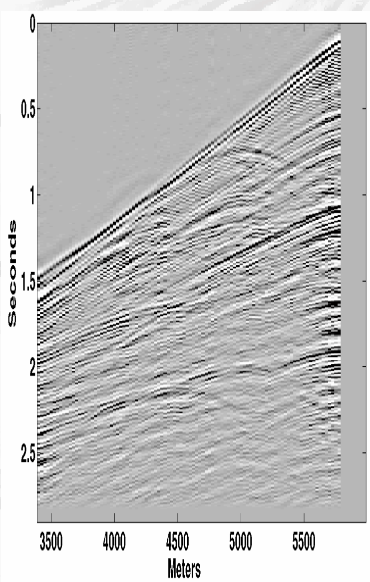
Marmousi Movie
Finite Difference Simulation



Marmousi Data



Marmousi Data



240 shots

96 receivers/shot

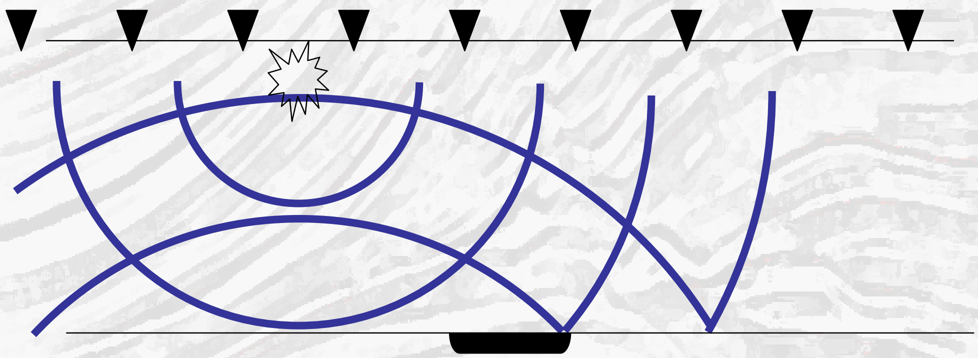
726 samples/receiver

8 bytes/samples

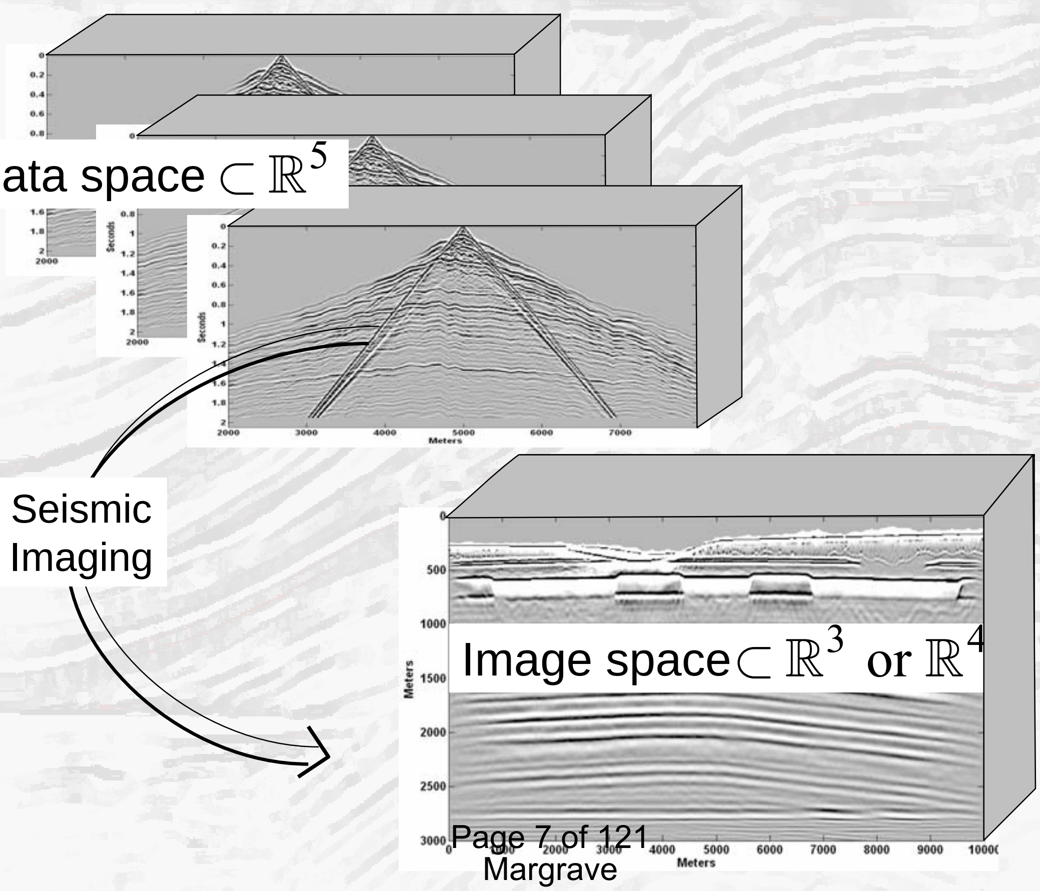
Dataset size= 240*96*726*8 ~ 134 Mbytes

Real datasets have 1000's of shots, 1000's of receivers/shot, and 1000's of samples/receiver.

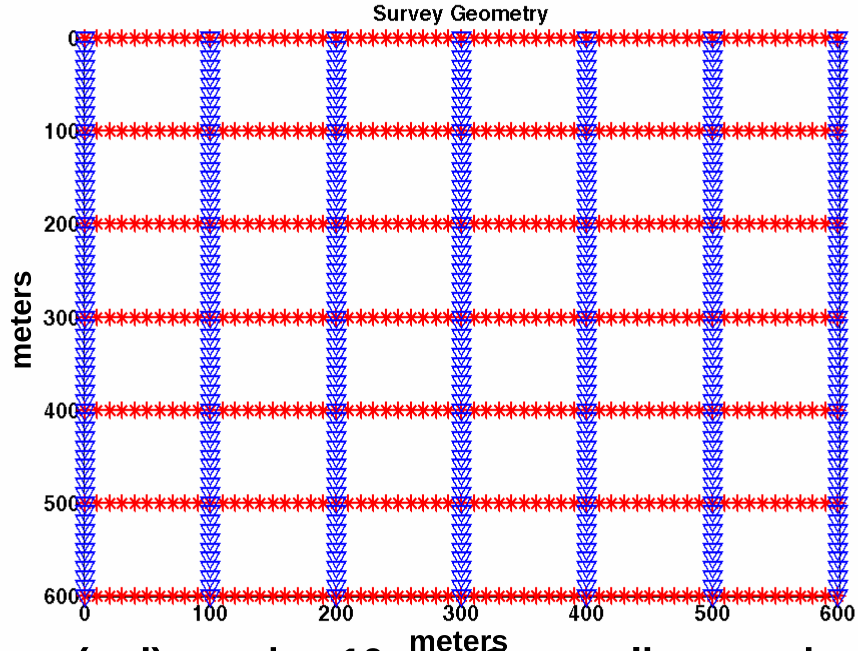
The Basic Seismic Experiment



A hydrocarbon target.
A regular array of detectors (1-C or 3-C).
A seismic source.
The target scatters energy to all receivers.



Typical Land Sampling Lattices



Source (red) spacing 10m --- Source line spacing 100m
Receiver (blue) spacing 10m --- Receiver line spacing 100m

Wave and Helmholtz Equations

There are many variations on the scalar wave equation but the canonical form is

$$\underbrace{\left[\nabla^2 - \frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2}\right]\Psi(\mathbf{x},\mathbf{x}_s,t) = 0}\text{ scalar wave equation}$$

If the wavefield obeys the wave equation, then its *temporal* Fourier transform

$$\underbrace{\psi(\mathbf{x},\mathbf{x}_s,\omega)}_{\text{Spectrum}} = \underbrace{\int_{\mathbb{R}} \overbrace{\Psi(\mathbf{x},\mathbf{x}_s,t)}^{\text{Wavefield}} e^{-i\omega t} d\omega}_{\text{Forward Fourier Transform}} \qquad \Psi(\mathbf{x},\mathbf{x}_s,t) = \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \psi(\mathbf{x},\mathbf{x}_s,\omega) e^{i\omega t} d\omega}_{\text{Inverse Fourier Transform}}$$

obeys the Helmholtz equation

$$\underbrace{\left[\nabla^2 + \frac{\omega^2}{v^2(\mathbf{x})}\right]\psi(\mathbf{x},\mathbf{x}_s,\omega) = 0}\text{ Helmholtz equation}$$

Exercise: The Helmholtz equation (!)

Consider the time-domain scalar wave equation

$$\left[\nabla^2 - \frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2}\right]\Psi(\mathbf{x},\mathbf{x}_s,t) = 0. \quad (1)$$

Express the wavefield as the inverse Fourier transform of it's temporal frequency spectrum as

$$\Psi(\mathbf{x},\mathbf{x}_s,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi(\mathbf{x},\mathbf{x}_s,\omega)e^{i\omega t}d\omega. \quad (2)$$

Show that equation (1) is then equivalent to

$$\left[\nabla^2 + \frac{\omega^2}{v^2(\mathbf{x})}\right]\psi(\mathbf{x},\mathbf{x}_s,\omega) = 0. \quad (3)$$

This is the source-free Helmholtz equation.

The Helmholtz equation (!)

solution

Substitution of equation (2) into (1) requires calculating the second time derivative

$$\frac{\partial^2}{\partial t^2}\Psi(\mathbf{x},\mathbf{x}_s,t) = \frac{1}{2\pi}\frac{\partial^2}{\partial t^2}\int_{\mathbb{R}} \psi(\mathbf{x},\mathbf{x}_s,\omega)e^{i\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \psi(\mathbf{x},\mathbf{x}_s,\omega)\frac{\partial^2}{\partial t^2}\left(e^{i\omega t}\right)d\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \psi(\mathbf{x},\mathbf{x}_s,\omega)(i\omega)^2 e^{i\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(-\omega^2\right)\psi(\mathbf{x},\mathbf{x}_s,\omega)e^{i\omega t}d\omega.$$

The spatial derivatives are not simplified in this case so equation (1) becomes

$$\frac{1}{2\pi}\int_{\mathbb{R}} \left[\left(\nabla^2 + \frac{\omega^2}{v^2(\mathbf{x})}\right)\psi(\mathbf{x},\mathbf{x}_s,\omega)\right]e^{i\omega t}d\omega = 0. \quad (4)$$

The Helmholtz equation

(!)

solution

$$\frac{1}{2\pi}\int_{\mathbb{R}}\left[\left(\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right)\psi(\mathbf{x},\mathbf{x}_s,\omega)\right]e^{i\omega t}d\omega=0. \quad (4)$$

This says that the inverse Fourier transform of the term in square brackets must vanish. The completeness of the Fourier transform means that the only way this can happen is if the term in square brackets must also vanish. That is, the zero signal has a zero spectrum and vice-versa. So we conclude

$$\left(\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right)\psi(\mathbf{x},\mathbf{x}_s,\omega)=0.$$

Example of a Typical Imaging Theory Kirchhoff Migration

•Assume a physics model: Balance simplicity and realism, define a small unknown perturbation of the model..

•Solve the forward scattering problem: Linearize the *Lippman-Schwinger* equation.

•Invert the forward scattering integral for the perturbation: integration over sources and receivers.

Forward Scattering (3D)

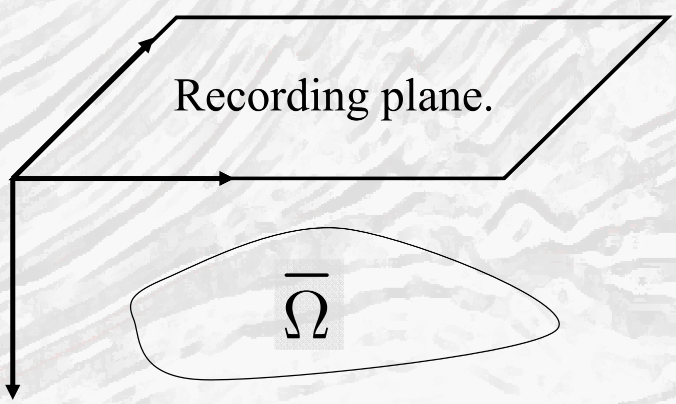
$$\left[\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right]\psi(\mathbf{x},\mathbf{x}_s,\omega)=-F(\omega)\delta(\mathbf{x}-\mathbf{x}_s)$$ Helmholtz problem

$$\lim_{r\to\infty}\left(r\left[\frac{\partial\psi}{\partial r}+\frac{i\omega}{v}\psi\right]\right)=0,\ r=|\mathbf{x}|$$ Radiation condition, outgoing waves at infinity

$$\frac{1}{v^2(\mathbf{x})}=\frac{1}{c^2(\mathbf{x})}\left(1+\alpha(\mathbf{x})\right)$$ Perturbation assumption

$$\text{supp}(\alpha)=\overline{\Omega},\quad \overline{\Omega}\subset z>0$$ Perturbation has compact support

Forward Scattering (3D)



Assume unbounded medium (i.e. recording plane is transparent).

Exercise: Radiation Condition

(!)

Consider the two monochromatic waves defined by

$$\psi_+ = \frac{1}{r} e^{i\omega r/v} \Rightarrow \Psi_+ = \frac{1}{r} e^{i\omega(t+r/v)},$$

$$\psi_- = \frac{1}{r} e^{-i\omega r/v} \Rightarrow \Psi_- = \frac{1}{r} e^{i\omega(t-r/v)}$$

Identify the direction of travel of each wave (as *t* increases, does *r* increase or decrease?). Which wave satisfies the radiation condition (consider *v* to be constant)

$$\lim_{r\to\infty}\left(r\left[\frac{\partial\psi}{\partial r} + \frac{i\omega}{v}\psi\right]\right) = 0, r = |\mathbf{x}|?$$

Radiation Condition

solution

(!)

We can determine the direction a wave moves by tracking a front of constant phase. Consider

$$\text{phase}(\Psi_+) = \omega(t + r/v)$$

Suppose this phase evaluates to a constant, q, at time t_1 and radius r_1. Then at a later time, t_2, and a different radius, r_2, then the wave front must satisfy

$$\theta = \omega(t_1 + r_1/v) = \omega(t_2 + r_2/v)$$

from which we deduce

$$r_2 = r_1 + (t_1 - t_2)v.$$

Since we chose $t_2 > t_1$, then it follows that $r_2 < r_1$ and so this wave moves in the direction of decreasing radius.

Radiation Condition

solution

(!)

After a similar analysis for the other wave we have the directions

$$\Psi_+ = \frac{1}{r} e^{i\omega(t+r/v)} \Rightarrow \text{incoming from infinity}$$

$$\Psi_- = \frac{1}{r} e^{i\omega(t-r/v)} \Rightarrow \text{outgoing to infinity}$$

By direct calculation of the partial derivatives we have

$$\frac{\partial \psi_-}{\partial r} = -\left(\frac{i\omega}{v} + \frac{1}{r}\right)\psi_- \Rightarrow \lim_{r\to\infty}\left(r\left[\frac{\partial \psi_-}{\partial r} + \frac{i\omega}{v}\psi_-\right]\right) = 0$$

$$\frac{\partial \psi_+}{\partial r} = \left(\frac{i\omega}{v} - \frac{1}{r}\right)\psi_+ \Rightarrow \lim_{r\to\infty}\left(r\left[\frac{\partial \psi_+}{\partial r} + \frac{i\omega}{v}\psi_+\right]\right) = 2\frac{i\omega}{v} e^{i\omega r/v} \neq 0$$

Warning: the form of the frequency domain radiation condition depends upon the Fourier transform sign convention chosen. Why?

Forward Scattering (3D)

$$\psi(\mathbf{x},\mathbf{x}_s,\omega) = \psi_I(\mathbf{x},\mathbf{x}_s,\omega) + \psi_S(\mathbf{x},\mathbf{x}_s,\omega)$$

Incident and reflected (scattered) fields

Incident field solves the background problem

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_I(\mathbf{x},\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s)$$

Forward Scattering (3D)

It results that the reflected field satisfies a perturbed Helmholtz equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega) = -\frac{\omega^2}{c^2(\mathbf{x})}\alpha(\mathbf{x})\psi(\mathbf{x},\mathbf{x}_s,\omega)$$

Note the appearance of the total field on the right. This is exact, no approximations.

Given measurements of the reflected field and knowledge of the background medium, we wish to solve for the perturbation $\alpha(\mathbf{x})$

Exercise: Derive the perturbed Helmholtz equation

(!)

Given:

$$\left[\nabla^2 + \frac{\omega^2}{v^2(\mathbf{x})}\right]\psi(\mathbf{x},\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s) \quad (1)$$

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_I(\mathbf{x},\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s) \quad (2)$$

$$\psi(\mathbf{x},\mathbf{x}_s,\omega) = \psi_I(\mathbf{x},\mathbf{x}_s,\omega) + \psi_S(\mathbf{x},\mathbf{x}_s,\omega) \quad (3)$$

Show that:

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega) = -\frac{\omega^2}{c^2(\mathbf{x})}\alpha(\mathbf{x})\psi(\mathbf{x},\mathbf{x}_s,\omega)$$

In the wavefield expressions, $\boldsymbol{x}_s$ is a constant and the Laplacian operates only on $\boldsymbol{x}$.

The perturbed Helmholtz equation (!)

Solution

Substitute (3) into (1)

$$\left[\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right]\left(\psi_I(\mathbf{x},\mathbf{x}_s,\omega)+\psi_S(\mathbf{x},\mathbf{x}_s,\omega)\right)=$$

$$\left[\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right]\psi_I(\mathbf{x},\mathbf{x}_s,\omega)+\left[\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega)=-F(\omega)\delta(\mathbf{x}-\mathbf{x}_s)$$

Subtract equation (2) from this

$$\left[\nabla^2+\frac{\omega^2}{v^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega)+\left[\frac{\omega^2}{v^2(\mathbf{x})}-\frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_I(\mathbf{x},\mathbf{x}_s,\omega)=0 \quad (4)$$

The perturbed Helmholtz equation (!)

Solution -2-

Recall the definition of the perturbation

$$\frac{1}{v^2(\mathbf{x})}=\frac{1}{c^2(\mathbf{x})}\left(1+\alpha(\mathbf{x})\right)\Rightarrow\frac{1}{v^2(\mathbf{x})}-\frac{1}{c^2(\mathbf{x})}=\frac{\alpha(\mathbf{x})}{c^2(\mathbf{x})}$$

Use this in equation (4) and rearrange

$$\left[\nabla^2+\frac{\omega^2}{c^2(\mathbf{x})}\left(1+\alpha(\mathbf{x})\right)\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega)=-\frac{\omega^2}{c^2(\mathbf{x})}\alpha(\mathbf{x})\psi_I(\mathbf{x},\mathbf{x}_s,\omega)$$

$$\left[\nabla^2+\frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega)=-\frac{\omega^2}{c^2(\mathbf{x})}\alpha(\mathbf{x})\left(\psi_I(\mathbf{x},\mathbf{x}_s,\omega)+\psi_S(\mathbf{x},\mathbf{x}_s,\omega)\right)$$

Since the last term on the right is the total field, this is the desired result.

Solution Strategy

- Convert the perturbed Helmholtz equation to an integral equation using Green's theorem.
- Try to invert the integral equation and solve for the perturbation.

Green's Theorem

Green's Theorem for the Laplacian

$$\int_D \left[a\nabla^2 b - b\nabla^2 a\right] d\mathbf{x} = \int_{\partial D} \left[a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right] d\sigma$$

Where "*a*" and "*b*" are arbitrary scalar fields. This can be derived from a generalization of the fundamental theorem of calculus to 3D.

Exercise: A Simple Green's Theorem (!)

The following equation is a simple manifestation of Green's theorem in 1D. a and b are ordinary functions of x and $[x_1,x_2]$ is an interval on the real line.

$$\int_{x_1}^{x_2}\left[a\frac{d^2b}{dx^2}-b\frac{d^2a}{dx^2}\right]dx=\left[a\frac{db}{dx}-b\frac{da}{dx}\right]\Bigg|_{x_1}^{x_2}$$

This can be derived by an application of integration by parts. See if you can do it before reading the solution on the next few slides.

A Simple Green's Theorem (!)

Solution

Recall the formula for integration by parts for two functions u and v:

$$\int_{x_1}^{x_2} u\,dv = uv\big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\,du$$

Use this to evaluate:

$$\int_{x_1}^{x_2} a\frac{d^2b}{dx^2}dx = ?$$

Let $u=a$ and $dv=\frac{d^2b}{dx^2}dx$

Then it follows that

$$\int_{x_1}^{x_2} a\frac{d^2b}{dx^2}dx = \underbrace{a\frac{db}{dx}\bigg|_{x_1}^{x_2}}_{uv\text{ term}} - \underbrace{\int_{x_1}^{x_2}\frac{da}{dx}\frac{db}{dx}dx}_{vdu\text{ term}} \qquad (1)$$

A Simple Green's Theorem

(!)

Solution -2-

Similarly we can find $$\int_{x_1}^{x_2} b\frac{d^2a}{dx^2}dx = b\frac{da}{dx}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{da}{dx}\frac{db}{dx}dx \quad (2)$$

Then, subtracting result (2) from (1) gives the desired solution:

$$\int_{x_1}^{x_2}\left[a\frac{d^2b}{dx^2} - b\frac{d^2a}{dx^2}\right]dx = \left[a\frac{db}{dx} - b\frac{da}{dx}\right]\Bigg|_{x_1}^{x_2}$$

This derivation is exactly analogous to what is required to derive Green's theorem in 3D. So we see that the theorem is simply a result of integral calculus and is a useful tool in physical problems although it has not "physics" itself.

Solution Strategy

Green's Theorem for the Laplacian

Math $\left\{ \int_D \left[g\nabla^2\psi_S - \psi_S\nabla^2 g\right]d\mathbf{x} = \int_{\partial D}\left[g\frac{\partial\psi_S}{\partial n} - \psi_S\frac{\partial g}{\partial n}\right]d\sigma \right.$

Physics $\left\{ \begin{array}{l} \left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]\psi_S(\mathbf{x},\mathbf{x}_s,\omega) = -\frac{\omega^2}{c^2(\mathbf{x})}\alpha(\mathbf{x})\psi(\mathbf{x},\mathbf{x}_s,\omega) \\ \left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]g(\mathbf{x},\mathbf{x}_g,\omega) = -\delta(\mathbf{x}-\mathbf{x}_g) \end{array} \right.$

Forward Scattering Lippman-Schwinger Equation

The surface integrals vanish due to the unbounded medium assumption and the radiation condition. One part of the volume integral collapses to the scattered field with the result

$$\psi_S\left(\mathbf{x}_g,\mathbf{x}_s,\omega\right)=\omega^2\int_{z>0}\frac{\alpha\left(\mathbf{x}\right)}{c^2\left(\mathbf{x}\right)}\psi\left(\mathbf{x},\mathbf{x}_s,\omega\right)g\left(\mathbf{x}_g,\mathbf{x},\omega\right)d\mathbf{x}$$

A Lippmann-Schwinger equation for the scattered field. Note the presence of the total field in the integral.

Forward Scattering Born Approximation

We approximate the total field with the incident field

$$\psi\left(\mathbf{x},\mathbf{x}_s,\omega\right)\approx\psi_I\left(\mathbf{x},\mathbf{x}_s,\omega\right),\ \left|\psi_S\left(\mathbf{x},\mathbf{x}_s,\omega\right)\right|<<\left|\psi_I\left(\mathbf{x},\mathbf{x}_s,\omega\right)\right|$$

$$\psi_S\left(\mathbf{x}_g,\mathbf{x}_s,\omega\right)=\omega^2\int_{z>0}\frac{\alpha\left(\mathbf{x}\right)}{c^2\left(\mathbf{x}\right)}\psi_I\left(\mathbf{x},\mathbf{x}_s,\omega\right)g\left(\mathbf{x}_g,\mathbf{x},\omega\right)d\mathbf{x}$$

The first-order Born approximation to the Lippmann-Schwinger scattering equation.

Forward Scattering Born Approximation

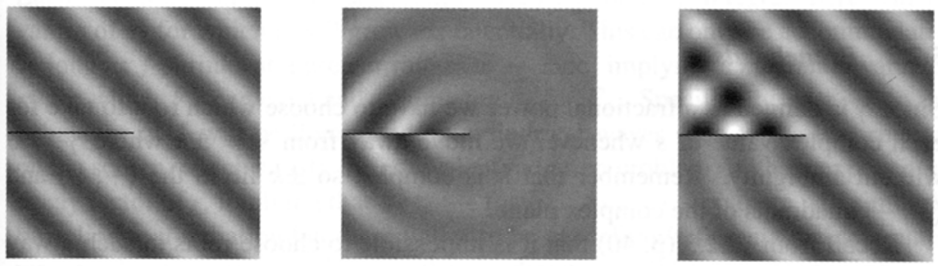


Fig. 5.2 The incident, scattered and total potential fields.

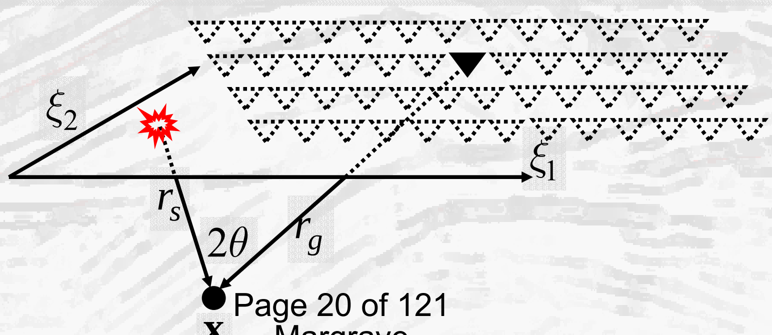
Taken from: Lecture Notes on the Mathematics of Acoustics, M.C.M. Wright (ed.), Imperial College Press, 2005

Inverse Born Scattering

Source Gather

Usually more approximations are required to invert linearized Lippmann-Schwinger equation.. For example, if we assume the geometry of a source gather, a constant background velocity, and approximate the incident field with a Green's function, then an approximate formula is

$$\alpha(\mathbf{x}) = \frac{4x_3}{\pi c} \int_A d\xi_1 d\xi_2 \frac{r_s \cos\theta}{r_g^2} \int_0^\infty d\omega \psi_S(\xi_1, \xi_2, \omega) e^{i\omega(r_s + r_g)/c}$$



Inverse Born Scattering

Source Gather

Dissecting the equation:

$$\alpha(\mathbf{x}) = \frac{4x_3}{\pi c}\int_A d\xi_1 d\xi_2 \frac{r_s \cos\theta}{r_g^2}\int_0^\infty d\omega \psi_S(\xi_1,\xi_2,\omega) e^{i\omega(r_s+r_g)/c}$$

$\psi_S(\xi_1,\xi_2,\omega)e^{i\omega r_g/c}$ The scattered data downward continued to the image point by phase shift.

$\left(\frac{1}{r_s}e^{-i\omega r_s/c}\right)^{-1}$ Green's function model of the incident field.

$\frac{\cos\theta}{r_g^2}$ A collection of geometric factors.

Exercise: Time shift by phase shift $(!)$

How do you time shift a signal in the frequency domain?

Consider a signal *g(t)* with Fourier transform given by

$$\hat{g}(\omega) = \int_{\mathbb{R}} g(t) e^{-i\omega t} dt$$

Show that the Fourier transform of *g(t+τ)* is

$$\underbrace{g(t+\tau)}_{\text{time domain}} \underbrace{\Leftrightarrow}_{\text{Fourier Pair}} \underbrace{\hat{g}(\omega)e^{i\omega\tau}}_{\text{Fourier domain}}$$

Time shift by phase shift (!)

solution

Denote the time shifted signal by $u(t) = g(t+\tau)$

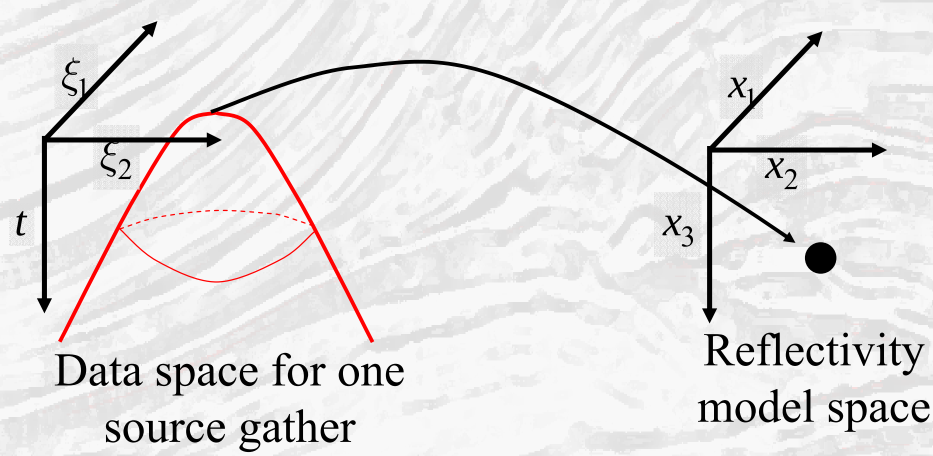
Then $\hat{u}(\omega) = \int_{\mathbb{R}} u(t) e^{-i\omega t} dt = \int_{\mathbb{R}} g(t+\tau) e^{-i\omega t} dt$

$$\hat{u}(\omega) \underbrace{=}_{\text{let } t+\tau = x} \int_{\mathbb{R}} g(x) e^{-i\omega(x-\tau)} dx = e^{i\omega\tau} \int_{\mathbb{R}} g(x) e^{-i\omega(x)} dx$$

finally $\hat{u}(\omega) = e^{i\omega\tau} \hat{g}(\omega)$

This is one of the most important properties of the Fourier transform. You can move things around by phase shifting the spectrum. **Warning:** The sign of the phase shift depends on the sign of the time shift AND on the Fourier transform convention. So you will almost always get it wrong the first time.

Inverse Born Scattering Kirchhoff Mapping



The summation along the hyperbolic surface is done by a phase shift that flattens the surface and then a sum over the spatial coordinates.

Inverse Born Scattering

Major Points

• Many assumptions required to get here.

• The perturbation at depth is estimated directly from an integration of the surface data. This integration requires phases and weights which must be estimated from raytracing.

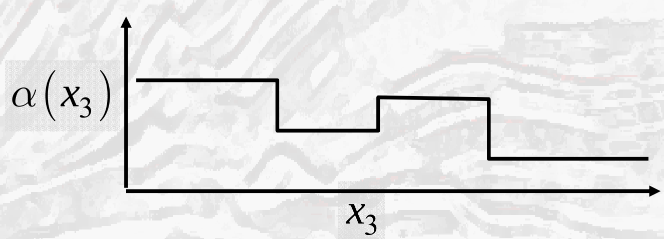
• Ultimately, this does note quite work because of the frequency bandwidth of seismic data.

Inverse Born Scattering

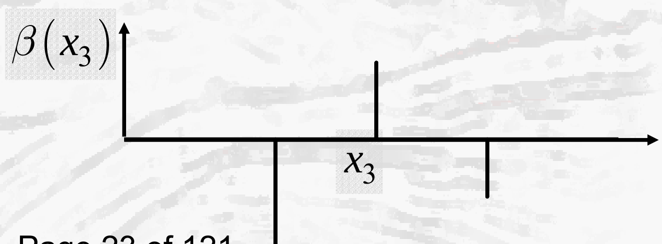
Seismic Frequency Band

• Typical: 10-100 Hz, missing both high and low frequencies.

$$\alpha(\mathbf{x}) = \frac{c^2}{v^2(\mathbf{x})} - 1$$



$$\beta(\mathbf{x}) = \frac{1}{2}\nabla\alpha(\mathbf{x}) \bullet \hat{n} \sim R$$



Inverse Born Scattering

Source Gather, Reflectivity Estimator

$$\beta(\mathbf{x}) = \frac{2x_3}{\pi c^2} \int_A d\xi_1 d\xi_2 \frac{r_s \cos\theta}{r_g^2} \int_0^\infty i\omega d\omega \psi_{S;s}(\xi_1, \xi_2, \omega) e^{i\omega(r_s + r_g)/c}$$

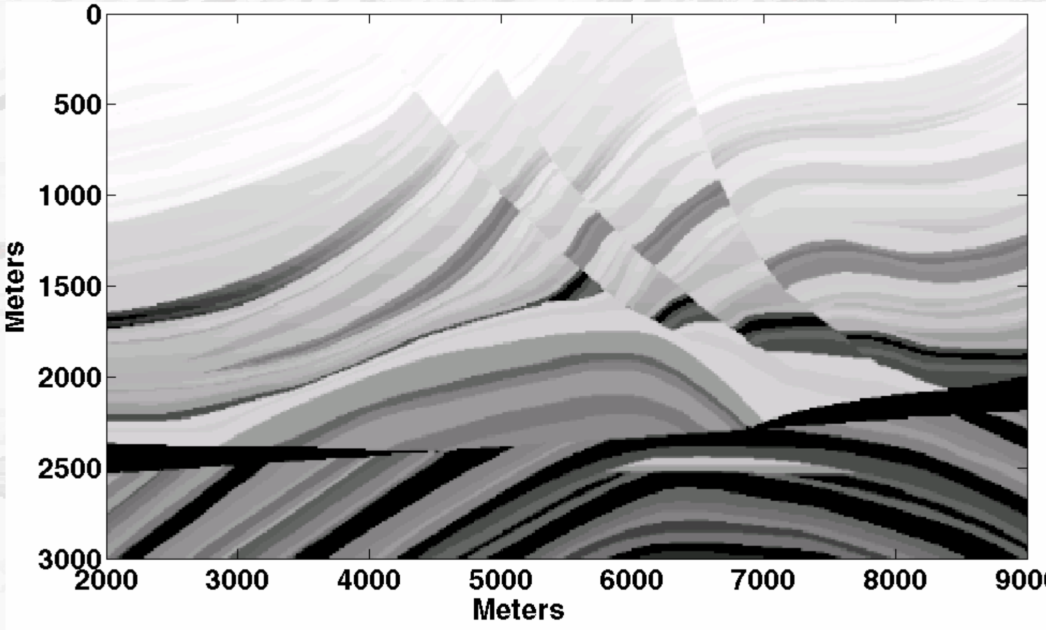
A similar integral as before but the linear frequency weighting means that the missing low frequencies are downweighted.

These methods are known as Kirchhoff Migration methods.

Kirchhoff Approach Summary

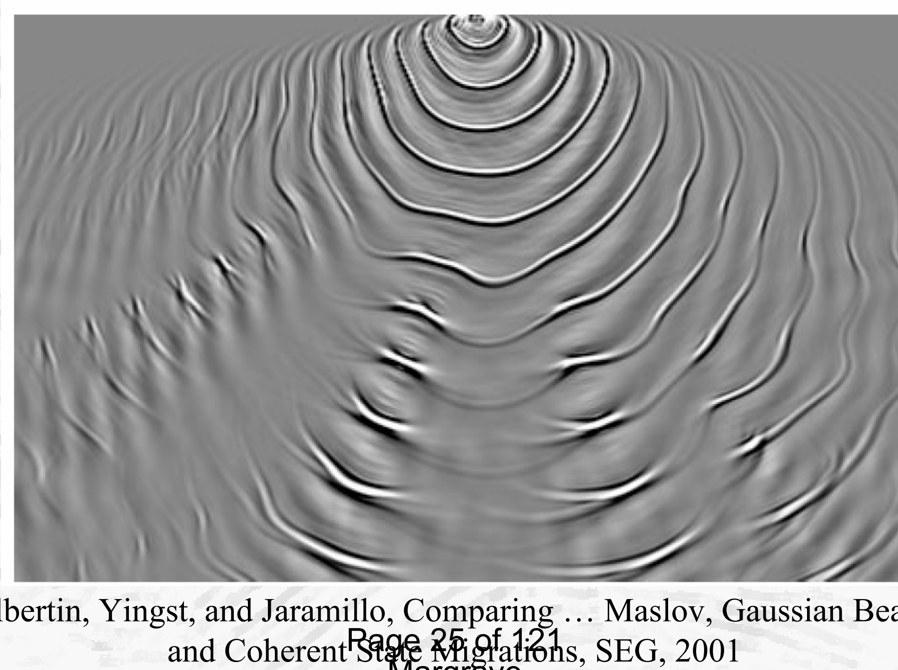
- Inverse scattering approach
- Ray theoretical assumptions made (high frequency)
- Stable ray tracing required
- Computationally simple but weights are subject to assumptions and are generally different from one application to the next
- Only a small subset of the seismic wavefield is captured in this approach

Marmousi Velocity Model



Marmousi Wavefronts

finite difference simulation



Albertin, Yingst, and Jaramillo, Comparing … Maslov, Gaussian Beam, and Coherent State Migrations, SEG, 2001

Marmousi Wavefronts

Kirchhoff (raytracing) simulation

Albertin, Yingst, and Jaramillo, Comparing … Maslov, Gaussian Beam, and Coherent State Migrations, SEG, 2001

Wavefield Extrapolation Methods

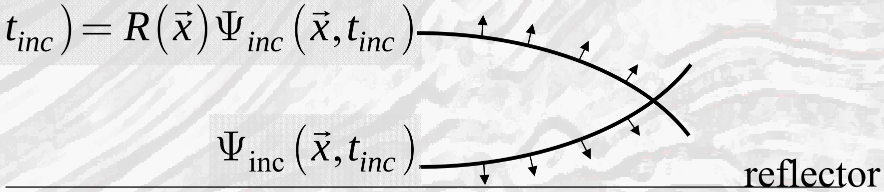
- Move away from the ray-theoretic inverse scattering approach towards a more complete simulation of wave propagation.
- In theory, these methods move toward wave propagation as a path integral along all possible paths rather than the few select, ray theoretical, paths.

Seismic Imaging Paradigm

A common seismic imaging methodology is derivable from first-order inverse Born scattering

$$\Psi_{refl}\left(\vec{x},t_{inc}\right)=R\left(\vec{x}\right)\Psi_{inc}\left(\vec{x},t_{inc}\right)$$

$$\Psi_{\text{inc}}\left(\vec{x},t_{inc}\right)$$



$$\frac{\Psi_{refl}\left(\vec{x},t_{inc}\right)}{\Psi_{inc}\left(\vec{x},t_{inc}\right)}=R\left(\vec{x}\right)$$ A reflectivity estimate.

Seismic Imaging Paradigm

Seismic imaging typically is done in the frequency domain and uses depth steps not time steps, so a more common imaging condition is:

$$R\left(x,y,\Delta z\right)=\sum_{\omega}\frac{\psi_{refl}\left(x,y,z=\Delta z,\omega\right)}{\psi_{inc}\left(x,y,z=\Delta z,\omega\right)}$$

Seismic Imaging Paradigm

So for each depth, we must calculate two fields:

$$\psi_{refl}\left(x,y,n\Delta z,\omega\right)$$

The reflected field comes from mathematically marching the recorded data down into the earth.

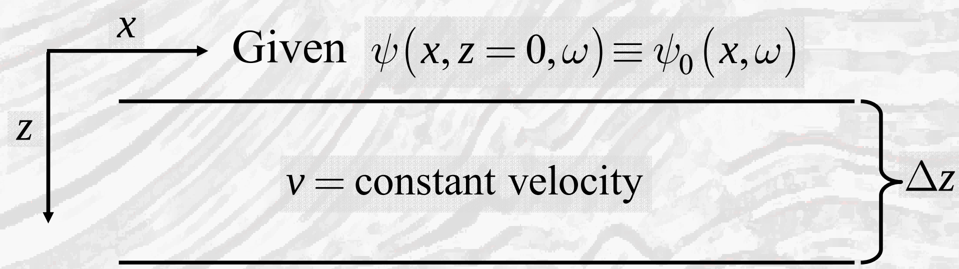
$$\psi_{inc}\left(x,y,n\Delta z,\omega\right)$$

The incident field comes from a mathematical model of the source wavefield that is also marched down.

In both cases, the wavefield marching is done through a "background" velocity field that is presumed known.

Wavefield Extrapolator

The Phase Shift Extrapolator



Given $\psi\left(x,z=0,\omega\right)\equiv\psi_0\left(x,\omega\right)$

$v=$ constant velocity

Find $\psi\left(x,\Delta z,\omega\right)$

Assume $$\left(\nabla^2+\frac{\omega^2}{v^2}\right)\psi\left(x,z,\omega\right)=0$$

Wavefield Extrapolator

The Phase Shift Extrapolator

$$\psi(x,\Delta z,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\hat{\psi}(k_x,0,\omega)}_{\text{wavefield in }(k_x,\omega)} \underbrace{\hat{W}(k,k_x,\Delta z)}_{\text{phase shift operator}} e^{-ik_x x}dk_x$$

$$\hat{W}(k,k_x,\Delta z,\omega)=\begin{cases}\exp\left(i\Delta z\sqrt{k^2-k_x^2}\right), k^2>k_x^2\\ \exp\left(-\Delta z\sqrt{k_x^2-k^2}\right), k^2<k_x^2\end{cases}$$

$$k^2=\frac{\omega^2}{v^2}$$

While valid only for constant velocity, this is still the "canonical form" to which all other methods aspire.

Wavefield Extrapolator

In the space-frequency domain

Since multiplication in the wavenumber domain is a convolution in the space domain, the phase-shift expression is equivalent to

$$\psi(x,\Delta z,\omega)=\int_{\mathbb{R}} \underbrace{\psi(x',0,\omega)}_{\text{wavefield in }(x,\omega)} \underbrace{W(k,x-x',\Delta z)}_{\substack{\text{Wavefield extrapolator}\\ \text{in }(x,\omega)\text{ domain}}} dx'$$

where

$$W(k,x-x',\Delta z)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}(k,k_x,\Delta z)e^{ik_x(x-x')}dk_x$$

Wavefield Extrapolator

as abstract operator

We often find it convenient to hide most of the details in an abstract wavefield extrapolation operator

$$\psi(x,\Delta z,\omega)=L_{W(\Delta z)}\psi(x,0,\omega)\equiv\int_{\mathbb{R}}\psi(x',0,\omega)W(k,x-x',\Delta z)\,dx'$$

For two steps we write

$$\psi(x,2\Delta z,\omega)=L_{W(2\Delta z)}\circ L_{W(\Delta z)}\psi(x,0,\omega)$$

Where $\circ$ symbolizes the *composition* of the operators which just means their sequential application. For N steps we write

$$\psi(x,N\Delta z,\omega)=L_{W(N\Delta z)}\cdots L_{W(2\Delta z)}\circ L_{W(\Delta z)}\psi(x,0,\omega)$$

$$\equiv\prod_{n=1}^{N}L_{W(n\Delta z)}\psi(x,0,\omega)$$

Exercise: Derive the phase shift extrapolation expression (!)

In 2D the Helmholtz equation is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{v^2}\right)\psi(x,z,\omega)=0$$

Define the spatially Fourier transformed wavefield

$$\hat{\psi}(k_x,z,\omega)=\int_{\mathbb{R}}\psi(x,z,\omega)e^{ik_x x}dx$$

In a similar fashion to the derivation of the Helmholtz equation we find that

$$\frac{\partial^2}{\partial z^2}\hat{\psi}(k_x,z,\omega)=-\left(\frac{\omega^2}{v^2}-k_x^2\right)\hat{\psi}(k_x,z,\omega)$$

Exercise: Derive the phase shift extrapolation expression (!)

It is customary to define $k_z^2 = k^2 - k_x^2, \quad k^2 = \frac{\omega^2}{v^2}$

So we must solve
$$\frac{\partial^2}{\partial z^2}\hat{\psi}(k_x,z,\omega) = -k_z^2\hat{\psi}(k_x,z,\omega) \quad (1)$$

This equation is actually an ODE and has the general solution
$$\hat{\psi}(k_x,z,\omega) = A(k_x,\omega)e^{ik_z z} + B(k_x,\omega)e^{-ik_z z} \quad (2)$$

where we define k_z explicitly by equation (3) and by √ we mean the positive square root.

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2}, k^2 \geq k_x^2 \\ -\sqrt{k_x^2 - k^2}, k^2 < k_x^2 \end{cases} \quad (3)$$
$$k^2 = \frac{\omega^2}{v^2}$$

Exercise: Derive the phase shift extrapolation expression (!)

In equation (2), the functions *A* and *B* are arbitrary functions of the Fourier coordinates and must be determined by the prescribed boundary conditions. This is actually a problem since we have two arbitrary functions and only one boundary condition, namely:

$$\psi(x,z,\omega) \equiv \psi_0(x,\omega) = \text{a known function}$$

Lacking a second boundary condition, we proceed with a simplifying assumption. We assume that the given wavefield contains waves moving only upward (in the –z direction). Using reasoning similar to that made in discussing the radiation condition, we can show that *A* represents the strength of upgoing waves and *B* represents downgoing waves. So we take

$$A(k_x,\omega) = \hat{\psi}_0((k_x,\omega)) \quad \text{and} \quad B(k_x,\omega) = 0$$

Exercise: Derive the phase shift extrapolation expression (!)

So we have our final solution

$$\hat{\psi}(k_x, z, \omega) = \hat{\psi}(k_x, 0, \omega) e^{ik_z z}$$

This works for any value of z if velocity remains constant but we are interested in the specific value z=Δz. So write:

$$\hat{\psi}(k_x, z, \omega) = \hat{\psi}(k_x, 0, \omega) \hat{W}(k, k_x, \Delta z)$$

where we define the wavefield extrapolation operator in the Fourier domain as

$$\hat{W}(k, k_x, \Delta z) = \begin{cases} \exp\left(i\Delta z\sqrt{k^2 - k_x^2}\right), k^2 > k_x^2 \\ \exp\left(-\Delta z\sqrt{k_x^2 - k^2}\right), k^2 < k_x^2 \end{cases}$$

$$k^2 = \frac{\omega^2}{v^2}$$

Exercise: Transform the phase-shift operator to the space-frequency domain (!)

The phase-shift wavefield extrapolator is

$$\psi(x, z, \omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\psi}(k_x, 0, \omega) \hat{W}(k, k_x, \Delta z) e^{-ik_x x} dk_x$$

To proceed, substitute $\hat{\psi}(k_x, z, \omega) = \int_{\mathbb{R}} \psi(x, z, \omega) e^{ik_x x} dx$

$$\psi(x, z, \omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \psi(x', z, \omega) e^{ik_x x'} dx'\right] \hat{W}(k, k_x, \Delta z) e^{-ik_x x} dk_x$$

Now interchange the order of integration

$$\psi(x, z, \omega) = \int_{\mathbb{R}} \psi(x', z, \omega) \left[\frac{1}{2\pi}\int_{\mathbb{R}} \hat{W}(k, k_x, \Delta z) e^{-ik_x (x - x')} dk_x\right] dx'$$

where we have been careful to construct the inner integral as an inverse Fourier transform over spatial coordinates.

Exercise: Transform the phase-shift operator to the space-frequency domain (!)

Now, introduce a new symbol for the inverse Fourier transform of the phase-shift operator (take its hat off) …

$$\psi(x,z,\omega)=\int_{\mathbb{R}}\psi(x',z,\omega)W(k,x-x',\Delta z)\,dx'$$

where we have defined

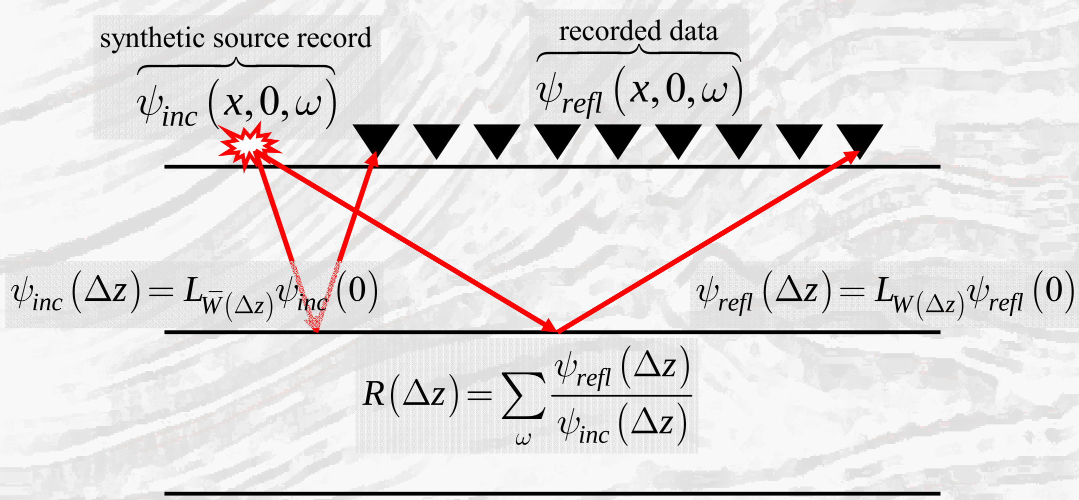
$$W(k,x-x',\Delta z)\equiv\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}(k,k_x,\Delta z)e^{-ik_x(x-x')}dk_x$$

So, in the space-frequency domain, wavefield extrapolation is a spatial convolution. Later we will see how to adapt this expression to variable velocity.

Wavefield Extrapolator Imaging

"wave equation migration"of shot records

First Step



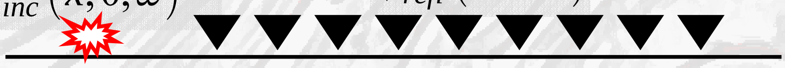
Wavefield Extrapolator Imaging

"wave equation migration" of shot records

Second Step

synthetic source record: $\psi_{inc}(x,0,\omega)$

recorded data: $\psi_{refl}(x,0,\omega)$



$\psi_{inc}(\Delta z)$ $\qquad$ $\psi_{refl}(\Delta z)$

$\psi_{inc}(2\Delta z) = L_{\bar{W}(2\Delta z)}\psi_{inc}(\Delta z)$ $\qquad$ $\psi_{refl}(2\Delta z) = L_{W(2\Delta z)}\psi_{refl}(\Delta z)$

$$R(2\Delta z) = \sum_{\omega} \frac{\psi_{refl}(2\Delta z)}{\psi_{inc}(2\Delta z)}$$

Wavefield Extrapolator Imaging

"wave equation migration" of shot records

Any Step

$$\psi_{refl}(x, z+\Delta z, \omega) = L_{W(z+\Delta z)}\psi_{refl}(x,z,\omega)$$

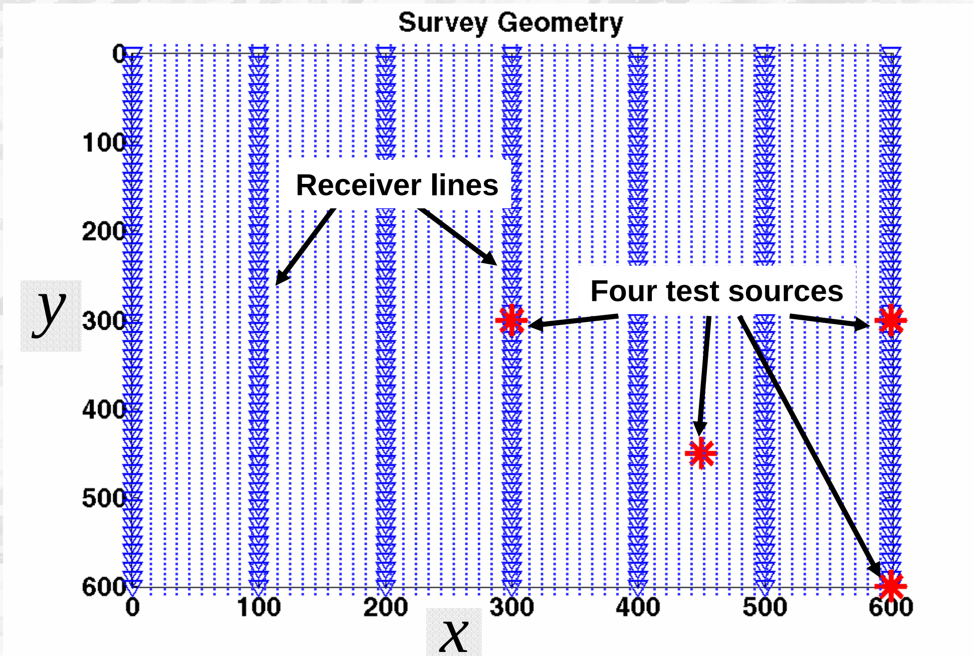
$$\psi_{refl}(x, N\Delta z, \omega) = L_{W(N\Delta z)} \circ \cdots L_{W(2\Delta z)} \circ L_{W(\Delta z)}\psi_{refl}(x,0,\omega)$$

$$\psi_{inc}(x, N\Delta z, \omega) = L_{\bar{W}(N\Delta z)} \circ \cdots L_{\bar{W}(2\Delta z)} \circ L_{\bar{W}(\Delta z)}\psi_{inc}(x,0,\omega)$$

$$R(x,z) = \sum_{\omega} \frac{\psi_{refl}(x,z,\omega)}{\psi_{inc}(x,z,\omega)}, \quad z \subset \{0, \Delta z, 2\Delta z, \ldots N\Delta z\}$$

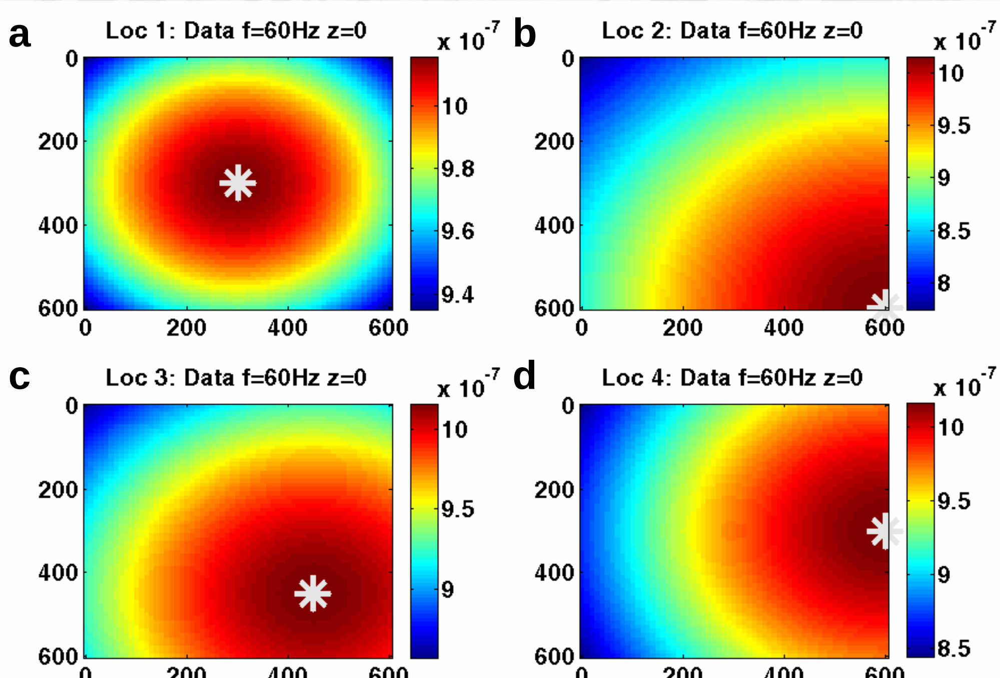
We can obtain such a reflectivity estimate for each depth and for each source position.

Simulation in 3D



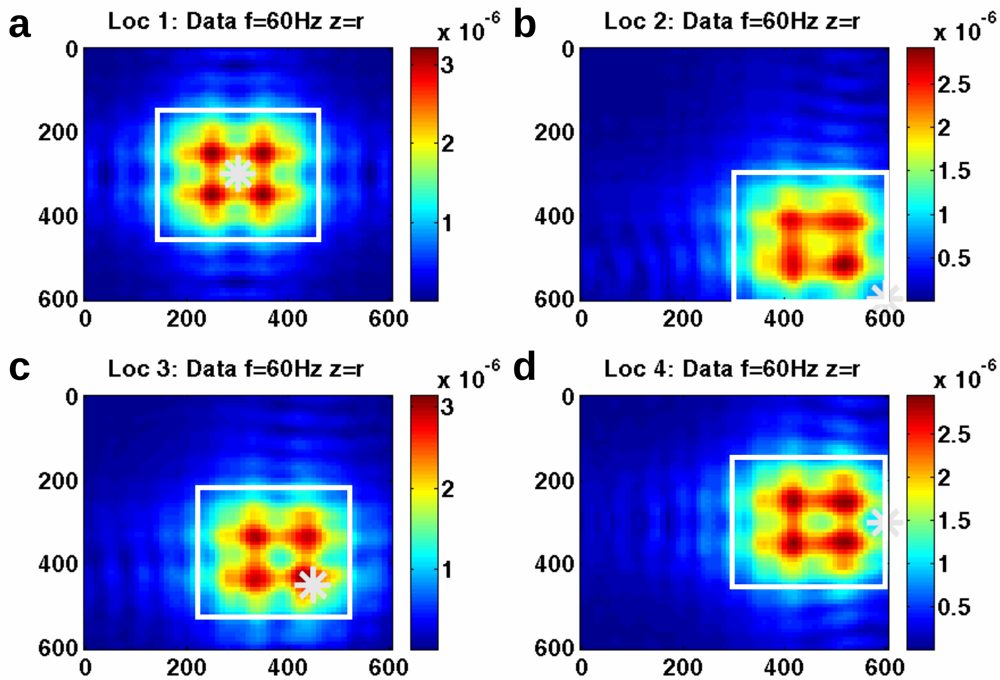
Receiver interval 10 meters and receiver line spacing 10 meters.

PP Data at z=0, 60 Hz

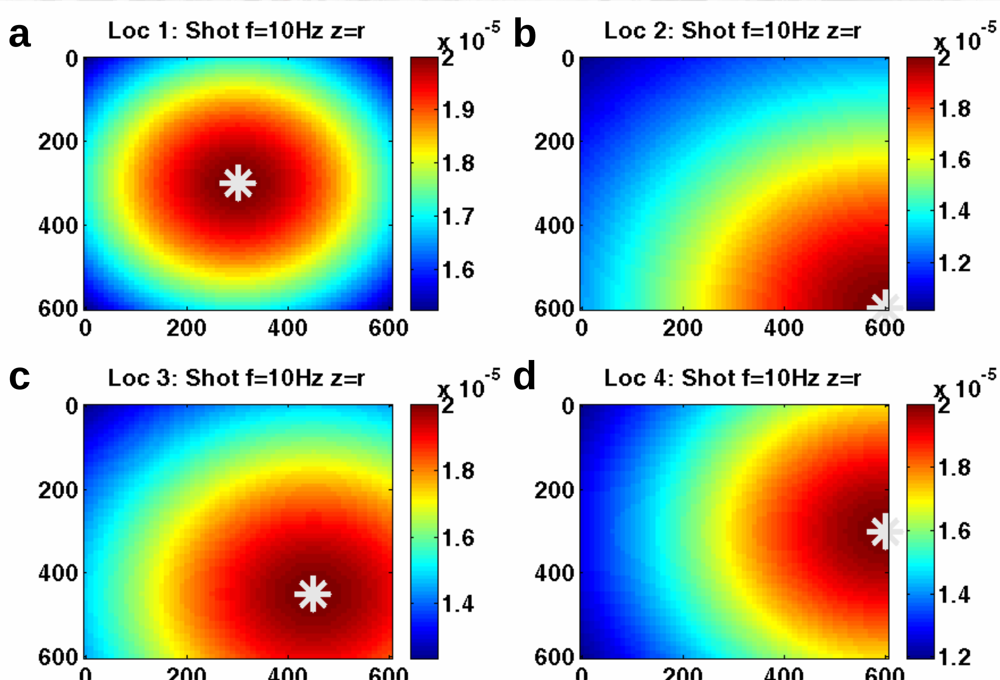


Horizontal reflector at 500 m depth with R=0.1

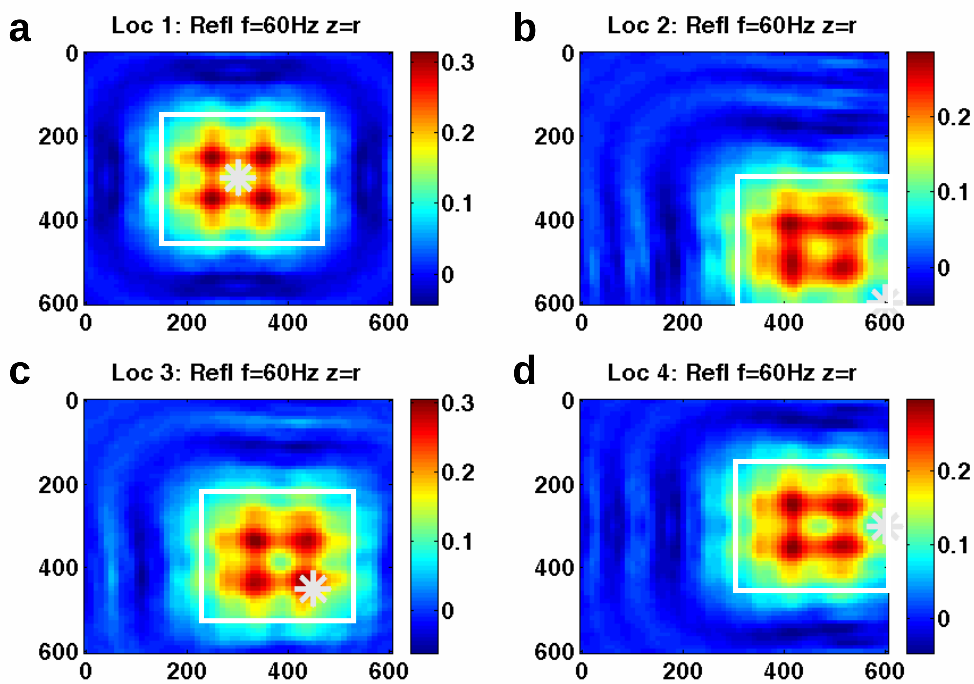
PP Data at z=500, 60 Hz



Source at z=500, 10 Hz



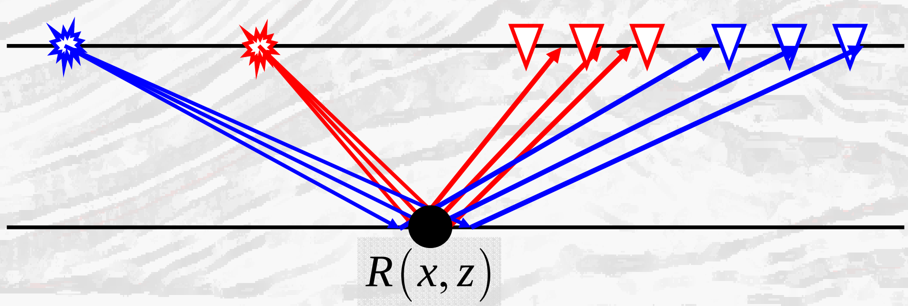
PP Reflectivity at 60 Hz



Wavefield Extrapolator Imaging

Compositing the individual shot records

Now we expand the notation to denote each individual source with an index. In principle, each source can provide a reflectivity estimate at each subsurface position "beneath" the survey.



So each the reflectivity estimates from each source have an angle dependency.

Wavefield Extrapolator Imaging

Compositing the individual shot records

Let the reflectivity estimate from the k^{th} source be

$$R_k(x,z)=\sum_{\omega}\frac{\psi_{refl}(x,x_k,z,\omega)}{\psi_{inc}(x,x_k,z,\omega)}$$

It is common to form a stacked reflectivity image (or migrated section) by summing the estimates from each source.

$$R_{stk}(x,z)=\sum_{k}R_k(x,z)$$

The ensemble of reflectivity estimates at a given x, considered as a function of k and z, is called a common image gather (CIG).

$$R_k(x,z)=\text{CIG at position } x.$$

Wavefield Extrapolator Imaging

Compositing the individual shot records

The stacked reflectivity image is

$$R_{stk}(x,z)=\sum_{k}R_k(x,z)=\sum_{k}\sum_{\omega}\frac{\prod_{n}L_{W(n\Delta z)}\psi_{refl}(x,x_k,0,\omega)}{\prod_{n}L_{W(n\Delta z)}\psi_{inc}(x,x_k,0,\omega)}$$

A natural question to ask is could we somehow move the stacking operator all the way to the right thereby compositing the data before all of the wave-equation stuff. This would save a lot of computational cost. To make life simpler, lets define

$$R_k(x,z)=O_{mig(x,z,x_k)}\psi_{refl}(x_k)\equiv\sum_{\omega}\frac{\prod_{n}L_{W(n\Delta z)}\psi_{refl}(x,x_k,0,\omega)}{\prod_{n}L_{W(n\Delta z)}\psi_{inc}(x,x_k,0,\omega)}$$

Wavefield Extrapolator Imaging

Compositing the individual shot records

The stacked reflectivity image is

$$R_{stk}(x,z) = \sum_k O_{mig(x,z,x_k)} \psi_{refl}(x_k)$$

So, is it possible that

$$R_{stk}(x,z) = \sum_k O_{mig(x,z,x_k)} \psi_{refl}(x_k) \stackrel{?}{=} O'_{mig(x,z)} \sum_k \psi_{refl}(x_k)$$

The answer to this is NO!, but it turns out that, with a great deal of effort we can do something like

$$R_{stk}(x,z) = \sum_k O_{mig(x,z,x_k)} \psi_{refl}(x_k)$$
$$\approx O'_{mig(x,z)} \sum_k O_{nmo(x,t,x_k)} \psi_{refl}(x_k)$$

Wavefield Extrapolator Imaging

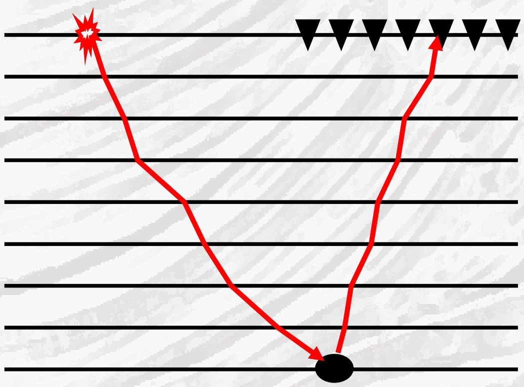
Compositing the individual shot records

So there is the possibility of a number of different imaging operators:

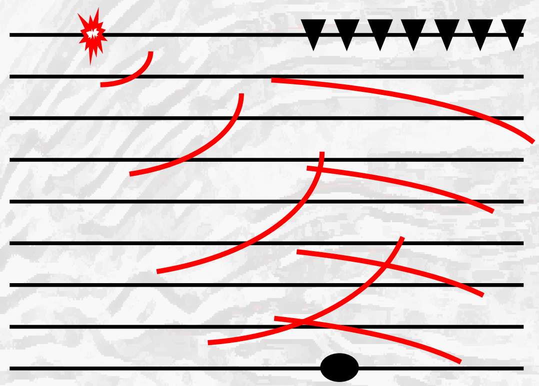
$$O_{mig(x,z,x_k)} = \text{Migration operator (pre-stack)}$$
$$O'_{mig(x,z)} = \text{Migration operator (post-stack)}$$
$$O_{nmo(x,t,x_k)} = \text{Normal moveout operator}$$

There is much more to this story than can be told here. The important thing is that O'_{mig} and O_{nmo} are the common choice today but too much is lost in the approximation. O_{mig} is the obvious choice for the future, but a great deal of work and research remains.

Kirchhoff versus WEM



Kirchhoff traces Snell rays to each point in the image.



Wavefield Extrapolation Migration uses mathematical operators to march entire wavefields to the image point.

Both methods are first-order Born approximations to the inverse scattering problem.

Final Points

Seismic Images are routinely produced but there are many outstanding problems.

The Kirchhoff method is derivable from Born scattering theory and is limited by ray theory.

The wavefield extrapolation method seems like a way forward but is computationally challenging and it is not clear what the limitations are.

Both methods are first-order Born approximations.

Determination of the background velocity model is a major concern.

No one knows anything about convergence.

Introduction to Phase Space Concepts in Seismic Imaging

Seismic Imaging Summer School
Calgary, 2006
Gary F. Margrave

Pacific Institute for the Mathematical Sciences

Canadian Society of Exploration Geophysicists CSEG Since 1949

mitacs

CREWES

POTSI PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

Outline

- Fourier Transforms
- Stationary Fourier Methods
- Phase Space
- Pseudodifferential Operators

Part 1

Fourier Transforms

Fourier Transform

Forward

Forward transform time→frequency

$$\psi(x,z=0,\omega)=\int_{\mathbb{R}}\Psi(x,z=0,t)\mathrm{e}^{-i\omega t}\,dt$$
$$=\widehat{\Psi}(x,z=0,\omega)$$

Forward transform space→wavenumber

$$\hat{\psi}(k_x,z=0,\omega)=\int_{\mathbb{R}}\psi(x,z=0,\omega)\mathrm{e}^{ik_x x}\,dx$$

Forward 2D transform over time and space

$$\hat{\psi}(k_x,z=0,\omega)=\int_{\mathbb{R}^2}\Psi(x,z=0,t)\mathrm{e}^{i(k_x x-\omega t)}\,dtdx$$
$$=\hat{\widehat{\Psi}}(k_x,z=0,\omega)$$

The use of a different sign convention for the space and time transforms is intentional. This is called the "symplectic" form of the Fourier transform.

Fourier Transform

Inverse

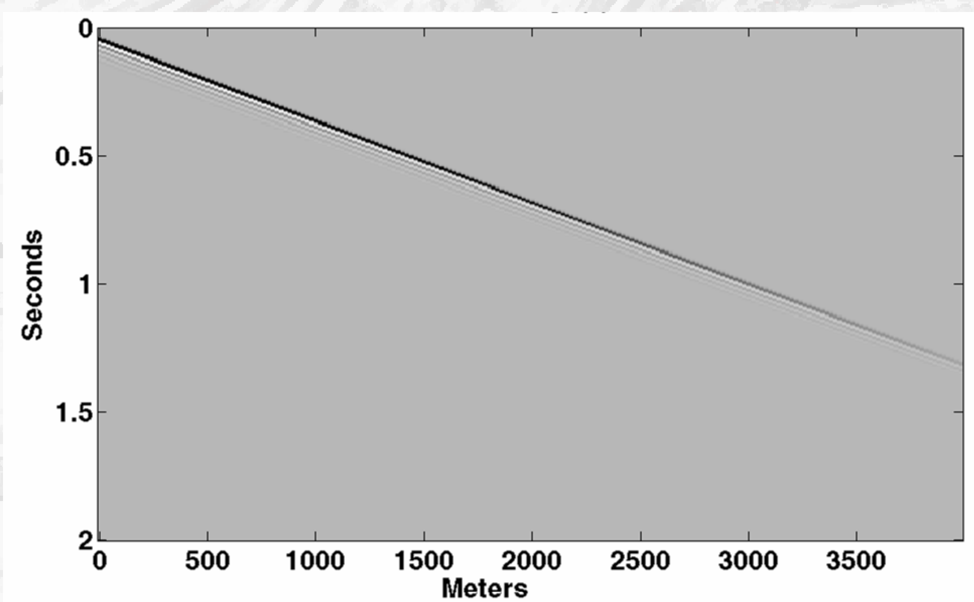
Inverse transform over wavenumber and frequency

$$\Psi\left(x, z=0, t\right) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{\psi}\left(k_x, z=0, \omega\right) \mathrm{e}^{i\left(\omega t - k_x x\right)}\, dk_x d\omega$$

Physical interpretation:

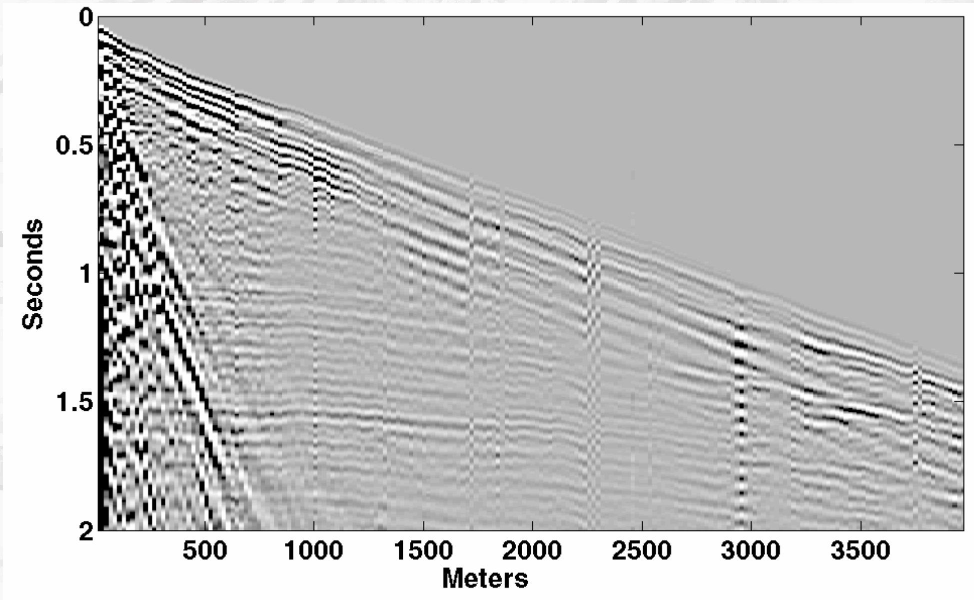
$\mathrm{e}^{i\left(\omega t - k_x x\right)}$	Basis vectors or fundamental waves, apparent velocity ω/k_x.
$\hat{\psi}\left(k_x, z=0, \omega\right)$	Amplitudes and phases of the fundamental waves

Synthetic First Break Event



Seismic Shot Record

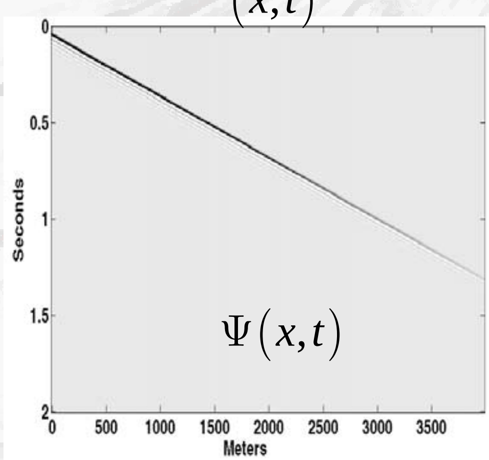
Gained and clipped



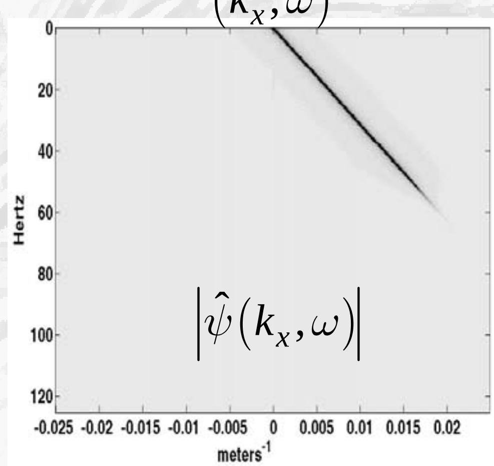
Fourier Transform

synthetic data

(x,t)



(k_x,ω)



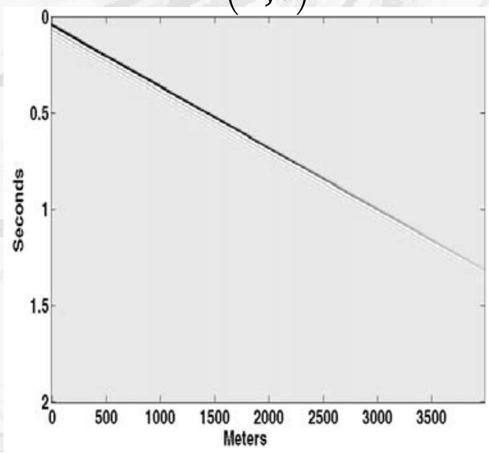
$$\hat{\psi}(k_x,\omega)=\int_{\mathbb{R}^2}\Psi(x,t)e^{i(k_x x-\omega t)}\,dxdt$$

opposing signs in exponent (symplectic)

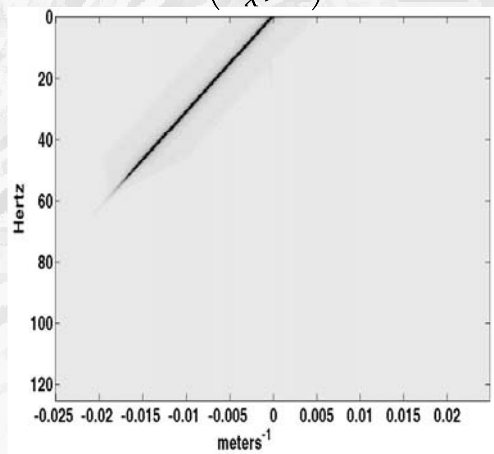
Fourier Transform

synthetic data

(x,t)



(k_x,ω)



$$\hat{\psi}(k_x,\omega) = \int_{\mathbb{R}^2} \Psi(x,t)\mathrm{e}^{-i(k_x x+\omega t)}\,dxdt$$

same signs in exponent

Exercise: 2D Transform of a linear event (!)

Model an ideal linear event using the Dirac Delta distribution:

$$\Psi(x,t) = \delta(px - t + c) \quad p,c \in \mathbb{R}$$

where the Delta distribution has the "sifting" property

$$f(u_0) = \int_{\mathbb{R}} \delta(u-u_0)\, f(u)\,du$$ for any f that we care about.

Show that the 2D (symplectic) Fourier transform of (x,t) is

$$\hat{\psi}(k_x,\omega) = 2\pi\delta(k_x - p\omega)e^{i\omega c}$$

use this to explain the preference stated in lecture for the symplectic Fourier transform. For $p \in [0,1]$ make a sketch showing where several typical events lie in both domains.

Exercise: 2D Transform of a linear event (!)
solution

We wish to calculate

$$\hat{\psi}(k_x,\omega)=\int_{\mathbb{R}^2}\delta(px-t+c)\mathrm{e}^{i(k_x x-\omega t)}\,dtdx$$

We can use the sifting property of the Delta function to collapse either the t or the x integral. We choose t:

$$\int_{\mathbb{R}^2}\delta(px-t+c)\mathrm{e}^{i(k_x x-\omega t)}\,dtdx=\int_{\mathbb{R}^2}\underbrace{\mathrm{e}^{i(k_x x-\omega(px+c))}}_{t\text{ becomes }px+c}dx$$

$$=e^{i\omega c}\int_{\mathbb{R}^2}\mathrm{e}^{i(k_x-\omega p)x}\,dx=2\pi\delta(k_x-\omega p)e^{i\omega c}$$

The last step is not obvious and is explained on the next slide.

Exercise: 2D Transform of a linear event (!)
solution

Using the sifting property of the Dirac distribution, we calculate its Fourier transform

$$\int_{\mathbb{R}}\delta(x-x_0)\mathrm{e}^{ik_x x}\,dx=e^{ik_x x_0}$$

Therefore, by the inverse Fourier transform, we must have

$$\delta(x-x_0)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ik_x x_0}\,\mathrm{e}^{-ik_x x}\,dk_x=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-ik_x(x-x_0)}dk_x$$

So we see that a complex exponential, whose phase is linear in the integration variable, yields a Dirac distribution when integrated over the real line. Applying this result gives the last step on the previous slide.

Exercise

So we have the Fourier correspondence:

$$\delta\left(px - t + c\right) \quad \Leftrightarrow \quad 2\pi\delta\left(k_x - p\omega\right)e^{i\omega c}$$

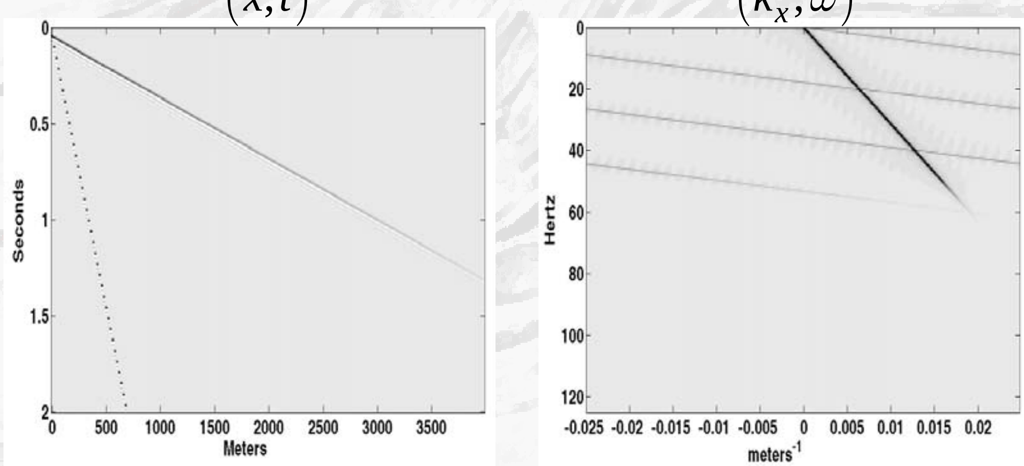
Important points

- All events with the same slope (p-value) in *(x,t)* have the same amplitude spectrum in *(k_x,ω)*.
- The slope of an event in *(x,t)* and the corresponding event in *(k_x,ω)* are inversely related.
- The value of p can be calculated directly from the ratio of k_x to ω in Fourier space.

Fourier Transform

synthetic data

(x,t) $\qquad\qquad$ (k_x,ω)

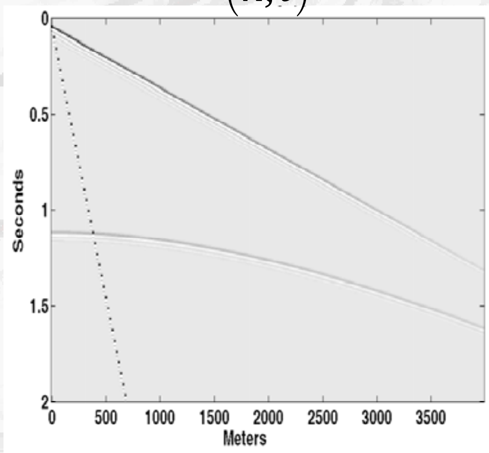


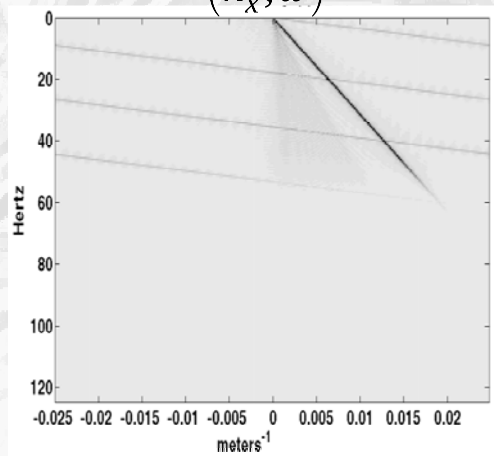
$$\hat{\psi}\left(k_x,\omega\right) = \int_{\mathbb{R}^2} \Psi\left(x,t\right)\mathrm{e}^{i\left(k_x x - \omega t\right)}\, dxdt$$

Fourier Transform

synthetic data

(x,t) (k_x,ω)



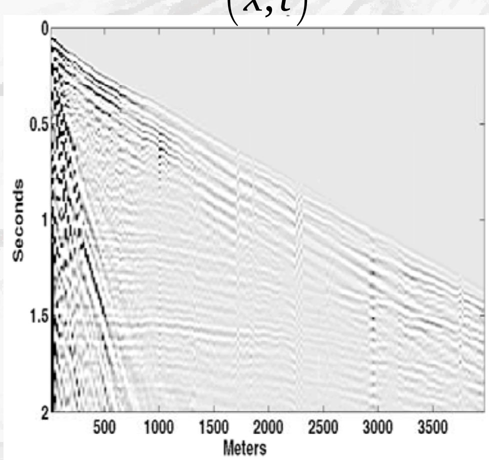


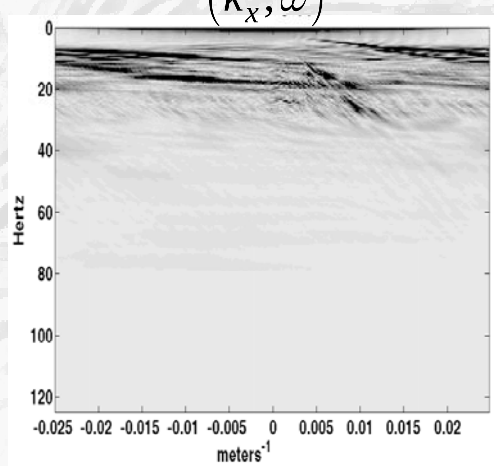
$$\hat{\psi}(k_x,\omega)=\int_{\mathbb{R}^2}\Psi(x,t)e^{i(k_x x-\omega t)}\,dxdt$$

Fourier Transform

real data

(x,t) (k_x,ω)

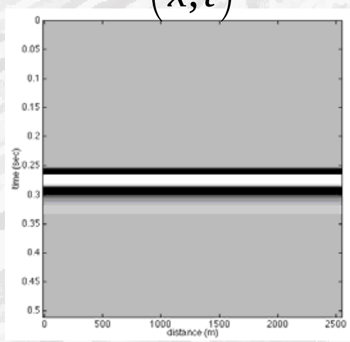


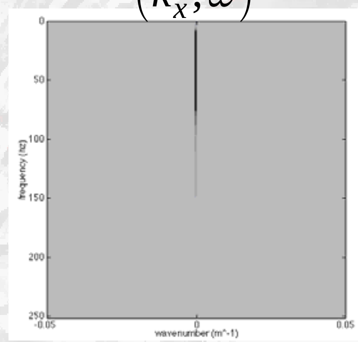


$$\hat{\psi}(k_x,\omega)=\int_{\mathbb{R}^2}\Psi(x,t)e^{i(k_x x-\omega t)}\,dxdt$$

Fourier Transform Pairs

$\left(x,t\right)$ $\left(k_x,\omega\right)$

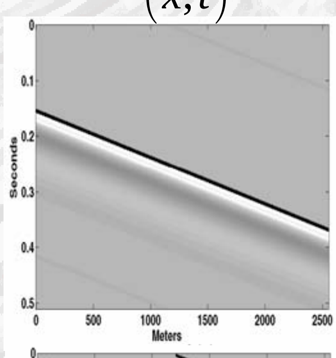


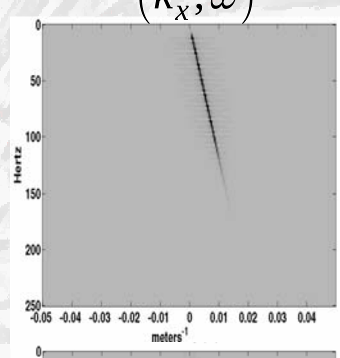


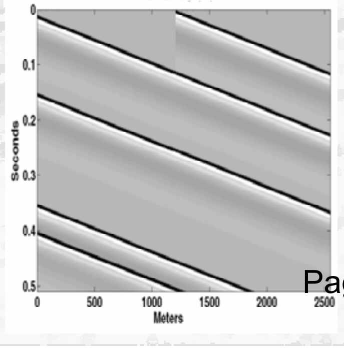
$\Psi\left(x,t\right)$ $\left|\hat{\psi}\left(k_x,\omega\right)\right|$

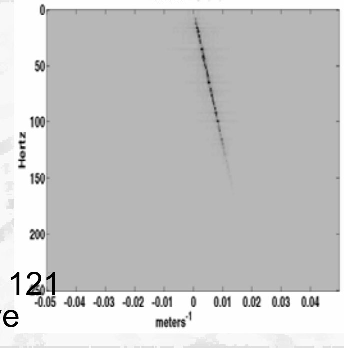
Fourier Transform Pairs

$\left(x,t\right)$ $\left(k_x,\omega\right)$



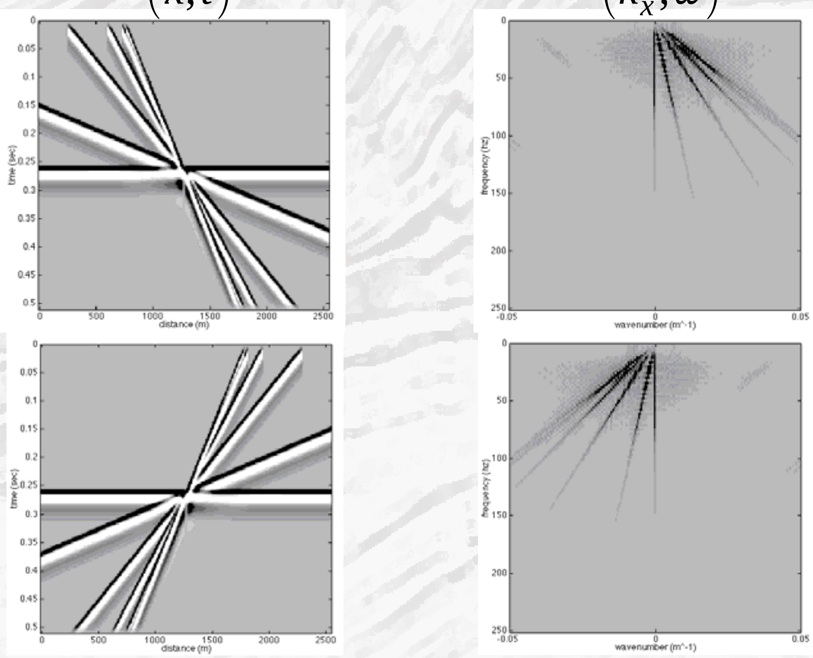






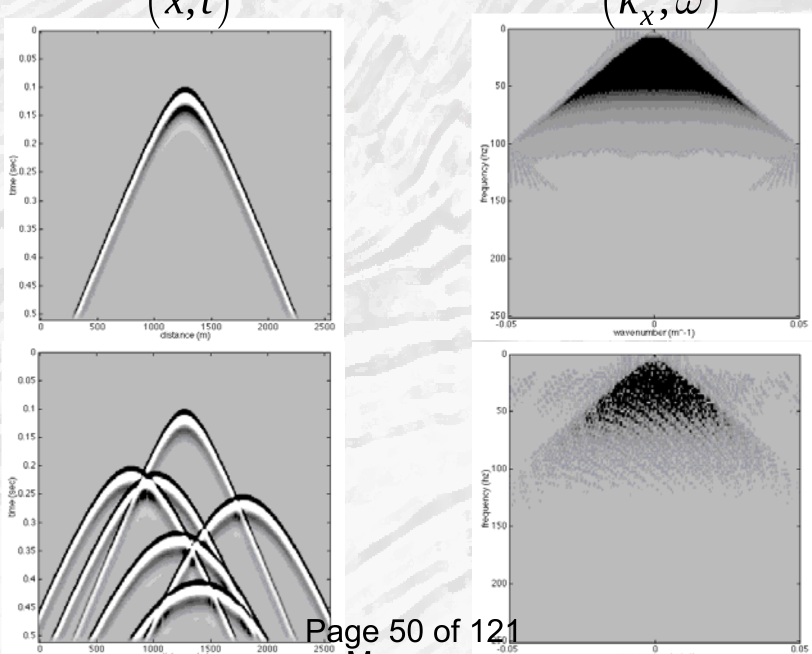
Fourier Transform Pairs

(x,t) (k_x,ω)



Fourier Transform Pairs

(x,t) (k_x,ω)



Part 2

Stationary Fourier Methods

Stationary Filters

A 1D stationary filter operation can be written

$$s(t) = \underbrace{\int_{\mathbb{R}} w(t-\tau)\, r(\tau)\, d\tau}_{\text{explicit integral}} \equiv \underbrace{(C_w r)(t)}_{\text{abstract operator}}$$

which is a convolution integral. In Seismology, for example, this is a prescription for generating a 1D synthetic seismogram when *r(t)* is called the reflectivity time series and *w(t)* is the source waveform or wavelet. The term *stationary* refers to the fact that *w(t)* appears in the integral dependent only upon the difference between input and output time. While this *translation independence* leads to beautiful mathematics, it fails to model a lot of physics.

Fourier Multipliers

Why we like stationarity

Every stationary convolution operator has a corresponding Fourier multiplier:

$$s(t) = (C_w r)(t) = \left(F^{-1} M_{\hat{w}} F r\right)(t)$$

or more simply

$$s = C_w r = F^{-1} M_{\hat{w}} F r$$

The "Convolution Theorem"

where:

$$M_a b \equiv ab$$

$$\hat{w} \equiv Fw$$

$$F = \text{the Fourier transform}$$

Fourier Multipliers
Inverse Operators

A Fourier multiplier has a simple inverse, if

$$s = F^{-1} M_{\hat{w}} F r$$

then

$$r = F^{-1} M_{\hat{w}^{-1}} F s$$

provided that $\left|\hat{w}\right| \neq 0$

$$F^{-1} M_{\hat{w}^{-1}} F s = F^{-1} M_{\hat{w}^{-1}} \underbrace{F F^{-1}}_{1} M_{\hat{w}} F r = F^{-1} \underbrace{M_{\hat{w}^{-1}} M_{\hat{w}}}_{1} F r = r$$

Fourier Multipliers
Inverse Operators

If $\hat{w} = 0$ somewhere in its domain, or is very small, then a common practice is to seek an approximate inverse such as

then

$$r \approx F^{-1} M_{\hat{w}_I} F s$$

where

$$\hat{w}_I = \frac{1}{\hat{w} + \mu \sup(\hat{w})}, \mu \in (0,1)$$

Fourier Multipliers
Square Root Operators

A Fourier multiplier has a square root operator. That is, if

$$s = F^{-1} M_{\hat{w}} F r$$

then $M_{\sqrt{\hat{w}}}$ is the square root multiplier in the sense that

$$F^{-1} M_{\sqrt{\hat{w}}} F F^{-1} M_{\sqrt{\hat{w}}} F = F^{-1} M_{\sqrt{\hat{w}}} M_{\sqrt{\hat{w}}} F = F^{-1} M_{\hat{w}} F$$

Generally this will require taking the square root of a complex-valued function so care must be taken to select the correct square-root branches.

Fourier Multipliers

Solution of PDE's

$$\frac{\partial^2\Psi}{\partial z^2}=\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\Psi$$

The constant-velocity wave equation rearranged.

$$\frac{\partial^2\Psi}{\partial z^2}=\frac{1}{4\pi^2}\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\underbrace{\int_{\mathbb{R}^2}\hat{\psi}(k_x,z,\omega)\mathrm{e}^{i(\omega t-k_x x)}\,dk_x d\omega}_{\Psi}$$

$$\frac{\partial^2\Psi}{\partial z^2}=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\alpha_2(k_x,\omega)\hat{\psi}(k_x,z,\omega)\mathrm{e}^{i(\omega t-k_x x)}\,dk_x d\omega$$

$$\alpha_2(k_x,\omega)=k_x^2-\frac{\omega^2}{v^2}$$

Fourier multiplier or symbol for the second z derivative.

Fourier Multipliers

Solution of PDE's

Now, we can deduce two alternative expressions for the first z derivative, as square root multipliers

$$\left(\frac{\partial\Psi}{\partial z}\right)^{\pm}=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\alpha_1^{\pm}(k_x,\omega)\hat{\psi}(k_x,z,\omega)\mathrm{e}^{i(\omega t-k_x x)}\,dk_x d\omega$$

$$\alpha_1^{\pm}(k_x,\omega)=\pm\sqrt{\alpha_2(k_x,\omega)}=\begin{cases}\pm i\,\mathrm{sign}(\omega)\sqrt{\dfrac{\omega^2}{v^2}-k_x^2}, & \dfrac{\omega^2}{v^2}\geq k_x^2\\ -\sqrt{k_x^2-\dfrac{\omega^2}{v^2}}, & k_x^2>\dfrac{\omega^2}{v^2}\end{cases}$$

These are examples of one-way wave equations. They are exact for v=constant and represent independent solutions to the full wave equation. However, this approach fails if v is not constant.

Exercise: Fourier Multipliers (!)

Solution of PDE's

Show that solutions to either of these one-way wave equations are also solutions to the two-way wave equation.

Let Ψ^+ satisfy $\dfrac{\partial \Psi^+}{\partial z} = F_2^{-1} M_{\alpha_1^+} F_2 \Psi^+$

Apply the first derivative twice $\dfrac{\partial}{\partial z}\left(\dfrac{\partial \Psi^+}{\partial z}\right) = \left[F_2^{-1} M_{\alpha_1^+} F_2\right] \dfrac{\partial}{\partial z} \Psi^+$

$$= F_2^{-1} M_{\alpha_1^+} F_2 F_2^{-1} M_{\alpha_1^+} F \Psi^+ = \underbrace{F_2^{-1} M_{\alpha_2} F_2 \Psi^+ = \frac{\partial^2 \Psi^+}{\partial z^2}}_{\text{The two-way equation}}$$

Operators and One-Way Wave Equations

$$\frac{\partial^2 \psi}{\partial z^2} = -\left[\frac{\omega^2}{v^2} + \frac{\partial^2}{\partial x^2}\right]\psi$$ The Helmholtz Operator

$$\frac{\partial^2 \psi}{\partial z^2} = \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_2(k_x,\omega)\hat{\psi}(k_x,z,\omega) e^{-ik_x x}\, dk_x}_{\text{The Helmholtz Operator realized as a Fourier Multiplier}} = \underbrace{F^{-1} M_{\alpha_2} F \psi}_{\text{abstract operator notation}}$$

$$\alpha_2(k_x,\omega) = k_x^2 - \frac{\omega^2}{v^2}$$ The Fourier multiplier or operator "symbol".

Operators and One-Way Wave Equations

Operator names:

$$\frac{\partial \psi^{\pm}}{\partial z} = \left[\pm i \sqrt{\frac{\omega^2}{v^2} + \frac{\partial^2}{\partial x^2}} \right] \psi^{\pm}$$

The Square Root Helmholtz Operator or one-way wave equation (frequency domain)

$$\frac{\partial \psi^{\pm}}{\partial z} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \alpha_1^{\pm}(k_x,\omega) \hat{\psi}^{\pm}(k_x,z,\omega) \mathrm{e}^{-ik_x x} \, dk_x}_{\text{Square Root Helmholtz Operator as Fourier Multiplier}} = \underbrace{F^{-1} M_{\alpha_1^{\pm}} F \psi}_{\text{abstract operator notation}}$$

$$\alpha_1^{\pm}(k_x,\omega) = \pm\sqrt{\alpha_2(k_x,\omega)} = \begin{cases} \pm i \,\mathrm{sign}(\omega) \sqrt{\dfrac{\omega^2}{v^2} - k_x^2}, & \dfrac{\omega^2}{v^2} \geq k_x^2 \\ -\sqrt{k_x^2 - \dfrac{\omega^2}{v^2}}, & k_x^2 > \dfrac{\omega^2}{v^2} \end{cases}$$

One-Way Wave Equations

$$\frac{\partial \psi^{\pm}}{\partial z} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \alpha_1^{\pm}(k_x,\omega) \hat{\psi}^{\pm}(k_x,z,\omega) \mathrm{e}^{-ik_x x} \, dk_x}_{\text{The Square Root Helmholtz Operator realized as a Fourier Multiplier}}$$

Solution: $\psi^{\pm}(x,z,\omega) = \dfrac{1}{2\pi} \int_{\mathbb{R}} A^{\pm}(k_x,\omega) \mathrm{e}^{\alpha_1^{\pm} z} \, \mathrm{e}^{-ik_x x} \, dk_x$

where $A^{\pm}(k_x,\omega) = \hat{\psi}^{\pm}(k_x, z=0, \omega)$ (boundary condition)

- Works in 2D or 3D
- Nonlocal operator (Two-way wave equation is local)
- Exact for homogeneous medium
- Not obvious what to do for variable velocity

Exercise: One Way Wave Equation (!)

Show that $\psi^{\pm}(x,z,\omega)=\dfrac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x,\omega)\mathrm{e}^{\alpha_1^{\pm}z}\,\mathrm{e}^{-ik_x x}\,dk_x$

(where *A* is arbitrary) solves the one-way wave equations on the previous slides. Then show that the + sign corresponds to waves traveling in the -z direction and the - sign gives waves traveling in the +z direction.

What happens with this approach when v depends on x?

Exercise: One Way Wave Equation solution (!)

We wish to show that $\psi^{\pm}=\dfrac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x,\omega)\mathrm{e}^{\alpha_1^{\pm}z}\,\mathrm{e}^{-ik_x x}\,dk_x \quad (1)$

Is a solution to

$$\frac{\partial\psi^{\pm}}{\partial z}=\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\pm}(k_x,\omega)\hat{\psi}^{\pm}(k_x,z,\omega)\mathrm{e}^{-ik_x x}\,dk_x \quad (2)$$

The z partial derivative of equation (1) is easy:

$$\frac{\partial}{\partial z}\left[\frac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x,\omega)\mathrm{e}^{\alpha_1^{\pm}z}\,\mathrm{e}^{-ik_x x}\,dk_x\right]=\frac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x,\omega)\left[\frac{\partial}{\partial z}\mathrm{e}^{\alpha_1^{\pm}z}\right]\mathrm{e}^{-ik_x x}\,dk_x$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\pm}A^{\pm}(k_x,\omega)\mathrm{e}^{\alpha_1^{\pm}z}\,\mathrm{e}^{-ik_x x}\,dk_x \quad (3)$$

Exercise: One Way Wave Equation solution

(!)

Now, since equation (1) is an inverse Fourier transform, it follows that

$$\hat{\psi}^{\pm}(k_x, z, \omega) = A^{\pm}(k_x, \omega) e^{\alpha_1^{\pm} z}$$

So that equation (3) reduces to

$$\frac{\partial}{\partial z}\left[\frac{1}{2\pi}\int_{\mathbb{R}} A(\xi_x, \omega) e^{\alpha_1^{\pm} z}\, e^{-ik_x x}\, dk_x\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\pm}\hat{\psi}^{\pm}(k_x, z, \omega) e^{-ik_x x}\, dk_x$$

which is equation (2).

Exercise: One Way Wave Equation solution

(!)

To determine the direction of travel of the solutions of equation (1) we write the corresponding time-domain solution

$$\Psi^{\pm}(x, z, t) = \frac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x, \omega) e^{\alpha_1^{\pm} z}\, e^{i(\omega t - k_x x)}\, dk_x d\omega \quad (4)$$

Now let

$$ik_z^{\pm} = \alpha_1^{\pm}(k_x, \omega) \Rightarrow k_z^{\pm} = \begin{cases} \pm \operatorname{sign}(\omega)\sqrt{\dfrac{\omega^2}{v^2} - k_x^2}, & \dfrac{\omega^2}{v^2} \geq k_x^2 \\ i\sqrt{k_x^2 - \dfrac{\omega^2}{v^2}}, & k_x^2 > \dfrac{\omega^2}{v^2} \end{cases} \quad (5)$$

And equation (4) is

$$\Psi^{\pm}(x, z, t) = \frac{1}{2\pi}\int_{\mathbb{R}} A^{\pm}(k_x, \omega) e^{ik_z^{\pm} z}\, e^{i(\omega t - k_x x)}\, dk_x d\omega \quad (6)$$

Exercise: One Way Wave Equation solution

(!)

From equation (5) it is obvious that

$$k_z^+ \geq 0, \text{ and } k_z^- \leq 0, \text{ when } \frac{\omega^2}{v^2} \geq k_x^2 \text{ and } \omega \geq 0$$

Equation (6) expresses the wavefield as a superposition of basis waves whose phase is given by

$$\theta(x,z,t,k_x,\omega) = \omega t - k_x x + k_z^\pm z$$

We track a wavefront, as a surface of constant phase, by equating the phase at (t_1,z) to that at (t_2,z+δz) by

$$\omega t_1 + k_z^\pm z = \omega t_2 + k_z^\pm\left(z + \delta z^\pm\right) \Rightarrow \delta z^\pm = \frac{\omega(t_1 - t_2)}{k_z^\pm}$$

So, taking $\omega > 0, t_2 > t_1$ we have

$$\delta z^+ < 0 \Rightarrow \Psi^+ \text{ is upgoing}$$

$$\delta z^- > 0 \Rightarrow \Psi^- \text{ is downgoing}$$

One Way Wave Equation

A Convenient Solution

$$\psi^\pm(x,z,\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\psi}^\pm(k_x, z=0, \omega)\, e^{ik_z^\pm z}\, e^{-ik_x x}\, dk_x$$

$$ik_z^\pm = \alpha_1^\pm(k_x,\omega) \Rightarrow k_z^\pm = \begin{cases} \pm\text{sign}(\omega)\sqrt{k^2 - k_x^2}, & k^2 \geq k_x^2 \\ i\sqrt{k_x^2 - k^2}, & k_x^2 > k^2 \end{cases} \qquad k^2 = \frac{\omega^2}{v^2}$$

This is a very convenient and accurate method of wavefield extrapolation, but what can we do if velocity varies?

Problem

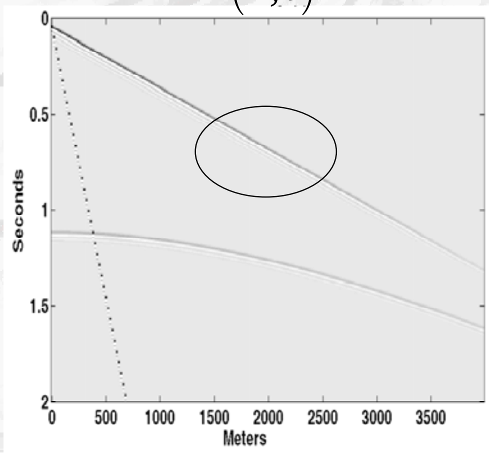
- We need wavefield analysis and filtering methods that adapt rapidly to spatial and temporal variations in the wavefield but still retain high fidelity.
- Raytracing offers rapid adaptation but poor fildelity.
- Fourier methods give high fidelity but poor spatial adaptivity.

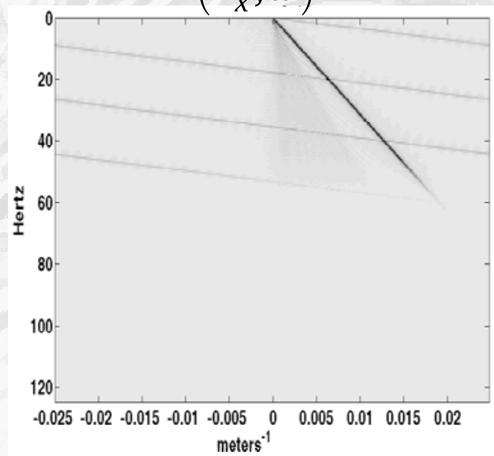
Part 3

Phase Space

Local Fourier Transforms

(x,t) (k_x,ω)

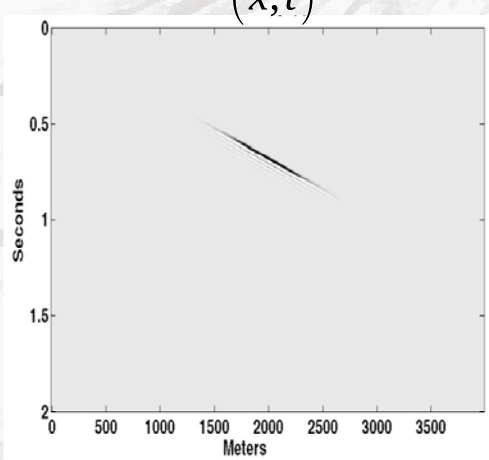


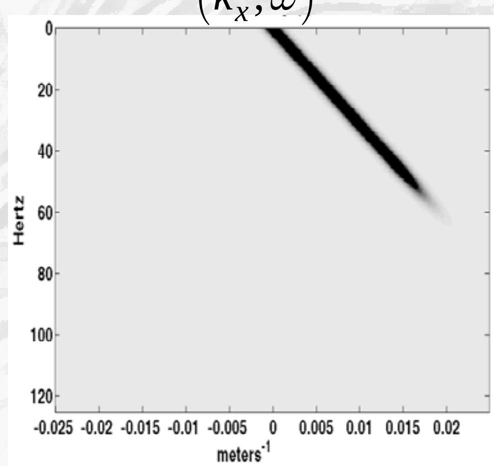


Apply a 2D Gaussian window in (x,t)

Local Fourier Transforms

(x,t) (k_x,ω)

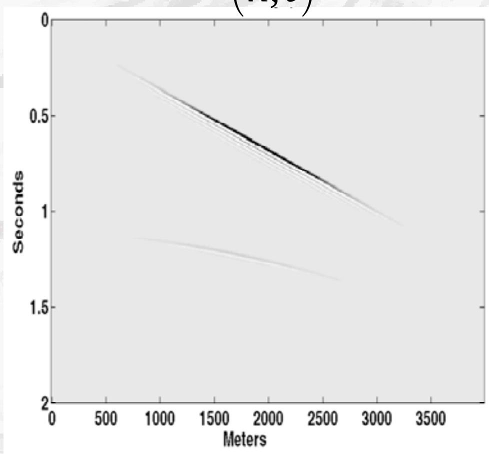


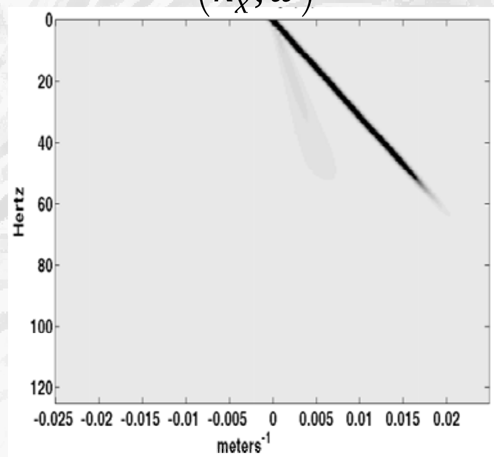


Localization in one domain causes blurring in the other

Local Fourier Transforms

(x,t) (k_x,ω)

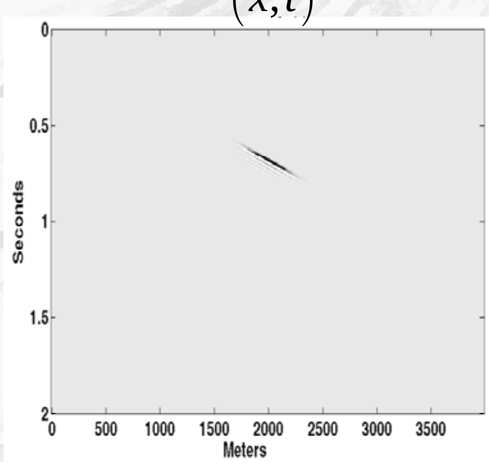


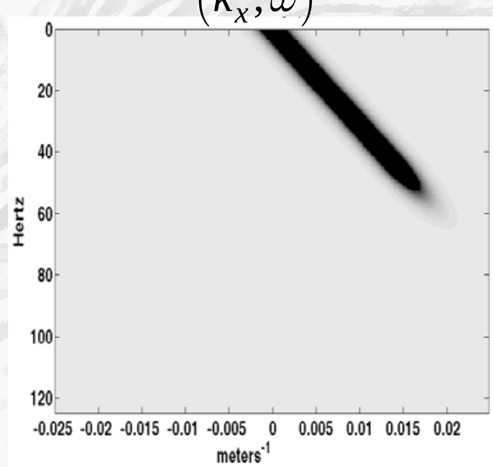


A larger window causes less blurring but is, of course, less local.

Local Fourier Transforms

(x,t) (k_x,ω)

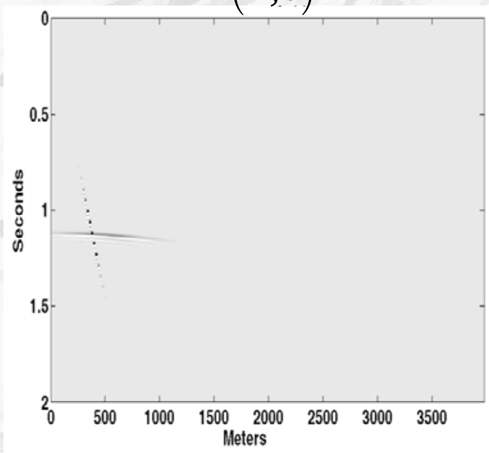


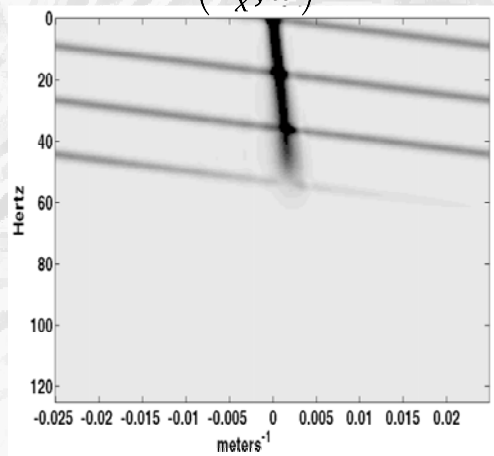


An even smaller window causes extreme blurring.

Local Fourier Transforms

(x,t) (k_x,ω)





Localizing somewhere else shows us a different spectrum.

Uncertainly Principle

Localization in (x,t) causes loss of detail in (k_x,ω). That is, we cannot precisely define the (k_x,ω) values at a precise (x,t) position. As Heisenberg showed in the context of quantum mechanics, this implies:

(uncertainty in (x,t))(uncertainty in (k_x,ω)) ≥ a constant

This is often stated as the time-width band-width theorem.

Question: Just what is meant by “uncertainty” in such a statement?

Time-width Band-width Theorem

Given any convenient measure of width, the time-width and bandwidth of a signal are inversely proportional.

$$E = \int_{\mathbb{R}} \left|s(x)\right|^2 dx$$

$$x_0 = \left[\int_{\mathbb{R}} x\left|s(x)\right|^2 dx\right] E^{-1} \qquad \xi_0 = \left[\int_{\mathbb{R}} \xi\left|\hat{s}(\xi)\right|^2 d\xi\right] E^{-1}$$

$$(\Delta x)^2 = \left[\int_{\mathbb{R}} (x-x_0)^2 \left|s(x)\right|^2 dx\right] E^{-1} \qquad (\Delta\xi)^2 = \left[\int_{\mathbb{R}} (\xi-\xi_0)^2 \left|\hat{s}(\xi)\right|^2 d\xi\right] E^{-1}$$

$$\Delta x \Delta \xi \geq (4\pi)^{-1}$$

The equality holds only for a Gaussian signal.

Time-limited Band-limited Theorem

If a signal, not identically zero, is compactly supported then its Fourier transform cannot be and vice-versa.

It follows that any finite length signal cannot be bandlimited.

Correspondence

- Associated with a neighborhood of a point in (x,t), there is a local Fourier spectrum. (Strictly speaking this depends upon the details of the localizing window.)
- Resolution in the local spectrum is directly proportional to the size (radius) of the neighborhood.

Phase Space

The phase space of a wavefield is the 8D manifold:

$$M : (x, y, z, t) \times \left(k_x, k_y, k_z, \omega\right)$$

Methods that have been devised to directly manipulate a field on its phase space include:

- **Ray tracing**
- **Pseudodifferential operators**
- **Gabor Multipliers**
- **Nonstationary filters**

Part 4

Pseudodifferential Operators

Helmholtz Operator

Variable Velocity

Construct the Helmholtz operator when v=v(x):

$$\frac{\partial^2\psi}{\partial z^2} = \frac{-1}{2\pi}\left[\frac{\omega^2}{v(x)^2} + \frac{\partial^2}{\partial x^2}\right]\int_{\mathbb{R}} \hat{\psi}(k_x, z, \omega) e^{-ik_x x}\, dk_x$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha_2(k_x, x, \omega)\hat{\psi}(k_x, z, \omega) e^{-ik_x x}\, dk_x$$

Helmholtz Operator

$$\alpha_2(k_x, x, \omega) = k_x^2 - \frac{\omega^2}{v(x)^2}$$

Helmholtz Symbol
(a function on phase space)

Superficially the Helmholtz operator appears the same as before; however, this integral is no longer an inverse Fourier transform but is instead an example of a pseudodifferential operator, specifically of the Kohn-Nirenberg (standard) calculus.

Pseudodifferential Operators

Kohn-Nirenberg standard form:

$$\underbrace{g_s(x)}_{\text{signal}} = \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\alpha(x,k_x)}_{\substack{\text{generalized}\\ \text{multiplier}}} \underbrace{\hat{h}(k_x)}_{\text{spectrum}} e^{-ik_x x} dk_x \equiv \left(F_\alpha^I \hat{h}\right)(x)$$

Kohn-Nirenberg anti-standard form:

$$\underbrace{\hat{g}_a(k_x)}_{\text{spectrum}} = \int_{\mathbb{R}} \underbrace{\alpha(x,k_x)}_{\substack{\text{generalized}\\ \text{multiplier}}} \underbrace{h(x)}_{\text{signal}} e^{ik_x x} dx \equiv (F_\alpha h)(k_x)$$

In general $g_a \neq g_s$, although you should be able to find an obvious case when they are equal.

Pseudodifferential Operators

Most of the time, we use the K-N standard form

$$g_s(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha(x,k_x) \hat{h}(k_x) e^{-ik_x x} dk_x \equiv \left(F_\alpha^I \hat{h}\right)(x)$$
$$= \left(F_\alpha^I F h\right)(x) \equiv T_\alpha h(x)$$

That is $\underbrace{T_\alpha}_{\text{abstract form}} = \underbrace{F_\alpha^I F}_{\substack{\text{Fourier integral}\\ \text{decomposition}}}$

So a standard pseudodifferential operator consists of an ordinary forward Fourier transform followed by the generalized multiplier-inverse transform.

Pseudodifferential Operators

(!)

These operators extend the idea of Fourier multipliers to the "nonstationary" setting.

Definition: The x dependence of the symbol will be called its nonstationary dependence.

Definition: A "stationary limit" of a pseudodifferential operator is any limiting form of the operator in which the nonstationary dependence of the symbol becomes constant.

Pseudodifferential Operators as generalizations of convolution

(!)

We have:

$$\left.\begin{aligned}\lim_{stat} F_{\alpha}^{I} &= F^{-1} M_{\alpha_s} \\ \lim_{stat} F_{\alpha} &= M_{\alpha_s} F\end{aligned}\right\}$$

The standard and anti-standard operators have the same stationary limits

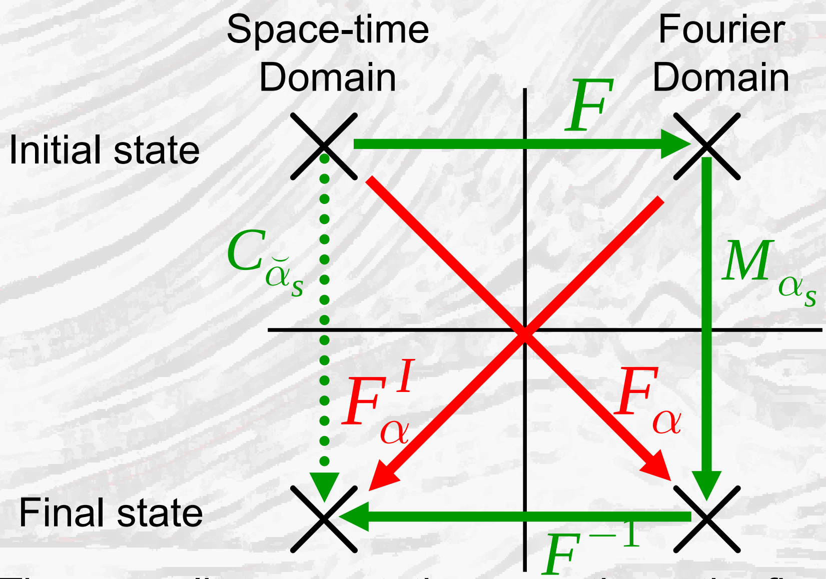
$$\left.\lim_{stat} T_{\alpha} = C_{\alpha_s}\right\}$$

The stationary limit is a convolution operator.

where $\lim_{stat} \alpha = \alpha_s$

(!)

Pseudodifferential Operators



The green lines are stationary paths to the final state while the red lines are nonstationary. In general, the red paths give a different result if the same symbol is used.

(!)

Spaces and Symbol Classes

Usually pseudodifferential operators can be extended to mappings:

$$T_\alpha : S' \to S'$$

Symbols are classified by the order of their polynomial growth at infinity:

We say $\alpha \in S_m$

if $$\frac{\partial^\rho \alpha}{\partial {k_x}^\rho} = O\left(\left[1 + |k_x|^2 \right]^{(m-\rho)/2} \right), \rho \in \mathbb{N}, m \in \mathbb{Z}$$

Symbols are also classified by their growth in x.

The Square-Root Helmholtz Operator

Back to the Helmholtz operator, in case of arbitrary v(x), we might still hope that

$$\frac{\partial \psi^{\pm}}{\partial z} \stackrel{?}{=} \frac{1}{2\pi} \int_{\mathbb{R}} \pm\sqrt{\alpha_2(k_x, x, \omega)}\, \hat{\psi}^{\pm}(k_x, z, \omega) \mathrm{e}^{-ik_x x}\, dk_x$$

It turns out that this is still a useful approximate one-way wave equation but its solutions are not exact solutions to the two-way equation.

The Square-Root Helmholtz Operator

Let, $\alpha_1(k_x,x,\omega)$ be the exact symbol of the square root Helmholtz operator for upgoing waves

$$\frac{\partial \psi}{\partial z} = \frac{1}{2\pi} \int_{\mathbb{R}} \alpha_1(k_x, x, \omega)\, \hat{\psi}(k_x, z, \omega) \mathrm{e}^{-ik_x x}\, dk_x \equiv T_{\alpha_1} \psi$$

Then, the following composition equation must be satisfied

$$\frac{\partial^2 \psi}{\partial z^2} = T_{\alpha_2} \psi = \left(T_{\alpha_1} \circ T_{\alpha_1}\right) \psi$$

That is, two applications of $T_{\alpha 1}$ must give $T_{\alpha 2}$ which is known exactly. In a generalized sense we are asking for the operator square root of a particular pseudodifferential operator.

Pseudodifferential Operators Composition Theorem

Let T_α T_β be two *elliptic* pseudodifferential operators with suitably nice symbols. Then

$$T_\beta \circ T_\alpha = T_\gamma \qquad \alpha \in S_m, \beta \in S_n \Rightarrow \gamma \in S_{m+n}$$

where γ has the asymptotic expansion

$$\gamma \sim \alpha\beta - i\frac{\partial\beta}{\partial\xi}\frac{\partial\alpha}{\partial x} - \frac{1}{2}\frac{\partial^2\beta}{\partial\xi^2}\frac{\partial^2\alpha}{\partial x^2}\cdots$$

This expansion is the generalization of the convolution theorem to the setting of pseudodifferential operators.

All of the higher order terms vanish in the stationary limit.

Pseudodifferential Operators

So, if we define $\alpha_1^{\text{lha}} = \sqrt{\alpha_2}$ $\left(\text{and } \alpha_1^{\text{-lha}} = -\sqrt{\alpha_2}\right)$

Then $T_{\alpha_1^{\text{lha}}} \circ T_{\alpha_1^{\text{lha}}} \psi = T_\gamma \psi$

where

$$\gamma \sim \left(\alpha_1^{\text{lha}}\right)^2 - i\frac{\partial\alpha_1^{\text{lha}}}{\partial\xi}\frac{\partial\alpha_1^{\text{lha}}}{\partial x} + \cdots = \alpha_2 - i\frac{\partial\alpha_1^{\text{lha}}}{\partial\xi}\frac{\partial\alpha_1^{\text{lha}}}{\partial x} + \cdots$$

Thus, only in the homogeneous (stationary) case is the square-root symbol the exact symbol of the one-way wave equation. However, it is still a very powerful approximation.

It is still possible to find an exact factorization in certain cases (e.g. Fishman …).

Pseudodifferential Operators

(!)

A problem with attempting this factorization using pseudodifferential operator theory is that the theory assumes the relevant symbols are *elliptic*.

Definition: A pseudodifferential symbol is said to be elliptic if there exists a constant C such that:

$$\left|\alpha\left(x,k_x\right)\right| > C\left|k_x\right|, \forall x \in \mathbb{R}$$

Symbol $\alpha_2\left(k_x, x, \omega\right) = k_x^2 - \frac{\omega^2}{v\left(x\right)^2}$ is not elliptic.

Wavefield Extrapolators

Recall the one-way wave equation

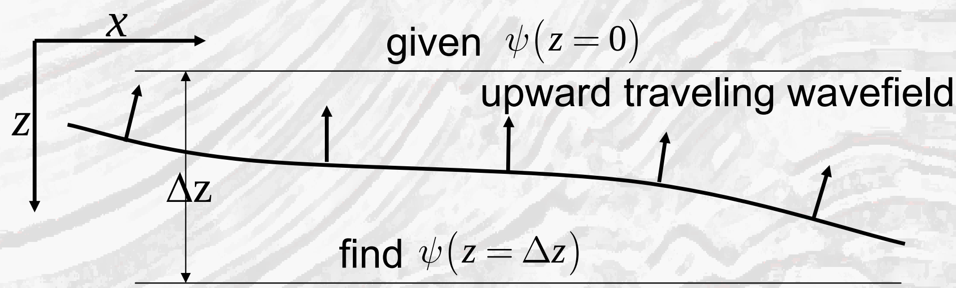
$$\frac{\partial \psi^+}{\partial z} = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\text{lha}}\left(k_x, x, \omega\right)\hat{\psi}^+\left(k_x, z, \omega\right)\mathrm{e}^{-ik_x x}\,dk_x$$

$$\alpha_1^{\text{lha}}\left(k_x, x, \omega\right) = \sqrt{\alpha_2\left(k_x, x, \omega\right)} \equiv ik_z\left(k_x, x, \omega\right)$$

$$k_z = \begin{cases} \sqrt{\dfrac{\omega^2}{v(x)^2} - k_x^2}, & \dfrac{\omega^2}{v(x)^2} \geq k_x^2 \\ i\sqrt{k_x^2 - \dfrac{\omega^2}{v(x)^2}}, & k_x^2 > \dfrac{\omega^2}{v(x)^2} \end{cases}$$

Wavefield Extrapolators

We wish to solve the wavefield extrapolation problem:



$$\frac{\partial \psi}{\partial z}=\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\text{lha}}\left(k_x,x,\omega\right)\hat{\psi}\left(k_x,z,\omega\right)\mathrm{e}^{-ik_x x}\,dk_x$$

one-way equation for upward traveling waves.

or $$\frac{\partial \psi}{\partial z}=T_{\alpha_1^{\text{lha}}}\psi\left(k_x,z,\omega\right)$$

Wavefield Extrapolators
The GPSPI formula

It turns out that pseudodifferential theory allows the following approximation

$$\hat{\psi}\left(x,\Delta z,\omega\right)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k\left(x\right),k_x,\Delta z\right)\hat{\psi}\left(k_x,0,\omega\right)e^{-ik_x x}dk_x$$

$$\hat{W}\left(k\left(x\right),k_x,\Delta z\right)=\begin{cases}\exp\left(i\Delta z\sqrt{k^2\left(x\right)-k_x^2}\right), k^2\left(x\right)\geq k_x^2\\ \exp\left(-\Delta z\sqrt{k_x^2-k^2\left(x\right)}\right), k^2\left(x\right)< k_x^2\end{cases}$$

$$k\left(x\right)=\frac{\omega}{v\left(x\right)}$$

This is known as the GPSPI (generalized phase shift plus interpolation) wavefield extrapolator.

Exercise: Derive The GPSPI formula (!)

Use Taylor series to derive the GPSPI formula from the approximate 1-way wave equation

$$\frac{\partial \psi}{\partial z} = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha_1^{\mathrm{lha}}\left(k_x, x, \omega\right)\hat{\psi}\left(k_x, z, \omega\right)\mathrm{e}^{-ik_x x}\, dk_x$$

$$\alpha_1^{\mathrm{lha}} = \begin{cases} i\sqrt{\dfrac{\omega^2}{v(x)^2} - k_x^2}, & \dfrac{\omega^2}{v(x)^2} \geq k_x^2 \\ -\sqrt{k_x^2 - \dfrac{\omega^2}{v(x)^2}}, & k_x^2 > \dfrac{\omega^2}{v(x)^2} \end{cases}$$

Carefully describe all approximations made.

Derive The GPSPI formula Solution (!)

Assume the starting depth is 0, and then write the wavefield one step down as a formal Taylor series

$$\psi(\Delta z) = \psi(0) + \Delta z \left.\frac{\partial \psi}{\partial z}\right|_{z=0} + \frac{(\Delta z)^2}{2}\left.\frac{\partial^2 \psi}{\partial z^2}\right|_{z=0} + \cdots + \frac{(\Delta z)^k}{k!}\left.\frac{\partial^k \psi}{\partial z^k}\right|_{z=0} + \cdots$$

which can be written symbolically as

$$\psi(\Delta z) = \psi_0 + \Delta z T_{\alpha_1^{\mathrm{lha}}}\psi_0 + \frac{(\Delta z)^2}{2} T_{\alpha_1^{\mathrm{lha}}} \circ T_{\alpha_1^{\mathrm{lha}}}\psi_0 + \ldots$$

Derive The GPSPI formula Solution

(!)

According to the composition theorem

$$\underbrace{\left(T_{\alpha_1^{\text{lha}}} \circ T_{\alpha_1^{\text{lha}}} \cdots\right)}_{n\text{ times}} \psi_0 \equiv T_\gamma \psi_0$$

is a pseudodifferential operator whose symbol has a first order approximation: $\gamma \sim \left(\alpha_1^{\text{lha}}\right)^n$

So, with an unknown error, we approximate the Taylor series as

$$\psi(\Delta z) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(1 + \Delta z \alpha_1^{\text{lha}} + \frac{\left(\Delta z \alpha_1^{\text{lha}}\right)^2}{2} + \ldots\right) \hat{\psi}_0 e^{-ik_x x} dk_x$$

The series in brackets converges to the exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

Derive The GPSPI formula Solution

(!)

Summing the series gives $\psi(\Delta z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\Delta z \alpha_1^{\text{lha}}} \hat{\psi}_0 e^{-ik_x x} dk$

or

$$\psi(x, \Delta z, \omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{W}\left(k(x), k_x, \Delta z\right) \hat{\psi}\left(k_x, 0, \omega\right) e^{-ik_x x} dk_x$$

$$\hat{W}\left(k(x), k_x, \Delta z\right) = \begin{cases} \exp\left(i\Delta z\sqrt{k^2(x) - k_x^2}\right), k^2(x) \geq k_x^2 \\ \exp\left(-\Delta z\sqrt{k_x^2 - k^2(x)}\right), k^2(x) < k_x^2 \end{cases}$$

$$k(x) = \frac{\omega}{v(x)}$$

This is known as the GPSPI (generalized phase shift plus interpolation) wavefield extrapolator.

Derive The GPSPI formula Solution

(!)

The GPSPI extrapolator

$$\psi(x,\Delta z,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k(x),k_x,\Delta z\right)\hat{\psi}(k_x,0,\omega)e^{-ik_x x}dk_x$$

Summary of approximations:

(1) $\alpha_1^{\pm}(k_x,x,\omega)\approx\pm\sqrt{\alpha_2(k_x,x,\omega)}$ True only for homogeneous medium.

(2) $\alpha_n\approx\alpha_1^n$ Only asymptotically valid even if the first derivative symbol is exact.

(3) $\alpha_2(k_x,x,\omega)$ is elliptic. A hidden assumption. Elliptic means bounded away from zero and this is false.

(4) The Taylor series converges. It does in some specific cases but we don't know in general.

The GPSPI Extrapolator

The GPSPI extrapolator

$$\psi(x,\Delta z,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k(x),k_x,\Delta z\right)\hat{\psi}(k_x,0,\omega)e^{-ik_x x}dk_x$$

Things we know (or think we do):

(1) Any explicit finite difference method is an approximation to GPSPI.

(2) "Screen" methods are approximations to GPSPI.

(3) GPSPI produces very high quality seismic images but it is computationally expensive.

(4) More accurate methods can be formulated simply as operators with different symbols.

Fishman's results

The exact one-way extrapolator, equivalently the one-way wave equation, for arbitrary v(x) can also be written this way, but the symbol becomes much more complicated.

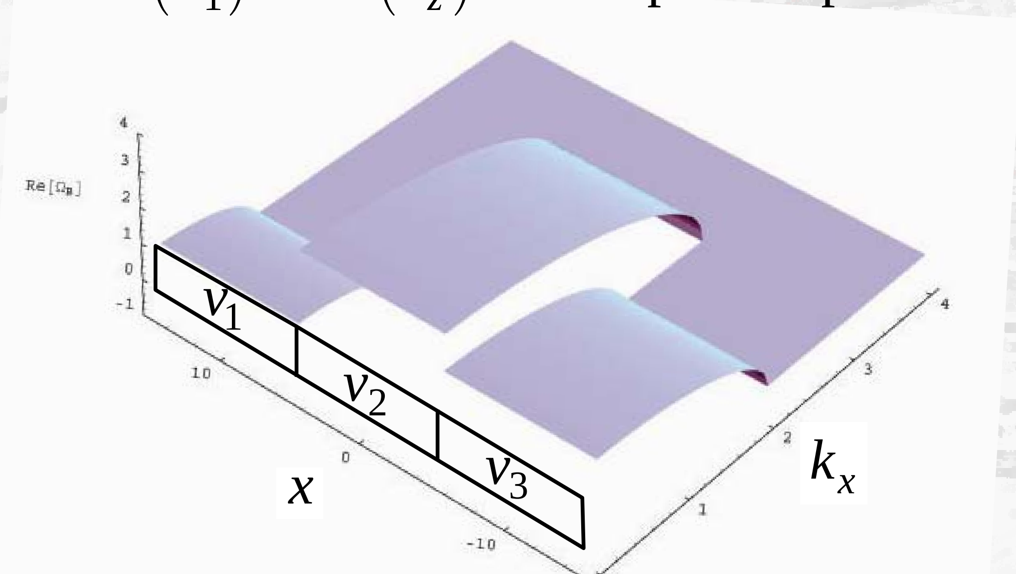
$$\psi(x,\Delta z,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}_{exact}(x,k_x,\Delta z)\hat{\psi}(k_x,0,\omega)e^{-ik_x x}dk_x$$

The cascade of many such operators in a wavefield marching scheme is a numerical computation of a *Path Integral.* This means that energy propagates along all possible paths not just the Snell paths.

To find the exact operator,

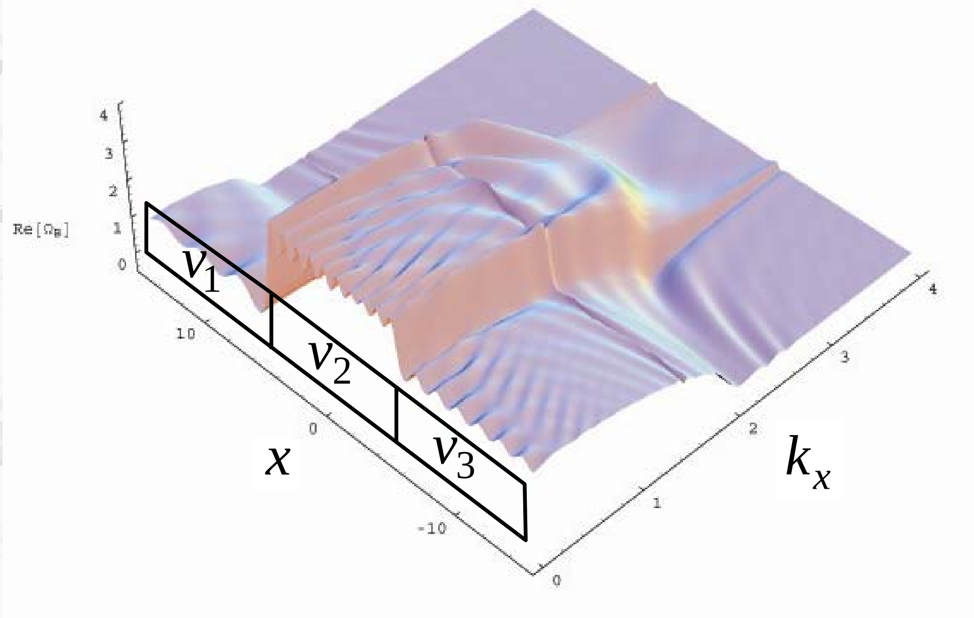
Fishman: Locally Homogeneous Approximation 3 block velocity function

$$\mathrm{Im}(\alpha_1)=\mathrm{Re}(k_z)=\text{extrapolator phase}$$



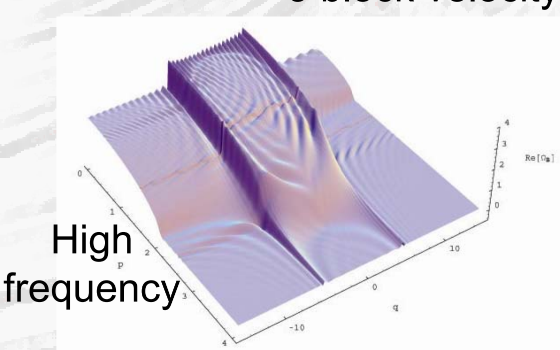
Fishman: Exact Operator Symbol
3 block velocity function

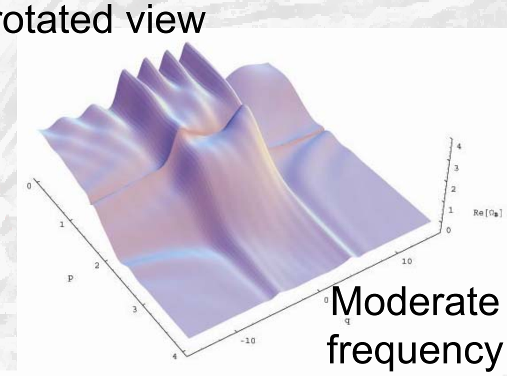
$$\mathrm{Im}(\alpha_1) = \mathrm{Re}(k_z) = \text{extrapolator phase}$$

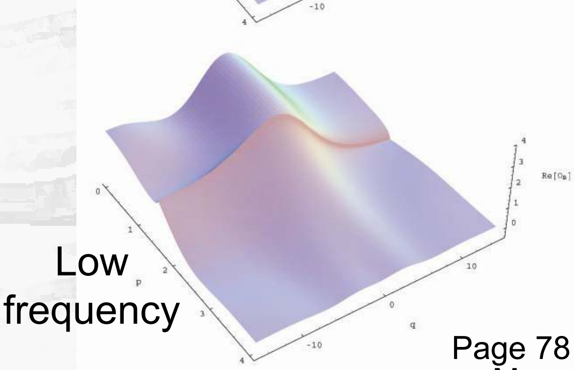


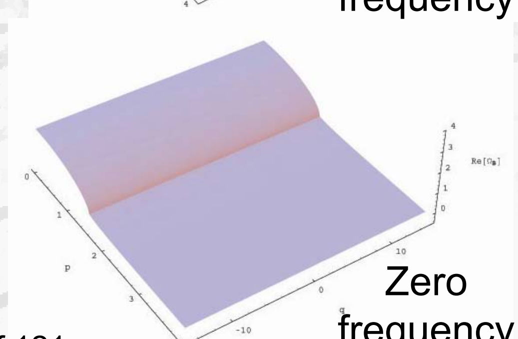
The Exact Operator Symbol
is Frequency Dependent

3 block velocity, rotated view









Exercise

Schwartz Kernel of a Pseudodifferential Operator

(!)

given: $$s(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,k_x)\hat{r}(k_x)e^{-ik_x x}dk_x$$

show by formal manipulation (don't worry about conversion etc) that this is equivalent to

$$s(x)=\int_{\mathbb{R}}A(x,x-y)r(y)dy \qquad A(x,x-y)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,k_x)e^{-ik_x(x-y)}dk_x$$

The quantity A(x,x-y) is called the Schwartz kernel of the pseudodifferential operator and the integral applying A is called a singular integral operator.

Singular Integral form of a ΨDO

(!)

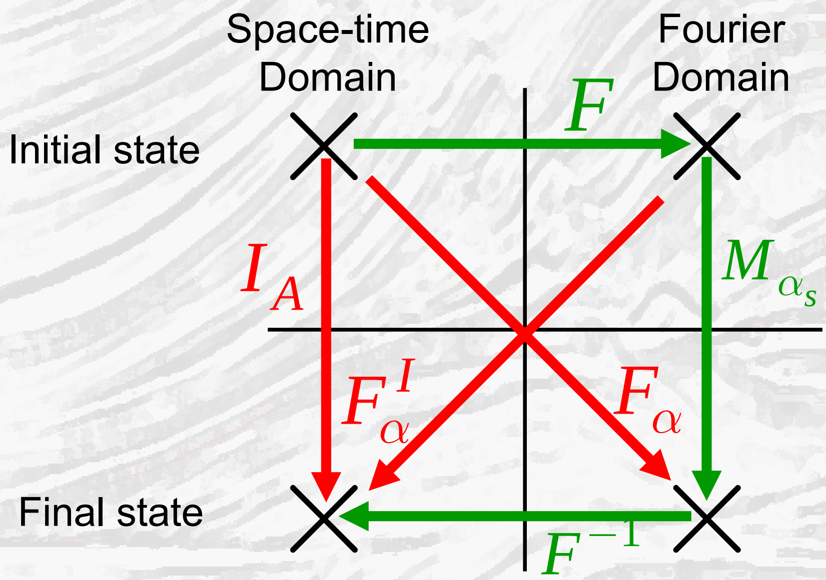
Given: $$s=T_\alpha r$$

then, with suitable circumstances, it follows that

$$s(x)=(I_A r)(s)\equiv\int_{\mathbb{R}}A(x,x-y)r(y)dy$$

where $$A(x,x-y)=\frac{1}{2\pi}\int_{\mathbb{R}}\alpha(x,k_x)e^{-ik_x(x-y)}dk_x$$

Pseudodifferential Operators



Exercise

(!)

Schwartz Kernel of a Fourier Multiplier

Given:

$$s = F^{-1} M_\alpha F r \qquad \alpha(k_x) : \mathbb{R} \to \mathbb{R}$$

show that the Schwartz kernal depends only on x-y (translation invariance) and that the resulting singular integral operator is just a convolution.

Conclusions

Fourier transforms provide a powerful method for stationary problems.

At every point in space-time, a localized Fourier space, with understandable properties, is easily defined.

The concept of phase space facilitates nonstationary extensions of Fourier theory.

Pseudodifferential operators generalize the concept of Fourier multipliers and convolutional operators to the nonstationary setting. They are generalized Fourier multipliers acting directly on phase space.

Phase Space Concepts in Seismic Imaging Part II

Seismic Imaging Summer School

Calgary, 2006

Gary F. Margrave

Outline

- A Pseudodifferential Operator Imaging Method
    - Separable Symbols and The Gabor Transform
- A Gabor Imaging Method

Part 1

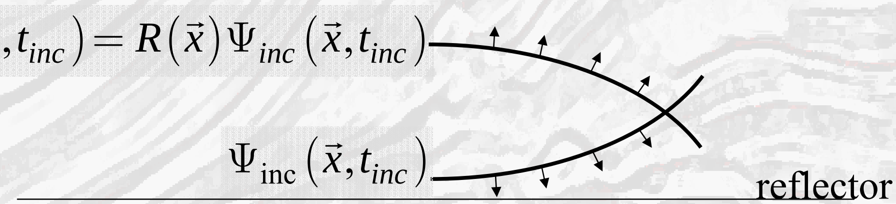
A Pseudodifferential Operator Imaging Method

Seismic Imaging Paradigm

A common seismic imaging methodology is derivable from first-order inverse Born scattering

$$\Psi_{refl}\left(\vec{x},t_{inc}\right)=R\left(\vec{x}\right)\Psi_{inc}\left(\vec{x},t_{inc}\right)$$

$$\Psi_{\mathrm{inc}}\left(\vec{x},t_{inc}\right)$$



$$\frac{\Psi_{refl}\left(\vec{x},t_{inc}\right)}{\Psi_{inc}\left(\vec{x},t_{inc}\right)}=R\left(\vec{x}\right)$$ A reflectivity estimate.

Seismic Imaging Paradigm

So for each depth, we must calculate two fields:

$$\psi_{refl}(x, y, n\Delta z, \omega)$$

The reflected field comes from mathematically marching the recorded data down into the earth.

$$\psi_{inc}(x, y, n\Delta z, \omega)$$

The incident field comes from a mathematical model of the source wavefield that is also marched down.

In both cases, the wavefield marching is done through a "background" velocity field that is presumed known.

Wavefield Extrapolator

Locally homogeneous approximation (GPSPI)

A K-N form FIO

$$\psi(x, z+\Delta z, \omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\psi}(k_x, z, \omega)\hat{W}\left(k(x), k_x, \Delta z\right)e^{-ik_x x}dk_x$$

$$\hat{W}\left(k(x), k_x, \Delta z\right) = \begin{cases} \exp\left(i\Delta z\sqrt{k^2(x) - k_x^2}\right), k^2(x) > k_x^2 \\ \exp\left(-\Delta z\sqrt{k_x^2 - k^2(x)}\right), k^2(x) < k_x^2 \end{cases}$$

Symbol (physics)

$$k^2(x) = \frac{\omega^2}{v(x)^2}$$

While a highly accurate approximation, this form is computationally challenging.

Wavefield Extrapolator Imaging

$$\psi_{refl}\left(x,z+\Delta z,\omega\right)=L_{W(z)}\left[\psi_{refl}\left(x,z,\omega\right)\right]$$

$$\psi_{refl}\left(x,z+\Delta z,\omega\right)=\underbrace{L_{W(z)}\circ\cdots L_{W(2\Delta z)}\circ L_{W(\Delta z)}\circ L_{W(0)}}_{\text{Hundreds of operators}}\left[\psi_{refl}\left(x,0,\omega\right)\right]$$

$$\psi_{inc}\left(x,z+\Delta z,\omega\right)=L_{W(z)}\circ\cdots L_{W(2\Delta z)}\circ L_{W(\Delta z)}\circ L_{W(0)}\left[\psi_{inc}\left(x,0,\omega\right)\right]$$

$$R\left(x,y,z+\Delta z\right)=\sum_{\omega}\frac{\psi_{refl}\left(x,z+\Delta z,\omega\right)}{\psi_{inc}\left(x,z+\Delta z,\omega\right)}$$

Possible routes to fast algorithms

- Approximate the operator using a compactly supported Schwartz kernel.
- Find a separable approximation to the K-N symbol (screen methods).
- Gabor methods (also separable).

Wavefield Extrapolators

In the space-frequency domain

$$\psi(x,z,\omega)=\int_{\mathbb{R}} W\left(k(x),x-x',z\right)\psi\left(x',z=0,\omega\right)dx'$$

where the Schwartz kernel is given by

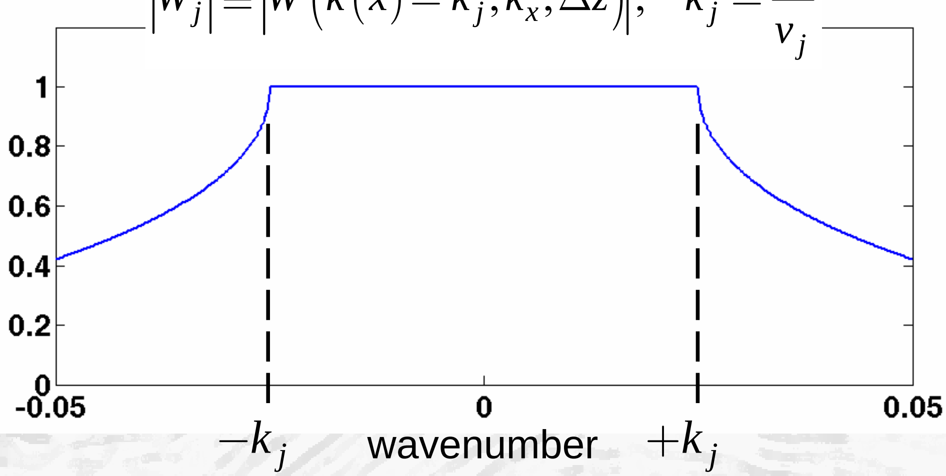
$$W\left(k(x),x-x',z\right)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{W}\left(k(x),k_x,z\right)e^{-ik_x(x-x')}dk_x$$

This can be an efficient algorithm if a suitably, compactly supported, approximation to *W* can be found.

Wavefield Extrapolators

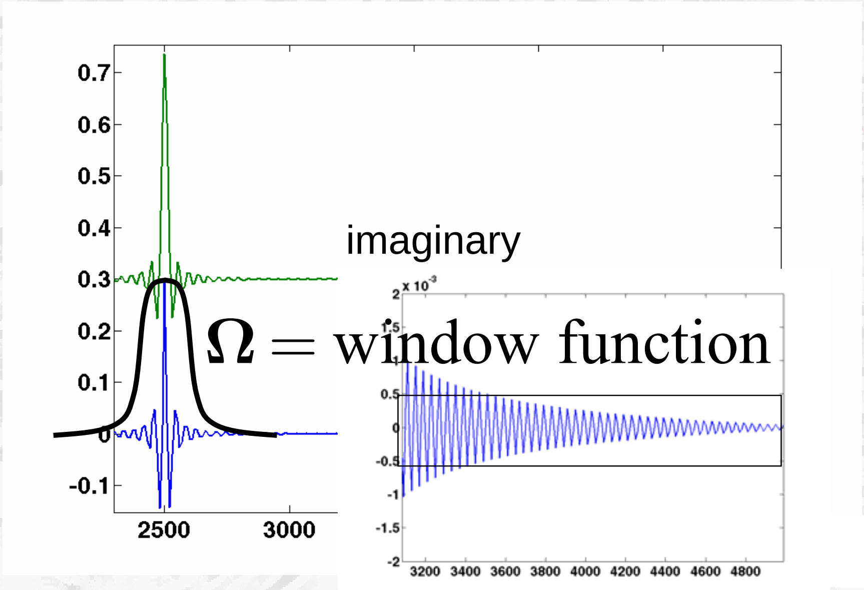
Consider the wavefield extrapolator in the wavenumber domain for some fixed velocity, v_j.

$$\left|\hat{W}_j\right|\equiv\left|\hat{W}\left(k(x)=k_j,k_x,\Delta z\right)\right|,\quad k_j=\frac{\omega}{v_j}$$



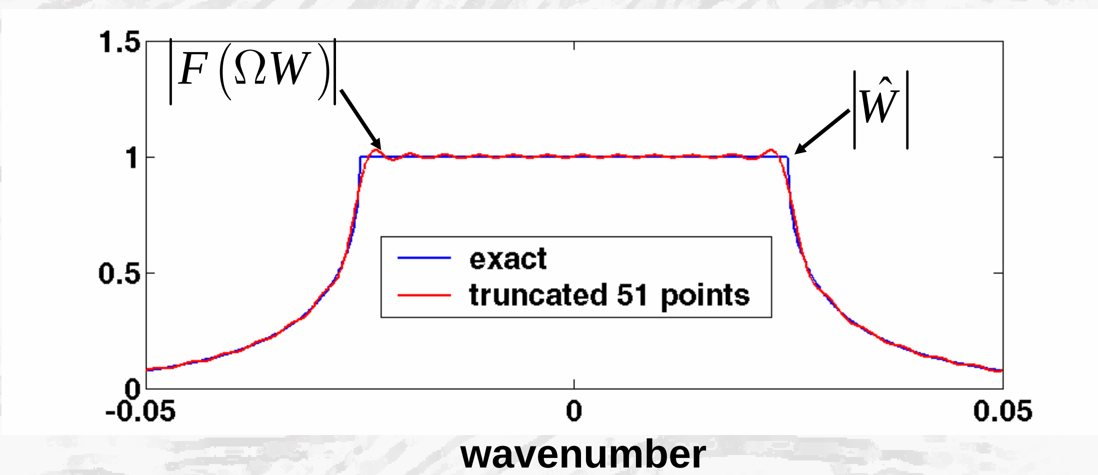
Wavefield Extrapolators

In the space-frequency domain



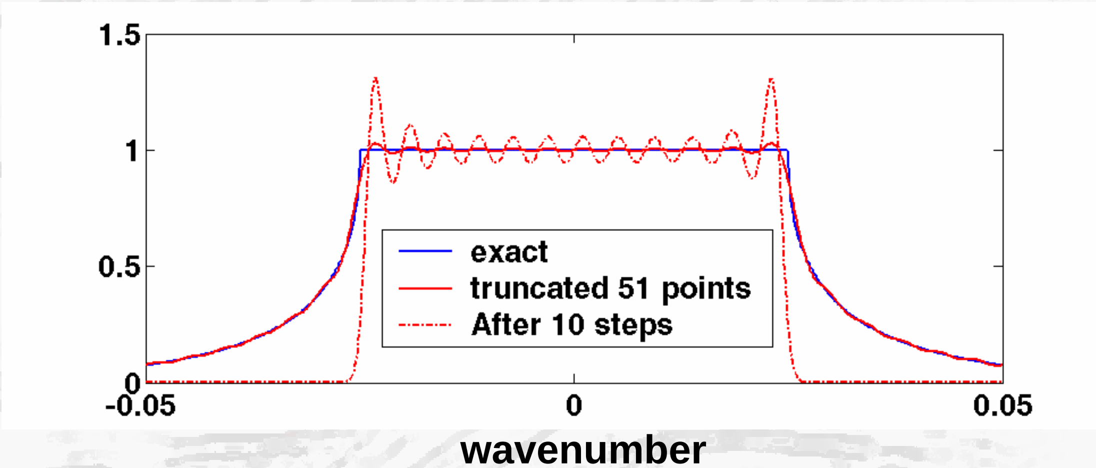
Wavefield Extrapolators

Back to the wavenumber domain



Wavefield Extrapolators

Back to the wavenumber domain



Stabilization by Wiener Filter

Two useful properties

$$\hat{W}_j(\Delta z) = \hat{W}_j(\Delta z/2)\hat{W}_j(\Delta z/2)$$

Product of two half-steps make a whole step.

$$\hat{W}_j^{-1} = \hat{W}_j^*, \quad k_j^2 > k_x^2$$

The inverse is equal to the conjugate in the wavelike region.

Stabilization by Wiener Filter

A windowed forward operator for a half-step

$$\tilde{W}_j\left(\Delta z/2\right) = \Omega W_j\left(\Delta z/2\right)$$

Solve by least squares for *WI*$_j$

$$\tilde{W}_j\left(\Delta z/2\right) \bullet WI_j = F^{-1}\left[\left|\hat{W}_j\left(\Delta z/2\right)\right|^{\eta}\right]$$

$$0 \le \eta \le 2$$

Stabilization by Wiener Filter

WI_j **is a band-limited inverse for** $\tilde{W}_j\left(\Delta z/2\right)$

Both have compact support

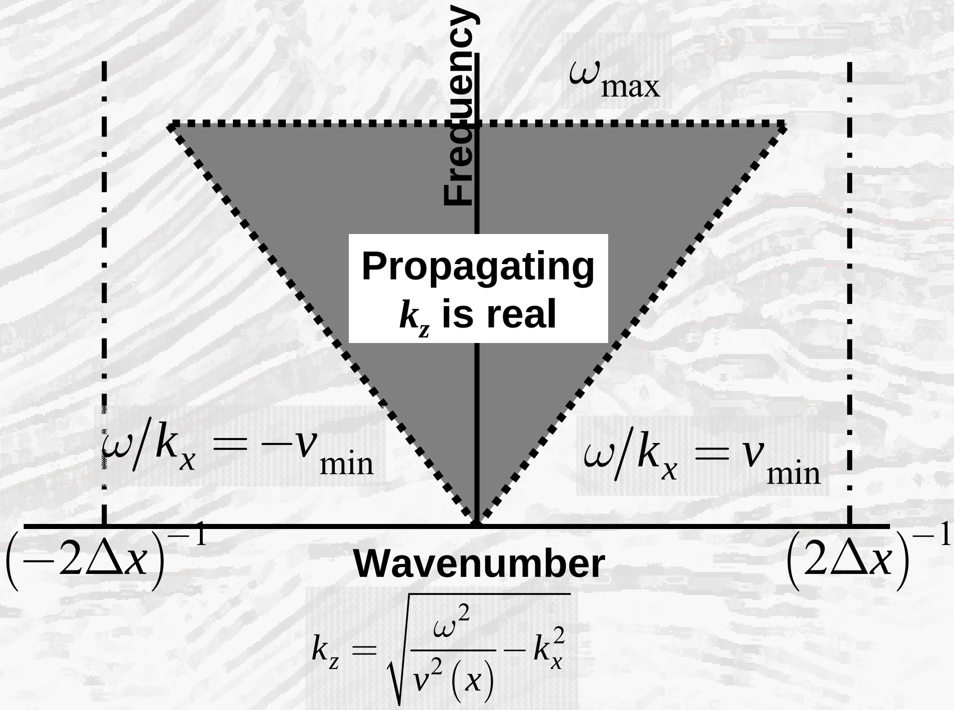
Form the FOCI approximate operator by

$$W_{Fj}\left(\Delta z\right) = WI_j^* \bullet \tilde{W}_j\left(\Delta z/2\right) \approx W_j\left(\Delta z\right)$$

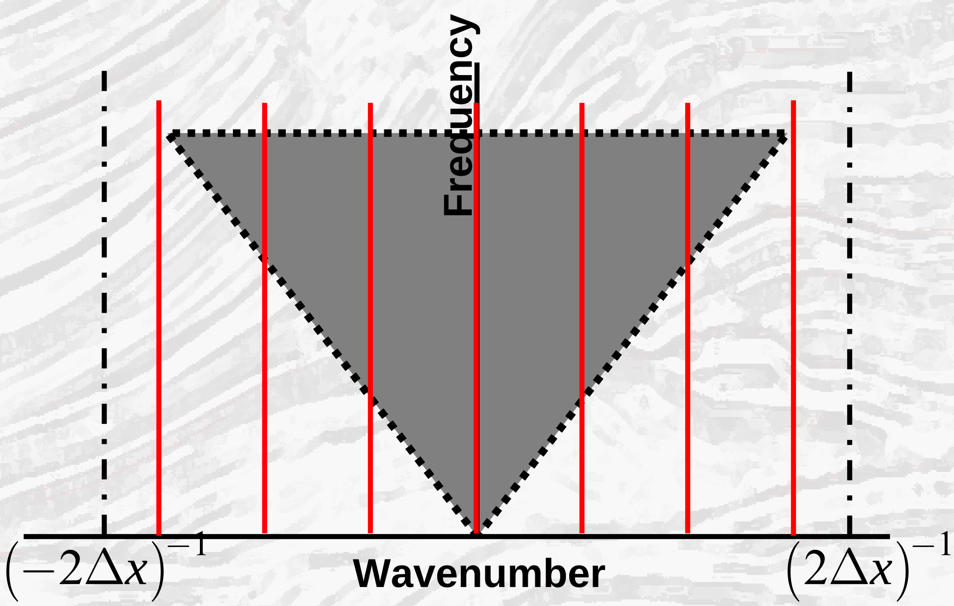
FOCI is an acronym for

Forward Operator with Conjugate Inverse.

Spatial Resampling

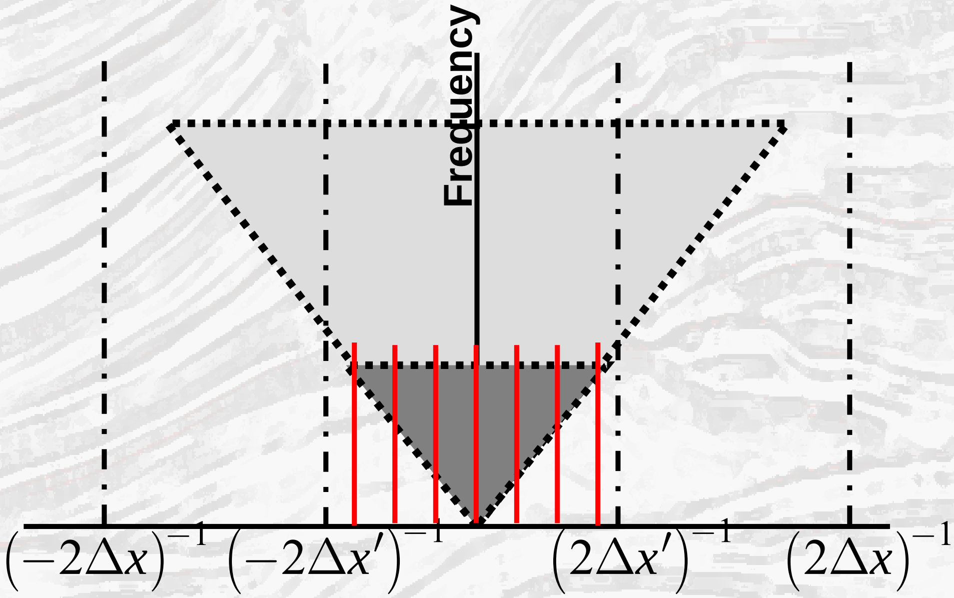


Spatial Resampling



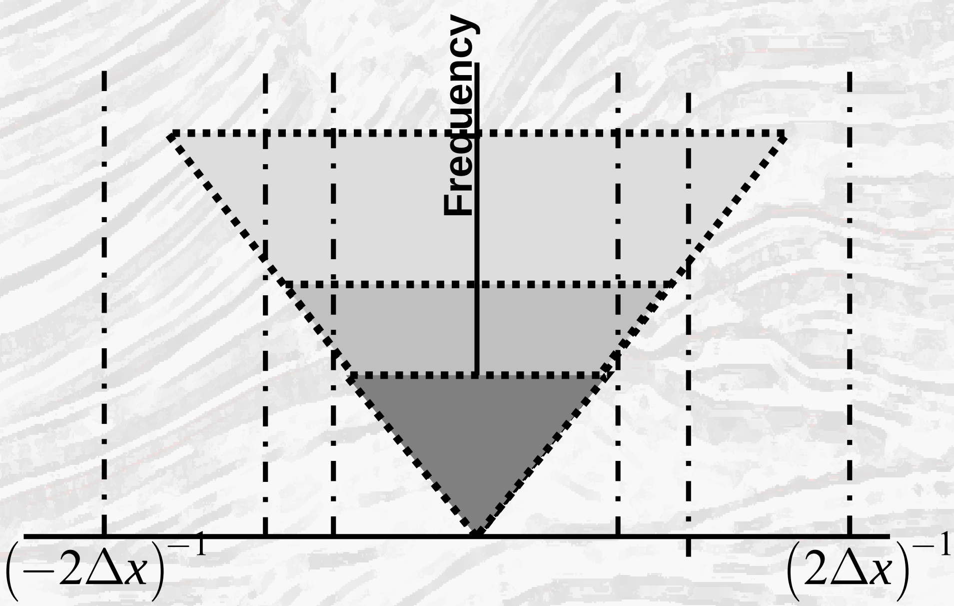
In red are the wavenumbers of a 7 point filter

Spatial Resampling



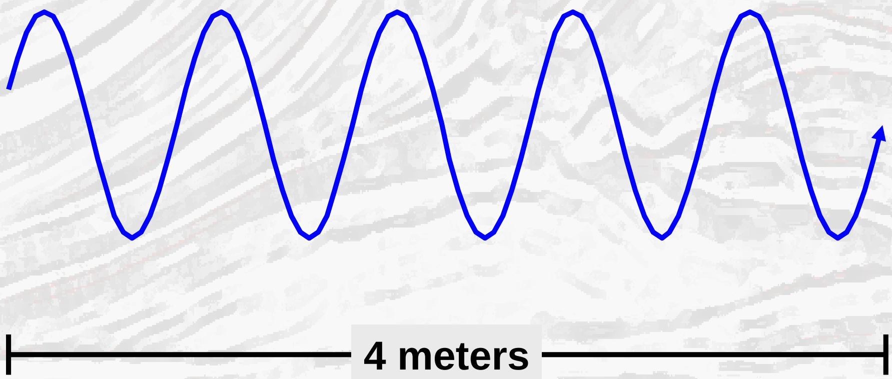
Downsampling for the lower frequencies uses the filter more effectively

Spatial Resampling

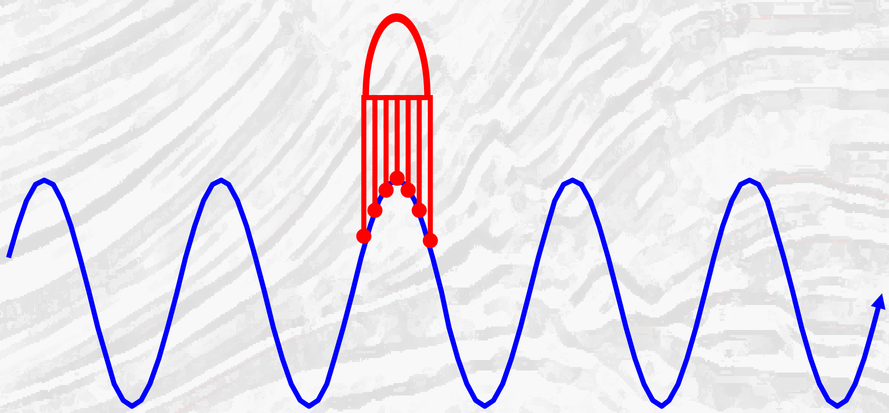


Spatial resampling is done in frequency "chunks".

Math Depot Analogy

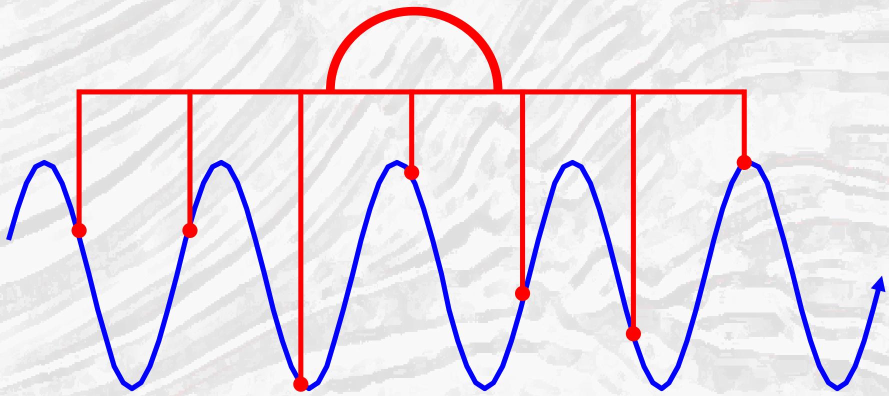


Sine Wave Carrier



This?

Sine Wave Carrier



Or This?

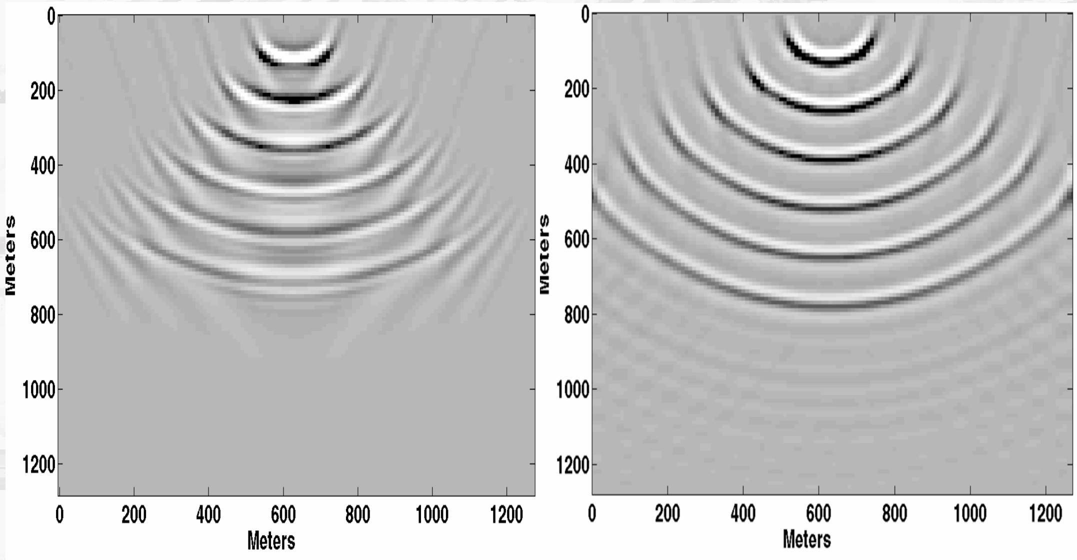
Spatial Resampling

Specific Results for Marmousi

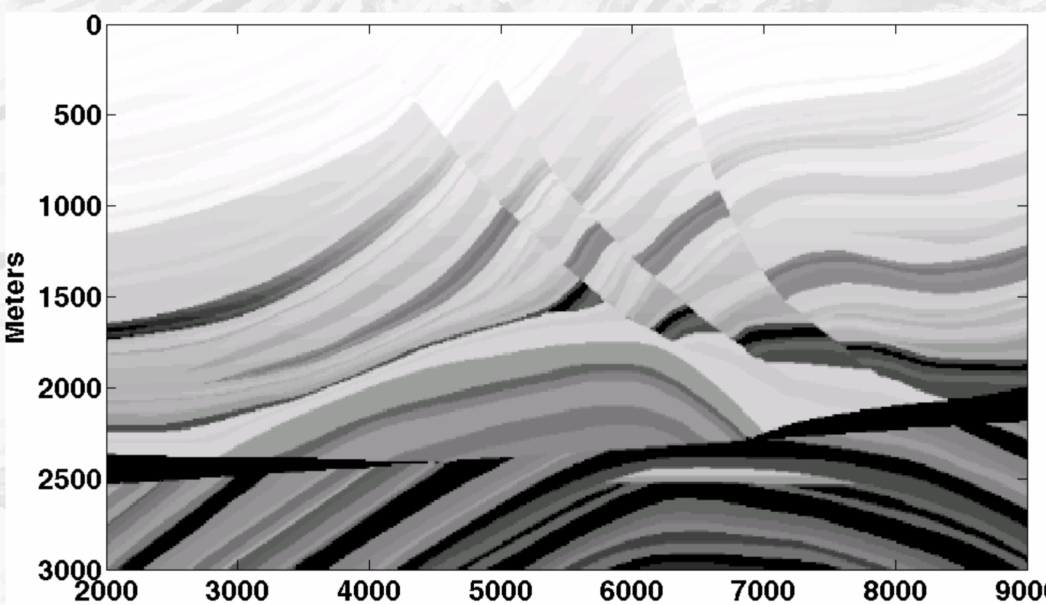
Partition frequencies (Hz)	Spatial sample size (m)	Number of traces
60→42	12.5	478
41.7→32.5	16	373
32.2→25.4	20.7	289
25.1→19.5	26.3	227
19.3→15.1	34.1	175
14.9→11.7	44.3	135
11.5→9.28	56.9	105
9.03→7.08	72	83
6.84→5.62	91.9	65
5.37→4.88		51

Spatial Resampling

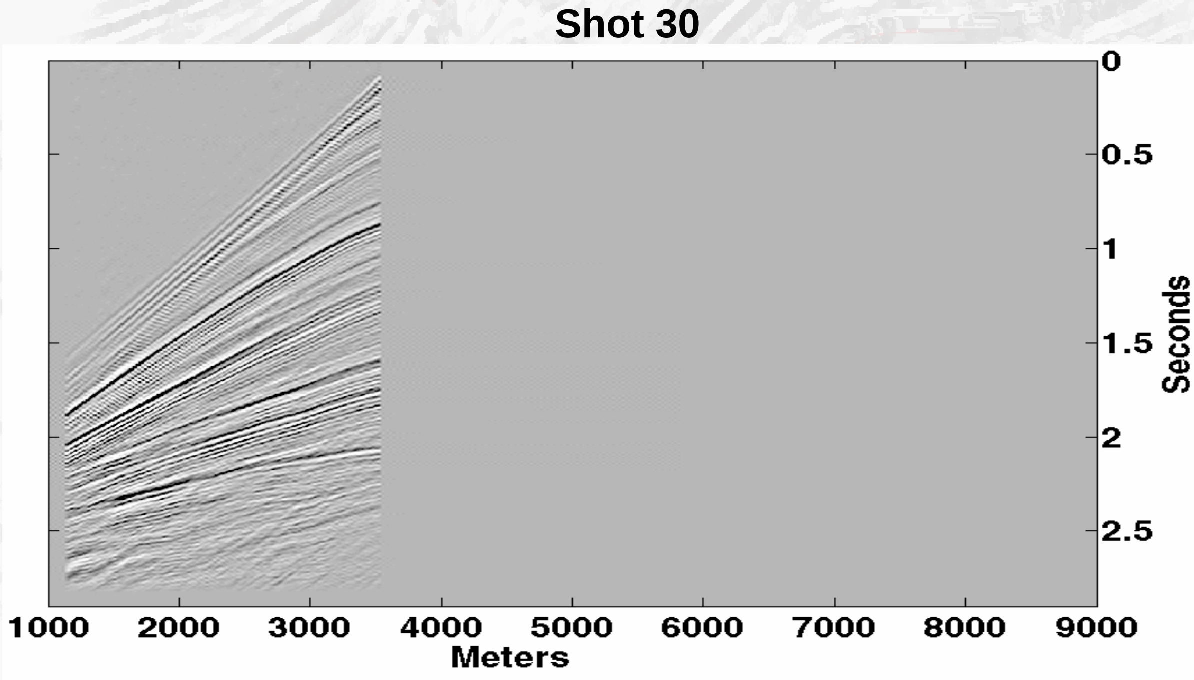
Without With



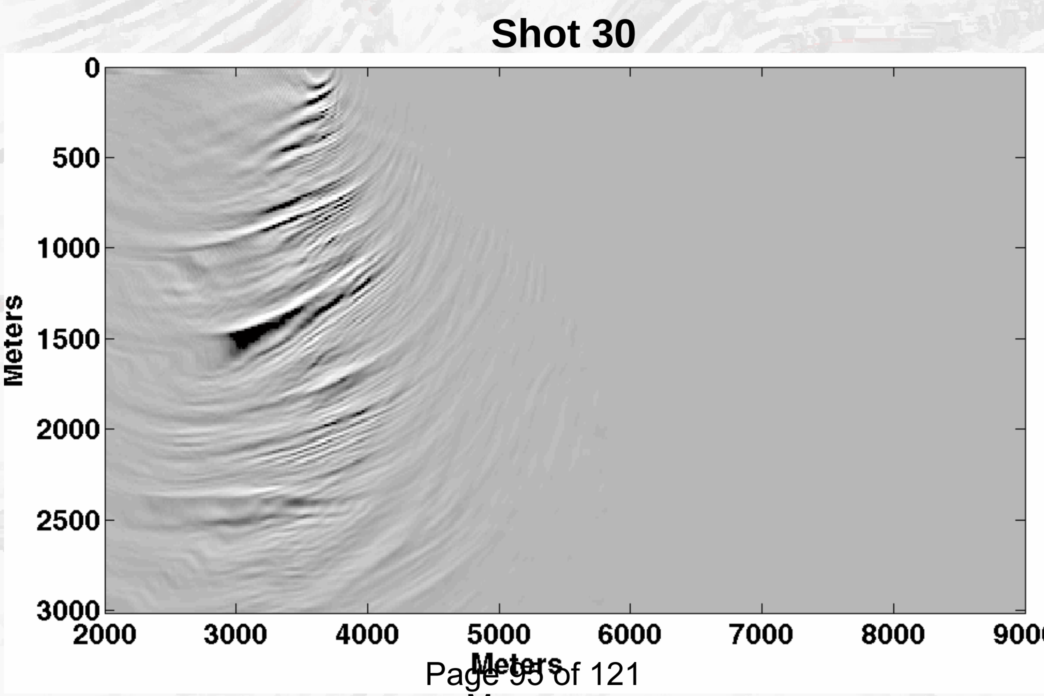
Marmousi Velocity Model



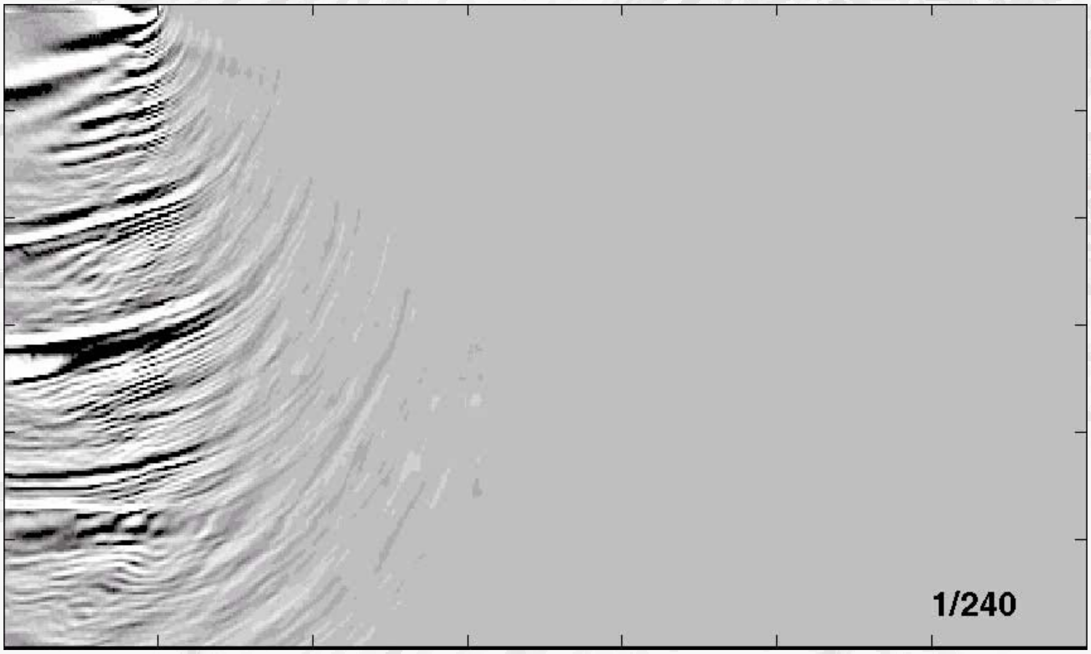
FOCI Pre-Stack Migration



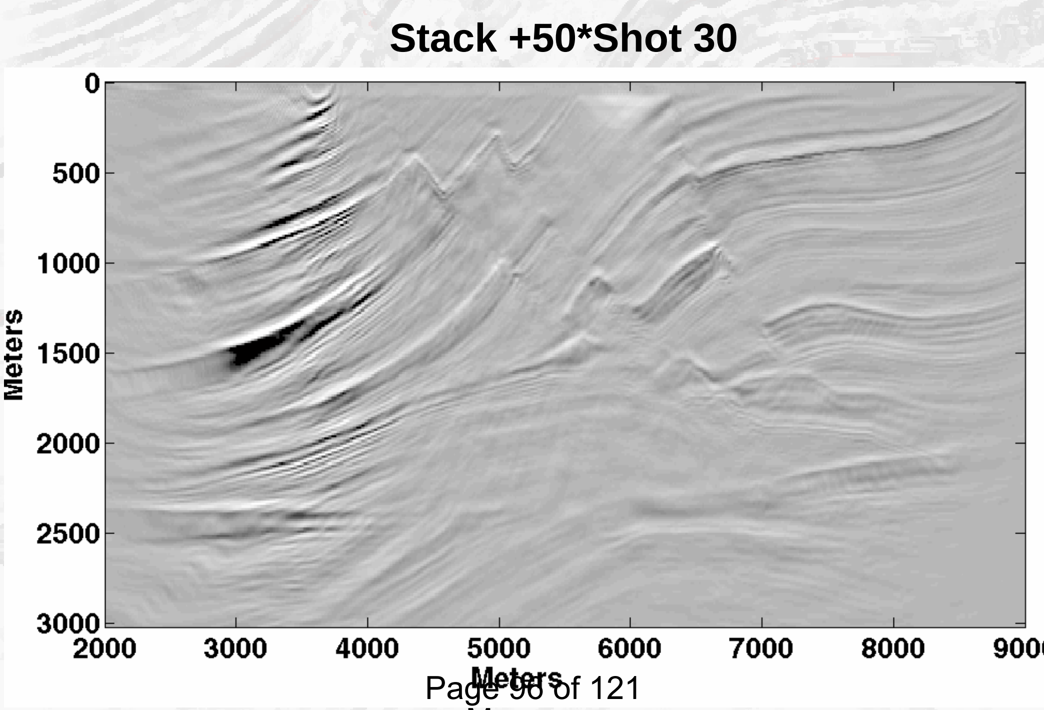
FOCI Pre-Stack Migration



Depth Migration Movie

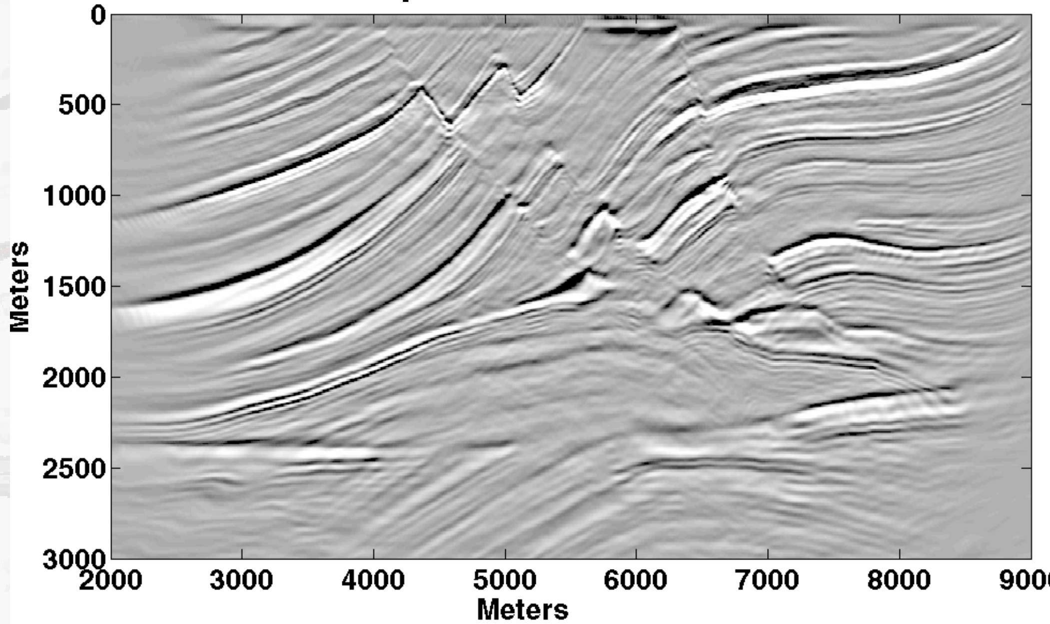


FOCI Pre-Stack Migration

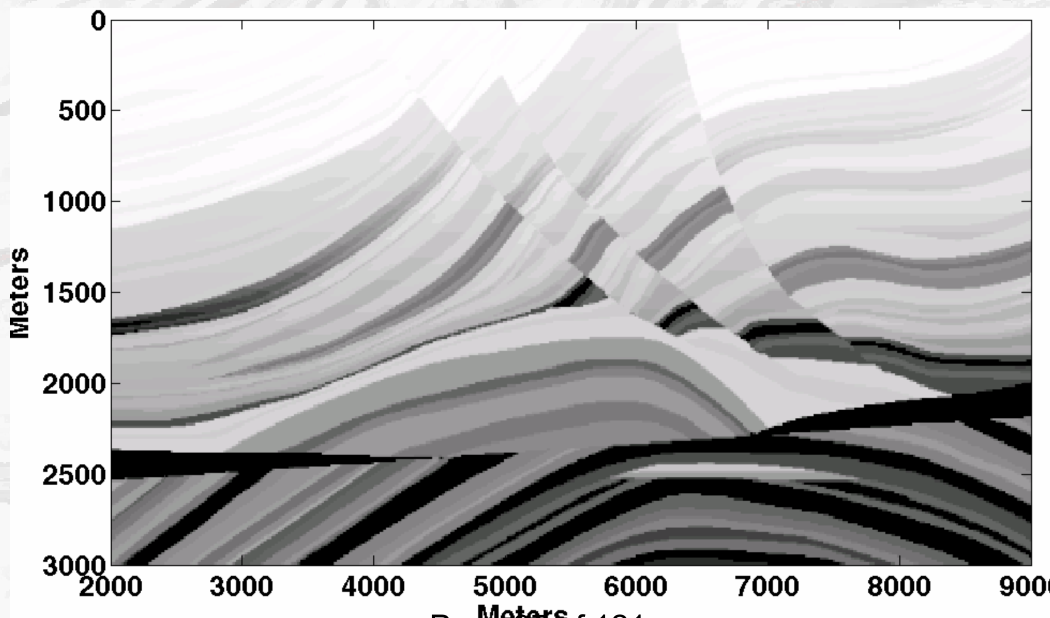


FOCI Pre-Stack Migration

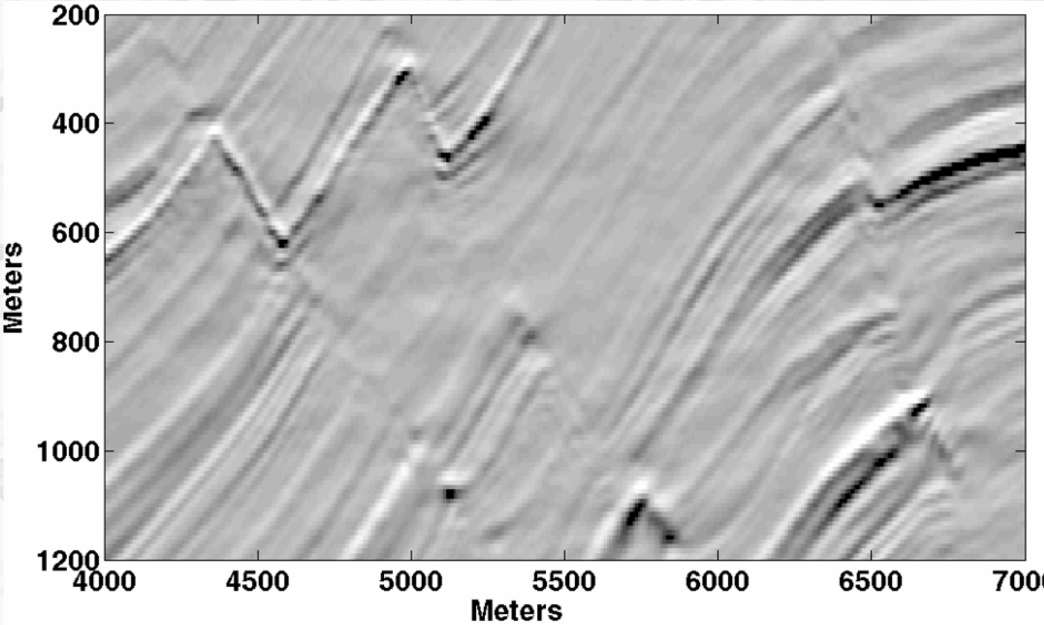
51 Point Operator, 15 Hours on 1 PC



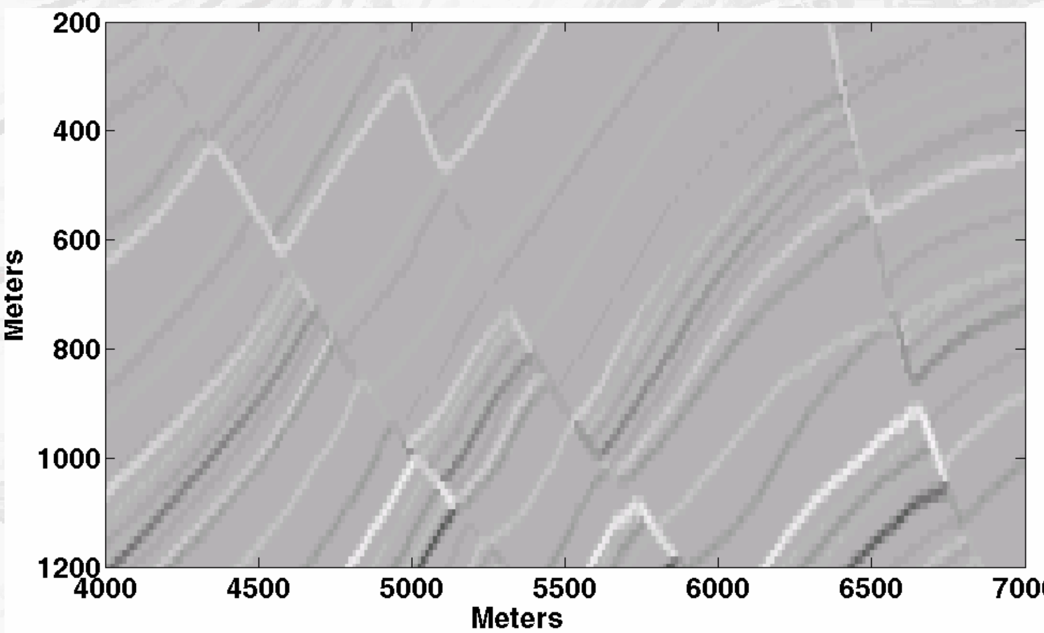
Marmousi Velocity Model



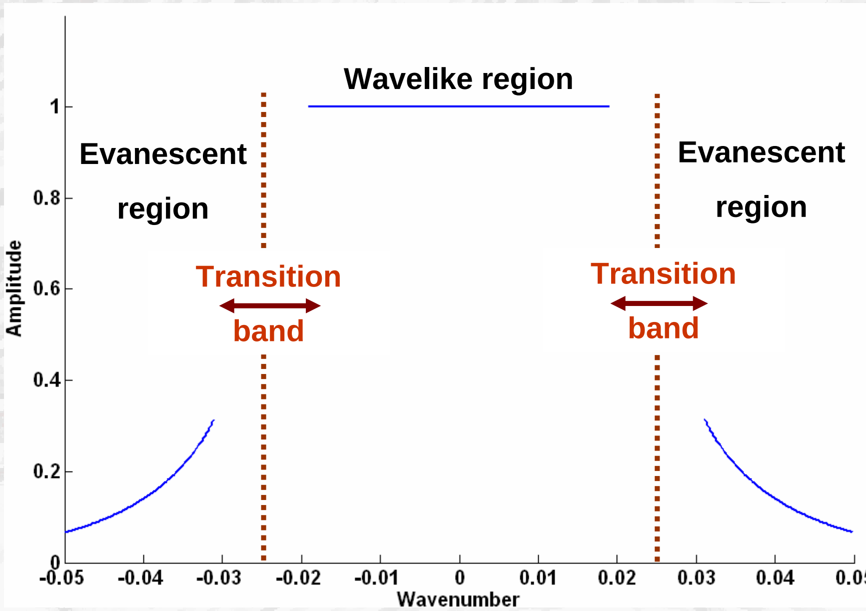
Detail of Pre-Stack Migration



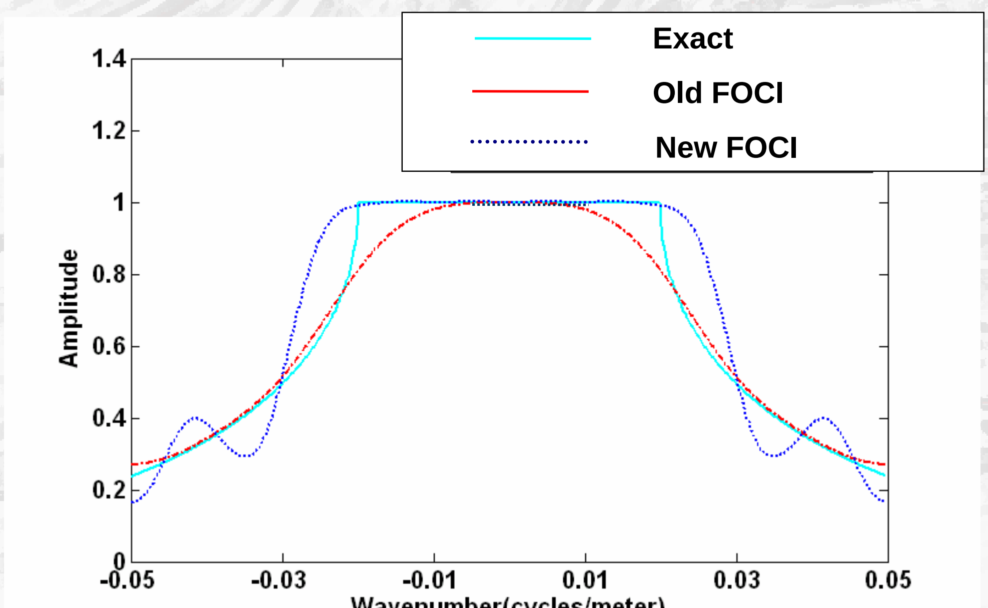
Marmousi Reflectivity Detail



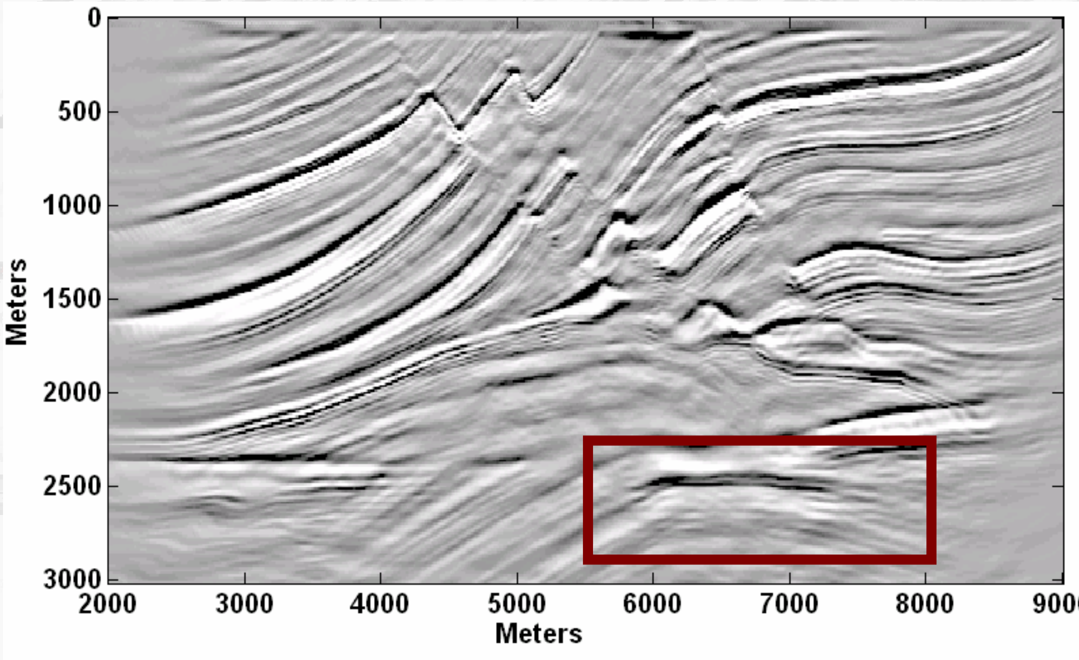
Improvements by Saleh Al-Saleh
Desired spectrum



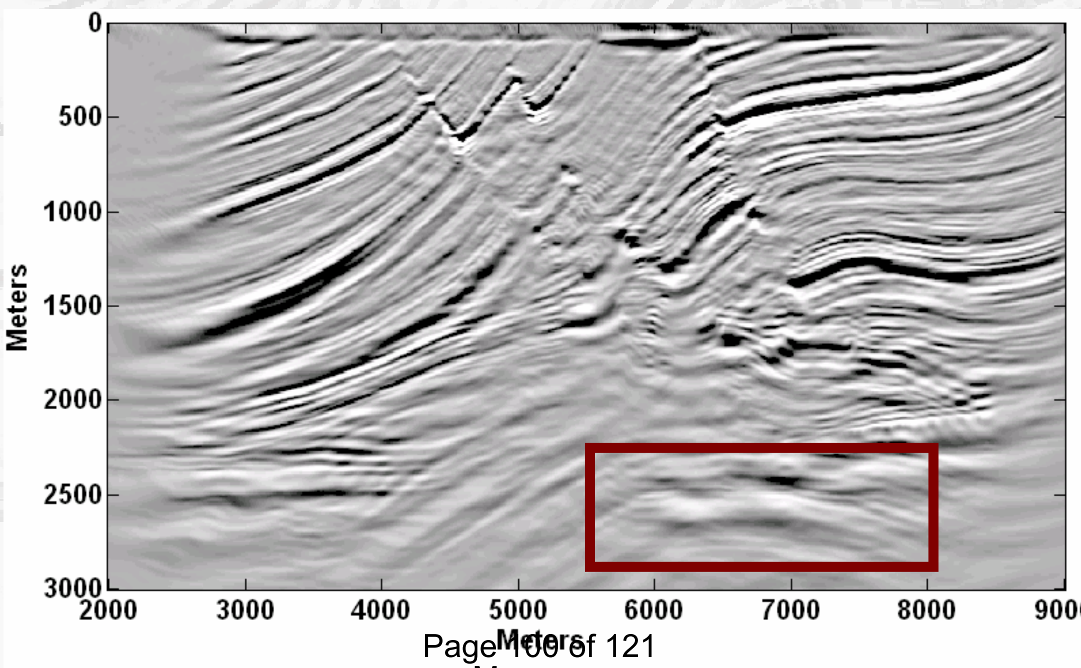
Improvements by Saleh Al-Saleh
Old/New



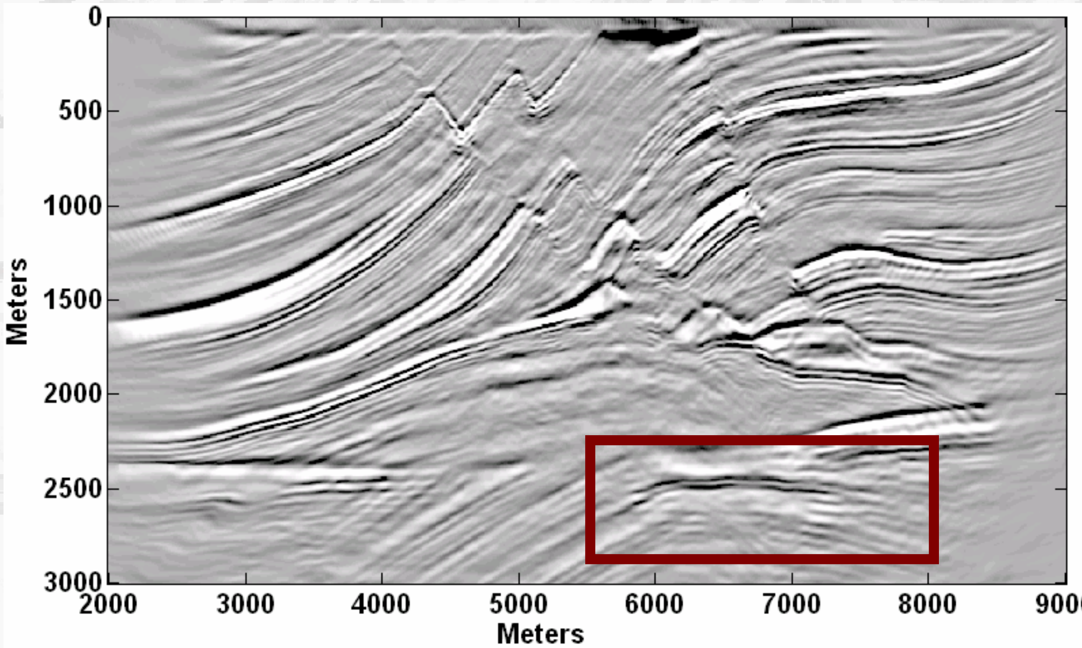
Old FOCI
operator = 51 points



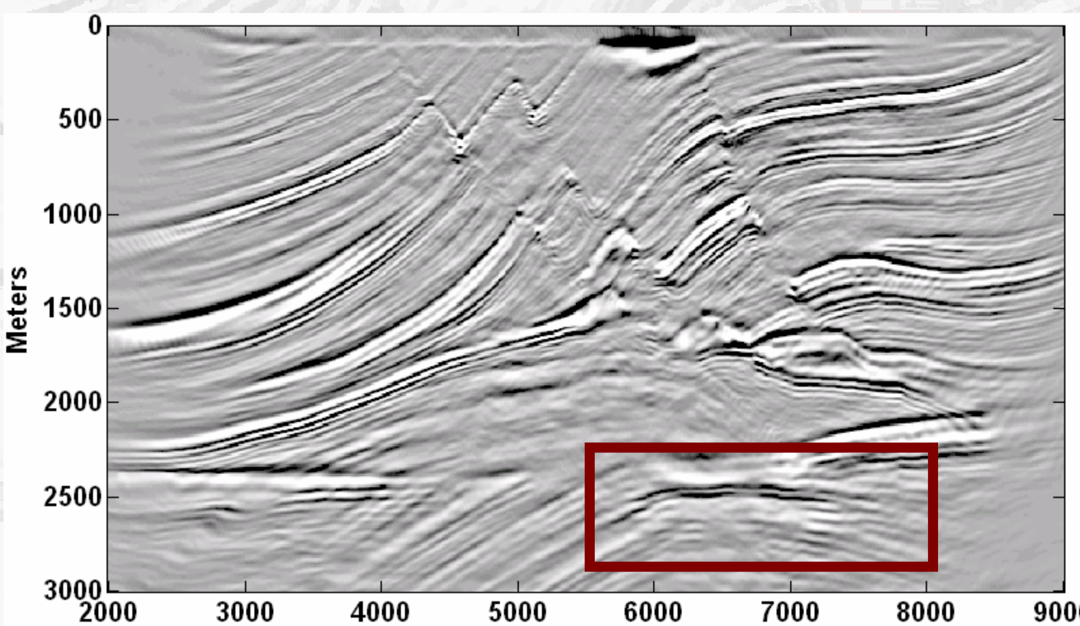
Old FOCI
operator = 15 points



New FOCI
operator = 15 points



New FOCI
operator = 9 points



Part 2

Separable Symbols
and
The Gabor Transform

Approximating PSDO's

Recall the standard form K-N Pseudodifferential operator (PSDO)

$$s(x) = (T_\alpha \psi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \alpha(x, k_x) \hat{\psi}(k_x) e^{-ixk_x} dk_x$$

If the symbol is taken to a stationary limit

$$\lim_{stat} \alpha(x, k_x) = \alpha_0(k_x)$$

Then the result is a simple Fourier multiplier

$$\lim_{stat} s = \underbrace{F^{-1} M_{\alpha_0} F}_{\text{a Fourier multiplier}} \psi$$

Piecewise Stationary Symbols

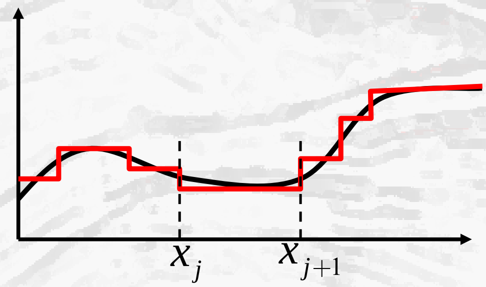
Consider an arbitrary symbol $\alpha(x,k_x)$

One can always find a partition of $\mathbb{R}, \{x_j\}, j \in \mathbb{Z}$

and corresponding functions $\{\alpha_j\}$ such that

$$\left\| \alpha(x,k_x) - \sum_{j\in\mathbb{Z}} \chi_j(x)\alpha_j(k_x) \right\|_{L2} < \varepsilon \quad \chi_j(x) = \begin{cases} 1, x \in [x_j, x_{j+1}) \\ 0, \text{otherwise} \end{cases}$$

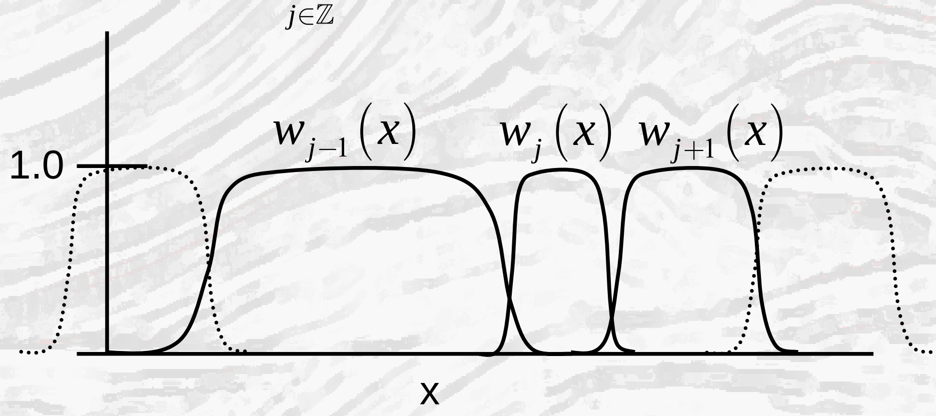
Piecewise constant approximation to a function



Piecewise Stationary Symbols

Suppose the symbol is separable such that

$$\alpha(x,k_x) = \sum_{j\in\mathbb{Z}} w_j(x)\alpha_j(k_x) \qquad w_j(x) \in C_0^\infty$$



Piecewise Stationary Symbols

Standard Calculus

A K-N standard operator is

$$(T_\alpha \psi)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \alpha(x,k_x)\hat{\psi}(k_x)e^{-ixk_x}dk_x$$

$$\text{Let} \quad \alpha(x,k_x) = \sum_{j\in\mathbb{Z}} w_j(x)\alpha_j(k_x)$$

$$(T_\alpha \psi)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\sum_{j\in\mathbb{Z}} w_j(x)\alpha_j(k_x)\right]\hat{\psi}(k_x)e^{-ixk_x}dk_x$$

$$(T_\alpha \psi)(x) = \sum_{j\in\mathbb{Z}} w_j(x)\underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} \alpha_j(k_x)\hat{\psi}(k_x)e^{-ixk_x}dk_x}_{\text{Ordinary Fourier Multipliers}}$$

Piecewise Stationary Symbols

Standard and Anti-Standard Calculus

So the operator reduces to a windowed superposition of Fourier multipliers

$$T_\alpha \psi = \sum_{j\in\mathbb{Z}} w_j F^{-1} M_{\alpha_j} F\psi$$

It is left as an exercise to show that the anti-standard operator reduces to

$$T_\alpha^a \psi = \sum_{j\in\mathbb{Z}} F^{-1} M_{\alpha_j} F w_j \psi$$

The only difference is the position of the window function! Both formulae are special cases of the application of a Gabor multiplier with a Gabor transform.

Gabor Transform

Begin with a partition of unity (POU)

$$\sum_{j\in\mathbb{Z}} \Omega_j(x) = 1, \quad \Omega_j(x) \text{ are suitable bump functions}$$

Let $\underbrace{g_j(x) = \Omega_j^{\,p}(x)}_{\text{analysis window}}$ and $\underbrace{\gamma_j(x) = \Omega_j^{\,1-p}(x)}_{\text{synthesis window}}$, $p \in [0,1]$

Then, the Gabor transform is defined by

$$V_g\psi(j,k_x) = \underbrace{F(g_j\psi)(k_x)}_{\substack{\text{Forward Fourier for} \\ \text{a suite of windows}}} : L^2(\mathbb{R}) \to L^2(\mathbb{Z}\times\mathbb{R})$$

This particular Gabor transform is partially discrete by design. Fully discrete and fully analytic algorithms are easily derived.

Inverse Gabor Transform

Given $V_g\psi(j,k_x) = F(g_j\psi)(k_x) \in \mathbb{Z}\times\mathbb{R}$

The signal is recovered with a windowed inverse Fourier transform and a summation over windows.

$$V_\gamma^{-1}(V_g\psi) = \sum_{j\in\mathbb{Z}} \gamma_j F^{-1} F g_j \psi = \psi \sum_{j\in\mathbb{Z}} \gamma_j g_j = \psi$$

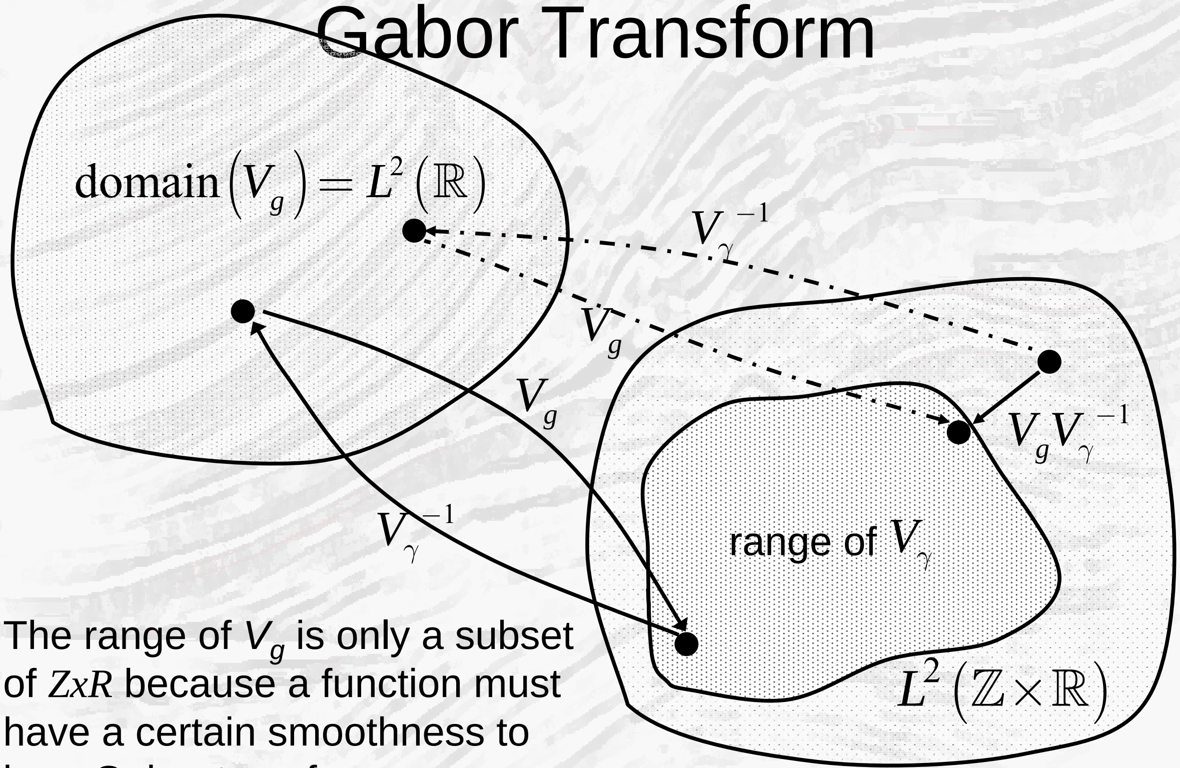
Note that:

$$V_\gamma^{-1} V_g = 1 \in L^2(\mathbb{R})$$

$$V_g V_\gamma^{-1} = P \neq 1 \in L^2(\mathbb{Z}\times\mathbb{R})$$

where P is a projection operator onto the range of the forward Gabor transform.

Gabor Transform



The range of V_g is only a subset of *ZxR* because a function must have a certain smoothness to be a Gabor transform.

Gabor Multipliers

Given $V_g\psi(j,k_x) = F(g_j\psi)(k_x) \in \mathbb{Z}\times\mathbb{R}$

$$\alpha(j,k_x) \in \mathbb{Z}\times\mathbb{R}$$

We define a Gabor multiplier through the operation

$$G_{\gamma g\alpha}\psi = \underbrace{V_\gamma^{-1}}_{\substack{\text{Inverse}\\ \text{Gabor}}} \; \underbrace{M_\alpha}_{\text{Multiplication}} \; \underbrace{V_g\psi}_{\substack{\text{Forward}\\ \text{Gabor}}}$$

Gabor Multipliers and K-N PSDO's

For a piecewise stationary symbol, we had for the standard KN operator

$$T_\alpha \psi = \sum_{j\in\mathbb{Z}} w_j F^{-1} M_{\alpha_j} F\psi$$

This can be written as a Gabor multiplier as

$$T_\alpha \psi = G_{\gamma g \alpha}\psi = V_\gamma^{-1} M_\alpha V_g \psi, \quad \gamma_k = w_k \text{ and } g_k = 1$$

Similarly, for the anti-standard operator

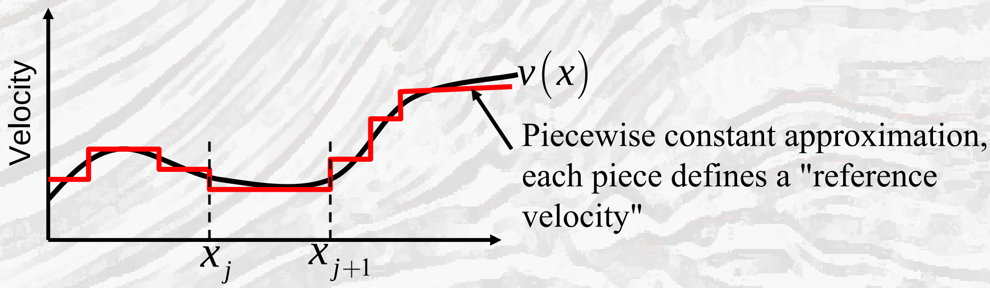
$$T_\alpha^a \psi = G_{\gamma g \alpha}\psi = V_\gamma^{-1} M_\alpha V_g \psi, \quad \gamma_k = 1 \text{ and } g_k = w_k$$

Part 3

A Gabor Imaging Method

Gabor Wavefield Extrapolation

Approximate the variable velocity GPSPI extrapolator as a windowed sum of constant-velocity operators



$$\sum_j \Omega_j(x) = 1$$

A Partition of Unity (POU) with each window localized for a "reference" velocity.

How to choose the optimal set of reference velocities and the corresponding windows?

Gabor Wavefield Extrapolation

Approximate the variable velocity GPSPI extrapolator as a windowed sum of constant-velocity operators

$$\hat{W}(k(x), k_x, \Delta z) \approx \sum_j \underbrace{\Omega_j(x)}_{\text{windows}} \underbrace{\hat{W}(k_j, k_x, \Delta z)}_{\text{constant velocity extrapolators}}$$

A usually better approximation is

$$\hat{W}(k(x), k_x, \Delta z) \approx \sum_j \underbrace{S_j(x)}_{\text{Split-step Fourier correction}} \Omega_j(x) \hat{W}(k_j, k_x, \Delta z)$$

where $S_j(x) = e^{ik(x)\Delta z}$ accounts for "residual" time shifts.

Gabor Wavefield Extrapolation

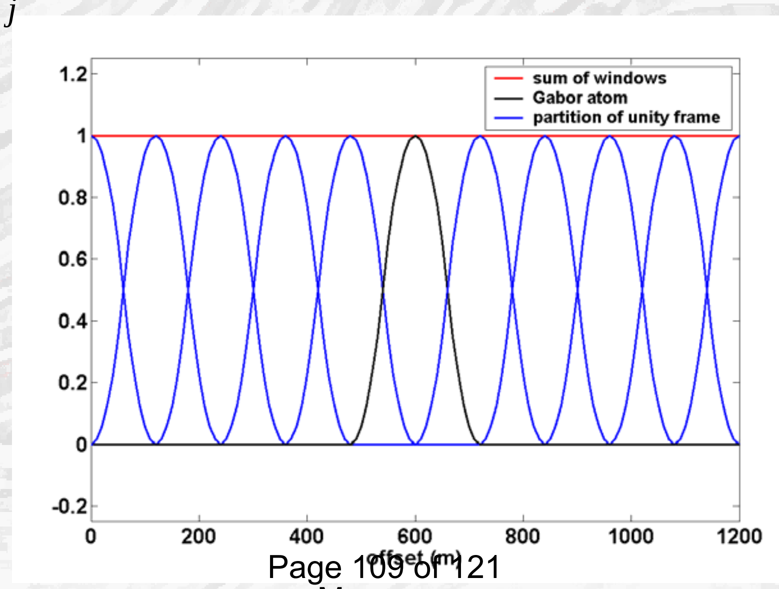
The Gabor approximation to GPSPI then becomes

$$\psi_P(x, z+\Delta z, \omega) = \frac{1}{2\pi}\sum_j \Omega_j(x) S_j(x) \ldots$$
$$\int_{\mathbb{R}} \hat{\psi}(k_x, z, \omega)\hat{W}\left(k_j, k_x, \Delta z\right) e^{-ik_x x} dk_x$$

Now, lets look at how to choose the POU.

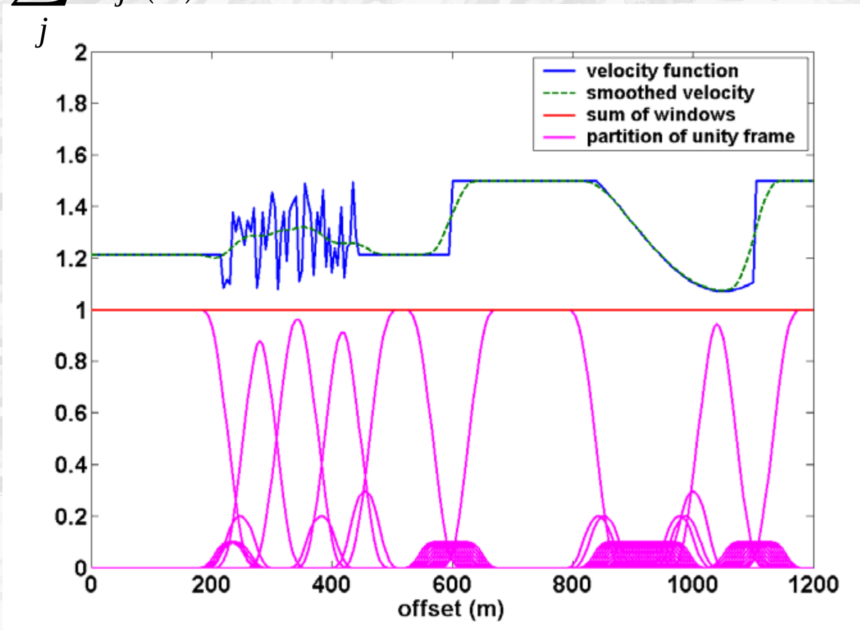
A uniform POU Gabor frame

$$\sum_j \Omega\left(x - x_j\right) = 1 \Rightarrow$$ All windows are translates of a mother window.



An adaptive POU frame

$$\sum_j \Omega_j(x) = 1 \Rightarrow \text{ Each window can be unique.}$$

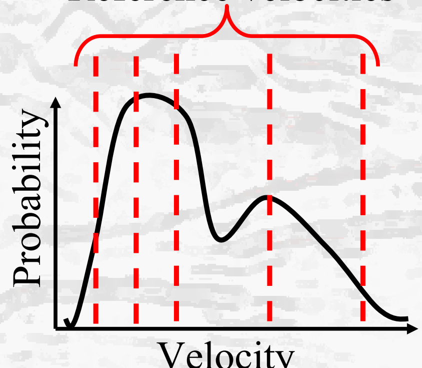


New Method

- Number of reference velocities chosen to give a defined maximum position error (relative to GPSPI).
- For each reference velocity define an indicator function:

$$I_j(x) = \begin{cases} 1, & |v(x) - v_j| = \min \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_j I_j(x) = 1$$



New Method

- Define a smallest "atomic window"
- Build the POU by a normalized convolution:

$$\Omega_j(x) = (I_j \bullet \Theta)(x)$$

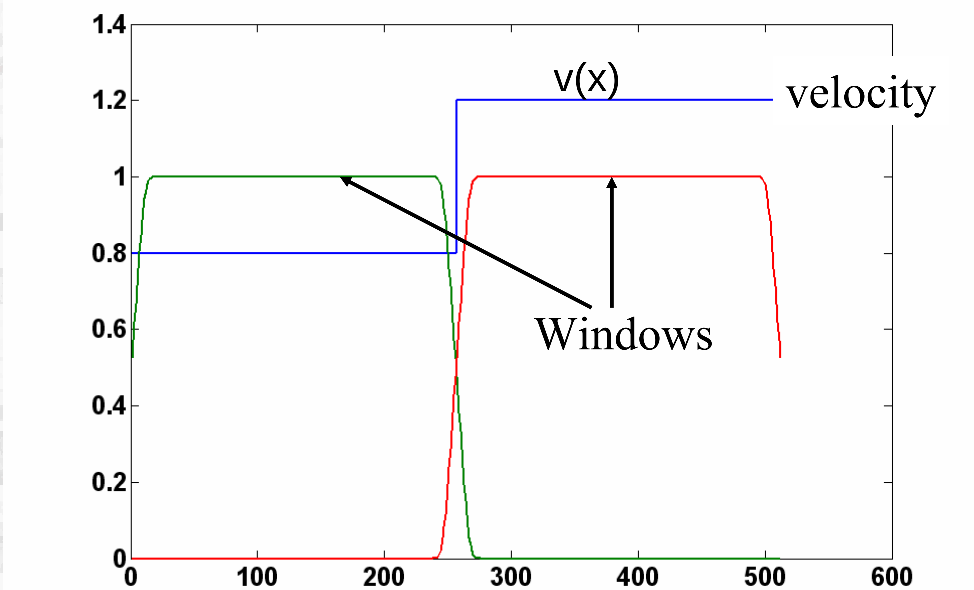
$$\Theta = \text{atomic window}$$

The POU is satisfied automatically

Works in any number of dimensions

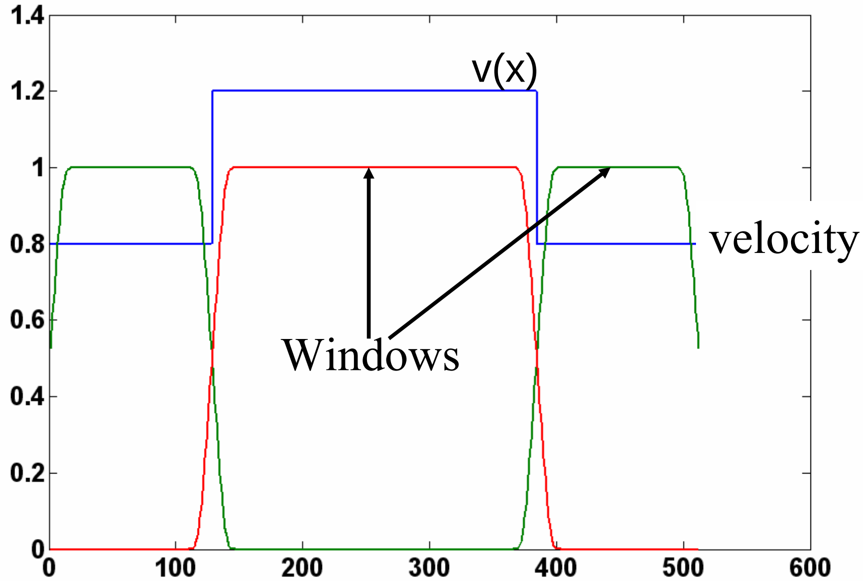
New Method

Example: v(x) is a step function and two reference velocities are chosen.



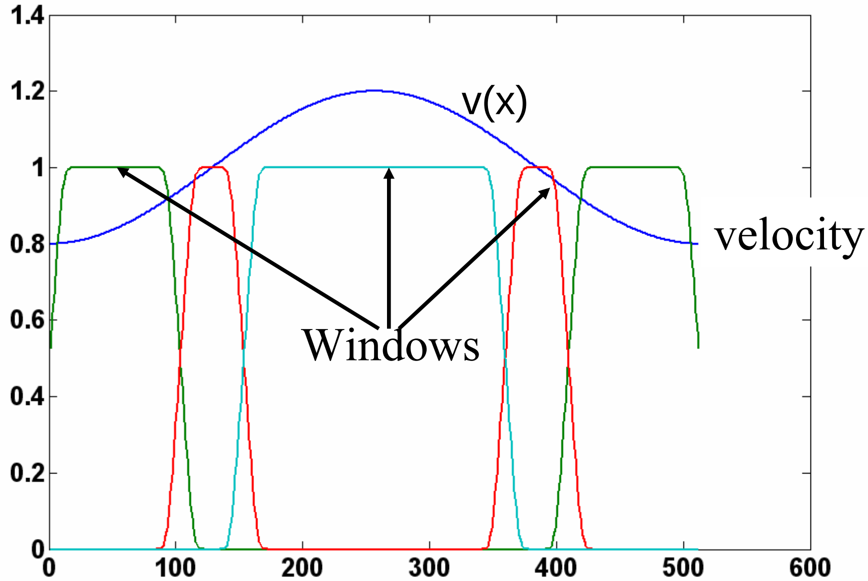
New Method

Example: v(x) is a step bump function
and two reference velocities are chosen.



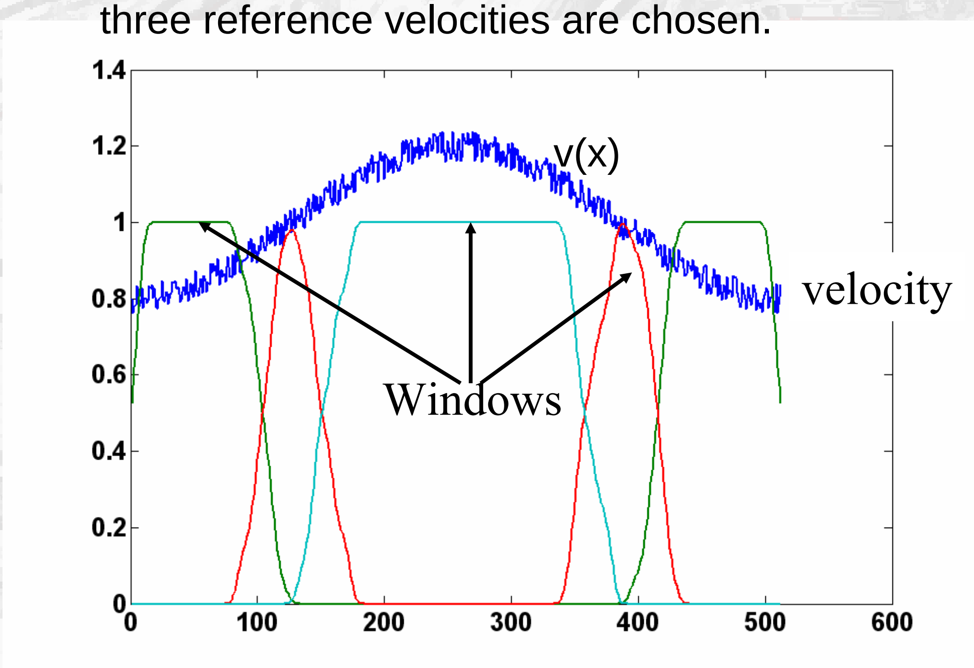
New Method

Example: v(x) is a smooth bump function
and three reference velocities are chosen.



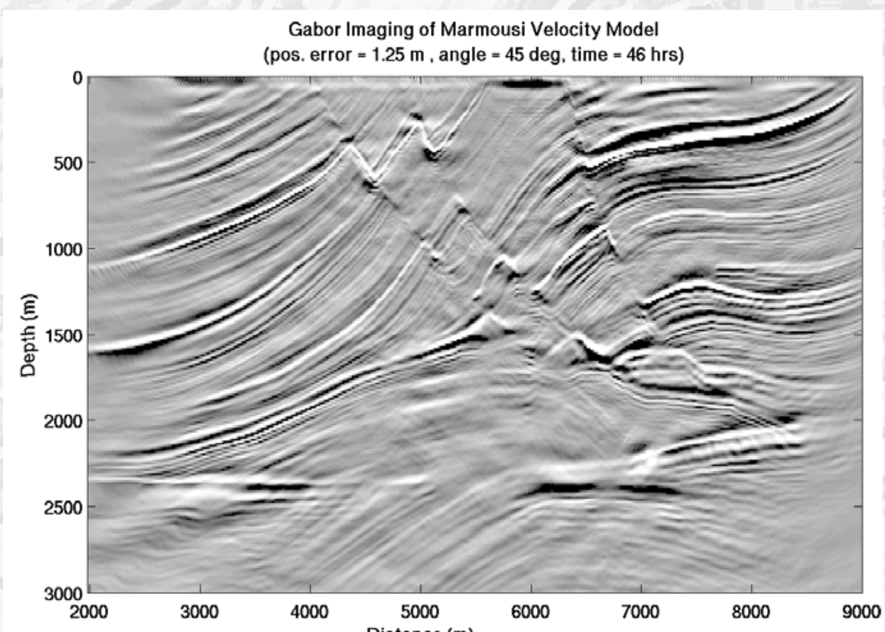
New Method

Example: v(x) is a ragged bump function and three reference velocities are chosen.

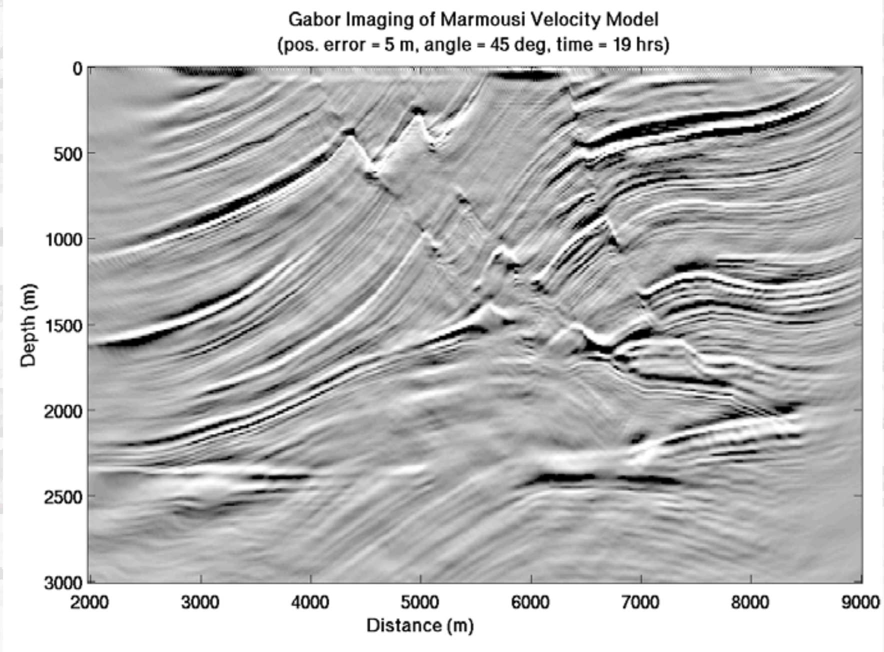


Gabor Test

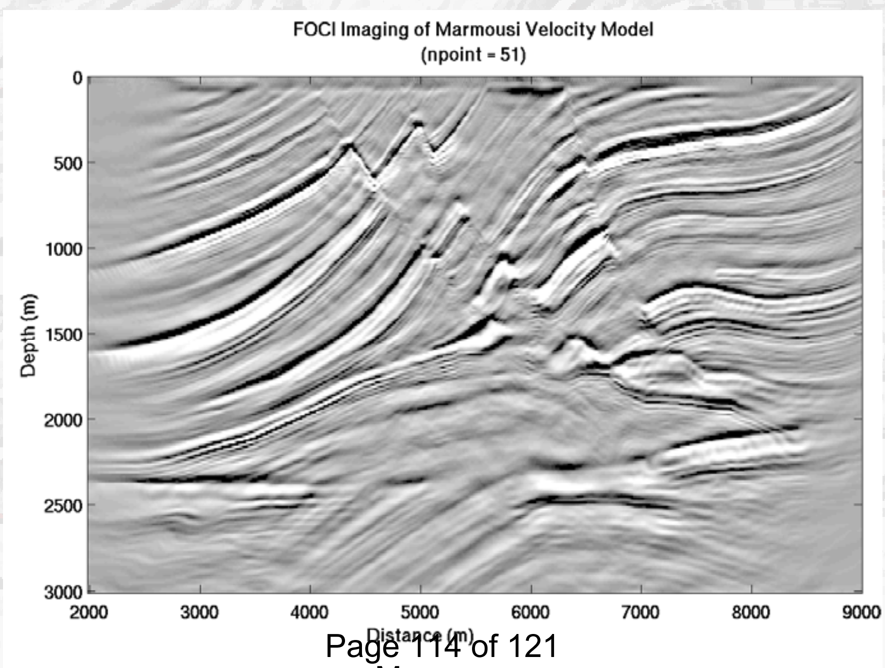
position error ~ 1.25m



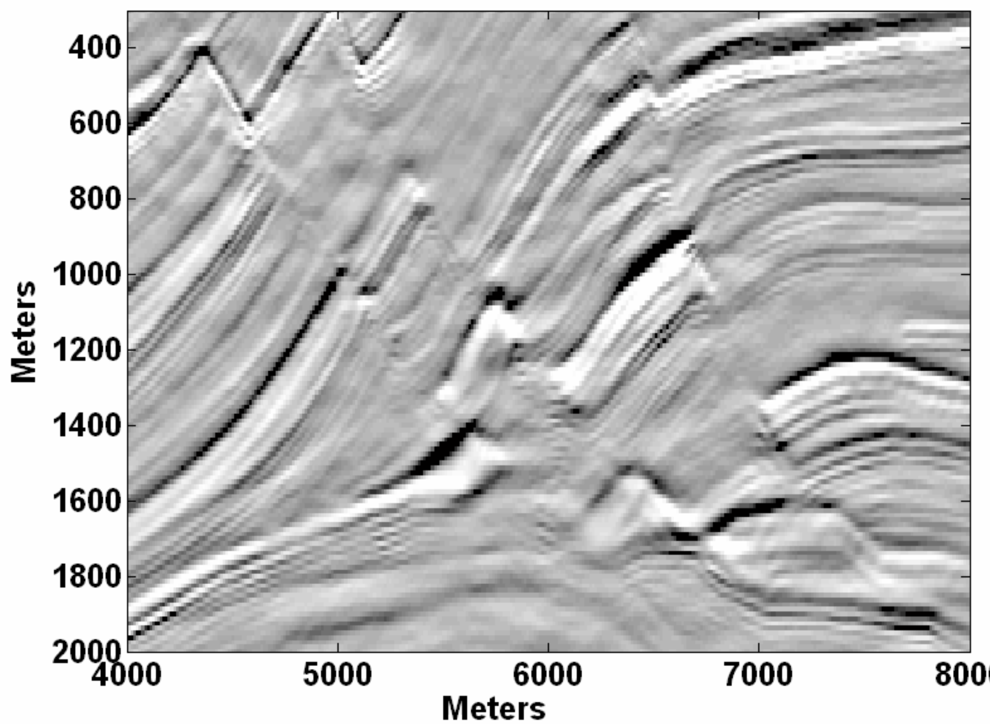
Gabor Test
position error ~ 5m



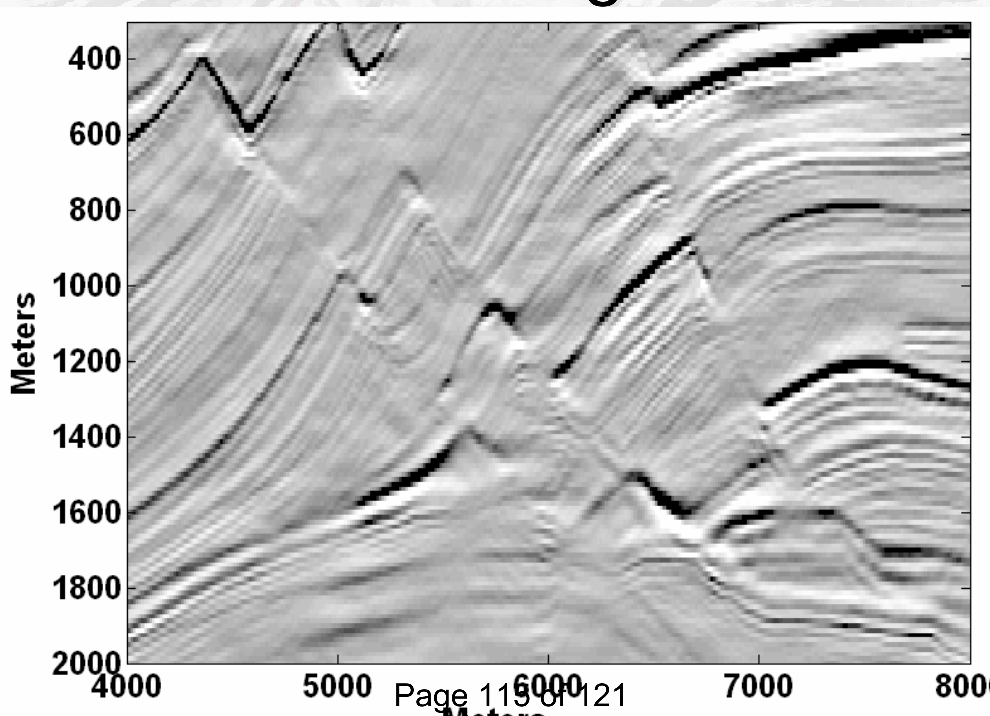
FOCI Result



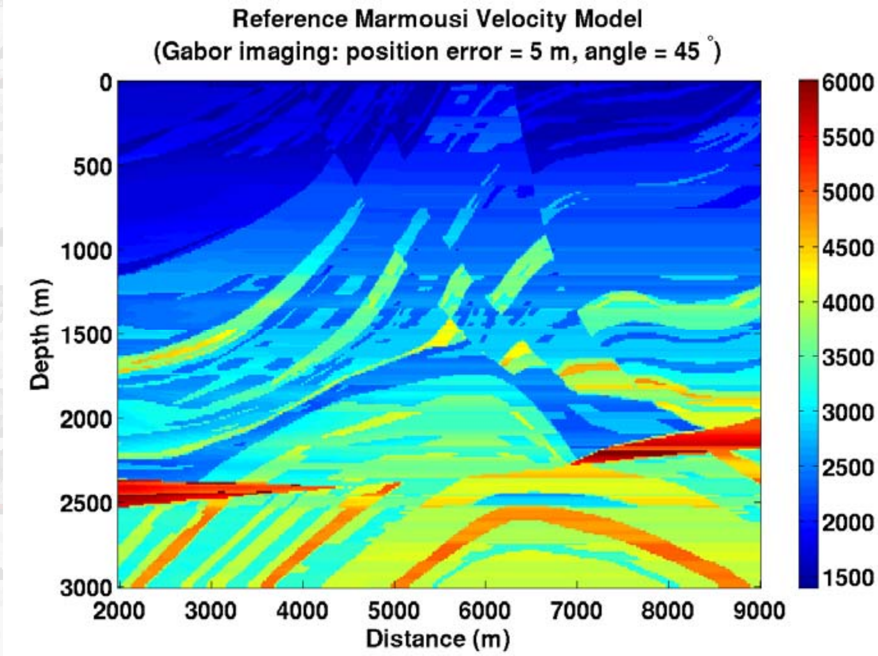
FOCI enlargement



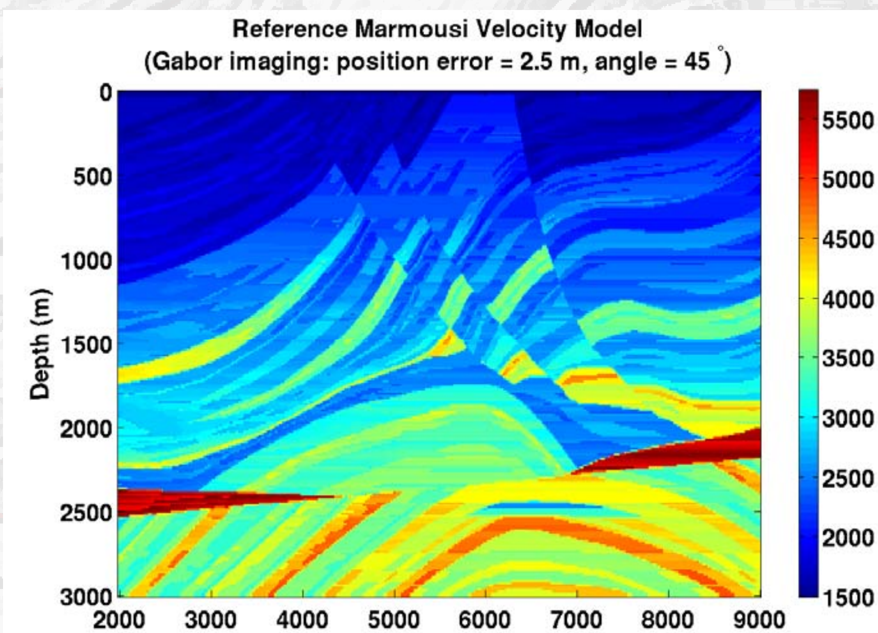
Gabor enlargement



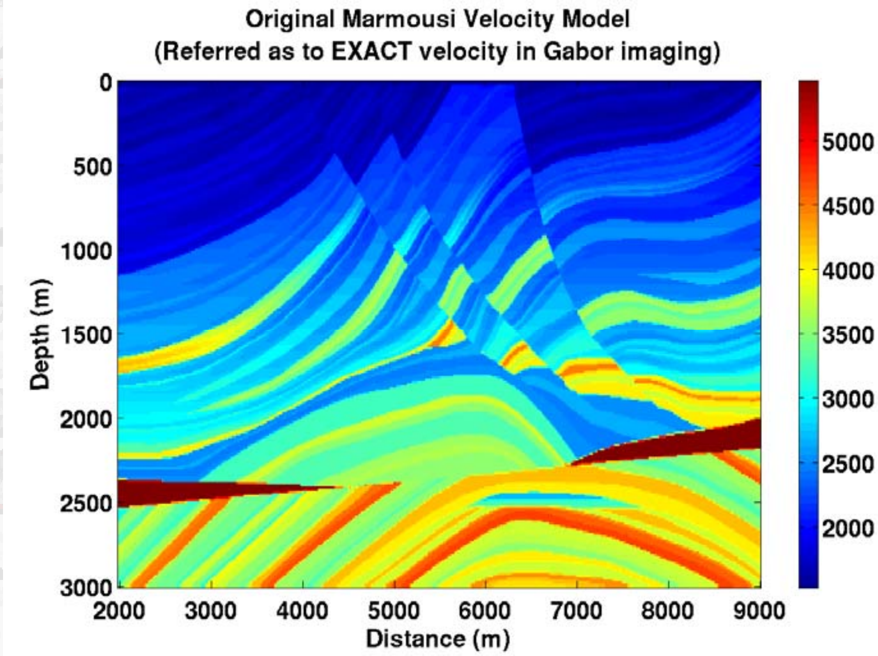
Marmousi Reference Velocities
position error ~ 5m



Marmousi Reference Velocities
position error ~ 2.5m



Marmousi Velocity Model



Conclusions

A fast, explicit wavefield extrapolator based on the GPSPI formula was presented.

The central problem of extrapolator stability was presented and addressed by designing two half-step operators with opposing instability.

Spatial resampling was described as a very useful imaging tool.

Gabor methods can be used to approximate pseudodifferential operators.

Gabor wavefield extrapolators, based on an adaptive POU, give promising wavefield extrapolation results.

Acknowledgements

We thank:

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MITACS

PIMS

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