The mathematics of PDEs and the wave equation

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Abstract

Abstract: We look at the mathematical theory of partial differential equations as applied to the wave equation. In particular, we examine questions about existence and uniqueness of solutions, and various solution techniques.

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1 Lecture One: Introduction to PDEs

A partial differential equation is simply an equation that involves both a function and its partial derivatives. In these lectures, we are mainly concerned with techniques to find a solution to a given partial differential equation, and to ensure good properties to that solution. That is, we are interested in the mathematical theory of the existence, uniqueness, and stability of solutions to certain PDEs, in particular the wave equation in its various guises.

Most of the equations of interest arise from physics, and we will use x, y, z as the usual spatial variables, and t for the the time variable. Various physical quantities will be measured by some function $u = u(x, y, z, t)$ which could depend on all three spatial variable and time, or some subset. The partial derivatives of u will be denoted with the following condensed notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$$

and so on.[1] The Laplace operator is the most physically important differential operator, which is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

1.1 Equations from physics

Some typical partial differential equations that arise in physics are as follows. **Laplace's equation**

$$\nabla^2 u = 0$$

which is satisfied by the temperature $u = u(x, y, z)$ in a solid body that is in thermal equilibrium, or by the electrostatic potential $u = u(x, y, z)$ in a region without electric charges. **The heat equation**

$$u_t = k\nabla^2 u$$

which is satisfied by the temperature $u = u(x, y, z, t)$ of a physical object which conducts heat, where k is a parameter depending on the conductivity of the object. **The wave equation**

$$u_{tt} = c^2 \nabla^2 u$$

which models the vibrations of a string in one dimension $u = u(x, t)$, the vibrations of a thin membrane in two dimensions $u = u(x, y, t)$ or the pressure vibrations of an acoustic wave in air $u = u(x, y, z, t)$. The constant c gives the speed of propagation for the vibrations. Closely related to the 1D wave equation is the fourth order[2] PDE for a vibrating beam,

$$u_{tt} = -c^2 u_{xxxx}$$

[1] We assume enough continuity that the order of differentiation is unimportant. This is true anyway in a distributional sense, but that is more detail than we need to consider.

[2] The order of a PDE is just the highest order of derivative that appears in the equation.

where here the constant c^2 is the ratio of the rigidity to density of the beam. An interesting nonlinear[3] version of the wave equation is the Korteweg-de Vries equation

$$u_t + cuu_x + u_{xxx} = 0$$

which is a third order equation, and represents the motion of waves in shallow water, as well as solitons in fibre optic cables.

There are many more examples. It is worthwhile pointing out that while these equations can be derived from a careful understanding of the physics of each problem, some intuitive ideas can help guide us. For instance, the Laplacian

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

can be understood as a measure of how much a function $u = u(x, y, z)$ differs at one point (x, y, z) from its neighbouring points. So, if $\nabla^2 u$ is zero at some point (x, y, z), then $u(x, y, z)$ is equal to the average value of u at the neighbouring points, say in a small disk around (x, y, z). If $\nabla^2 u$ is positive at that point (x, y, z), then $u(x, y, z)$ is smaller than the average value of u at the neighbouring points. And if $\nabla^2 u(x, y, z)$ is negative, then $u(x, y, z)$ is larger that the average value of u at the neighbouring points.

Thus, Laplace's equation

$$\nabla^2 u = 0$$

represents temperature equilibrium, because if the temperature $u = u(x, y, z)$ at a particular point (x, y, z) is equal to the average temperature of the neighbouring points, no heat will flow. The heat equation

$$u_t = k\nabla^2 u$$

is simply a statement of Newton's law of cooling, that the rate of change of temperature is proportional to the temperature difference (in this case, the difference between temperature at point (x, y, z) and the average of its neighbours). The wave equation

$$u_{tt} = c^2\nabla^2$$

is simply Newton's second law ($F = ma$) and Hooke's law ($F = k\Delta x$) combined, so that acceleration u_{tt} is proportional to the relative displacement of $u(x, y, z)$ compared to its neighbours. The constant c^2 comes from mass density and elasticity, as expected in Newton's and Hooke's laws.

1.2 Deriving the 1D wave equation

Most of you have seen the derivation of the 1D wave equation from Newton's and Hooke's law. The key notion is that the restoring force due to tension on the string will be proportional

[3]Nonlinear because we see u multiplied by u_x in the equation.

to the curvature at the point, as indicated in the figure. Then mass times acceleration ρu_{tt} should equal that force, ku_{xx}. Thus

$$u_{tt} = c^2 u_{xx}$$

where $c = \sqrt{k/\rho}$ turns out to be the velocity of propagation.

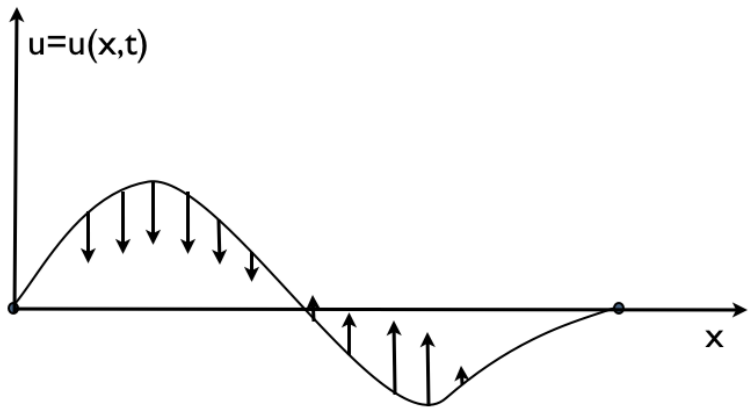

Figure 1: The restoring forces on a vibrating string, proportional to curvature.

Let's do it again, from an action integral.

Let $u = u(x,t)$ denote the deplacement of a string from the neutral position $u \equiv 0$. The mass density of the string is given by $\rho = \rho(x)$ and the elasticity given by $k = k(x)$. In paricular, in this derivation we do not assuming the the string is uniform. Consider a short piece of string, in the interval $[x, x + \Delta x]$. Its mass with be $\rho(x)\Delta x$, its velocity $u_t(x,t)$, and thus its kinetic energy, one half mass times velocity squared, is

$$\Delta K = \frac{1}{2}\rho \cdot (u_t)^2 \Delta x.$$

The total kinetic energy for the string is given by an integral,

$$K = \frac{1}{2}\int_0^L \rho \cdot (u_t)^2 \, dx.$$

From Hooke's law, the potential energy for a string is $(k/2)y^2$, where y is the length of the spring. For the stretched string, the length of the string is given by arclength $ds = \sqrt{1 + u_x^2}dx$ and so we expect a potential energy of the form

$$P = \int_0^L \frac{k}{2}(1 + u_x^2) \, dx.$$

[4] The action for a given function u is defined as the integral over time of the difference of these two energies, so

$$L(u) = \frac{1}{2}\int_0^T \int_0^L \rho \cdot (u_t)^2 - k \cdot [1 + (u_x)^2] \, dx \, dt.$$

[4]The one doesn't really need to be in there, but it doesn't matter for a potential energy.

Adding δ times a perturbation $h = h(x,t)$ to the function u gives a new action

$$L(u+\delta h) = L(u) + \delta \int_0^T \int_0^L \rho \cdot u_t \cdot h_t - k \cdot u_x h_x \, dx \, dt + \text{ higher order in } \delta.$$

The principle of least action says that in order for u to be a physical solution, the first order term should vanish for any perturbation h. Integration by parts (in t for the first term, in x for the second term, and assuming h is zero on the boundary) gives

$$0 = \int_0^T \int_0^L (-\rho \cdot u_{tt} + k \cdot u_{xx} + k_x \cdot u_x) \cdot h \, dx \, dt.$$

Since this integral is zero for all choices of h, the first factor in the integral must be zero, and we obtain the wave equation for an inhomogeneous medium,

$$\rho \cdot u_{tt} = k \cdot u_{xx} + k_x \cdot u_x.$$

When the elasticity k is constant, this reduces to usual two term wave equation

$$u_{tt} = c^2 u_{xx}$$

where the velocity $c = \sqrt{k/\rho}$ varies for changing density.

1.3 One way wave equations

In the one dimensional wave equation, when c is a constant, it is interesting to observe that the wave operator can be factored as follows

$$\left(\frac{\partial}{\partial t^2} - c^2 \frac{\partial}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right).$$

We could then look for solutions that satisfy the individual first order equations

$$u_t - c u_x = 0 \text{ or } u_t + c u_x = 0.$$

There are one way wave equations, and the general solution to the two way equation could be done by forming linear combinations of such solutions. The solutions of the one wave equations will be discussed in the next section, using characteristic lines $ct - x =$ constant, $ct + x =$ constant.

Another way to solve this would be to make a change of coordintates, $\xi = x - ct$, $\eta = x + ct$ and observe the second order equation becomes

$$u_{\xi\eta} = 0$$

which is easily solved.

In higher dimensions, one could hope to factor the second order wave equation in the form

$$\left(\frac{\partial}{\partial t^2} - c^2\nabla^2\right) = \left(\frac{\partial}{\partial t} - cD\right)\left(\frac{\partial}{\partial t} + cD\right),$$

where D is some first order partial differential operator (independent of t) which satisfies $D^2 = \nabla^2$. Good luck solving this one.[5] The operator D is called the Dirac operator; finding particular Dirac operators is a major intellectual achievement of modern mathematics and physics. The Atiyah-Singer index theorem is a deep result connecting the Dirac operator with the geometry of manifolds.

1.4 Solution via characteristic curves

One method of solution is so simple that it is often overlooked. Consider the first order linear equation in two variables,

$$u_t + cu_x = 0,$$

which is an example of a one-way wave equation. To solve this, we notice that along the line $x - ct =$ constant k in the x, t plane, that any solution $u(x, y)$ will be constant. For if we take the derivative of u along the line $x = ct + k$, we have,

$$\frac{d}{dt}u(ct + k, t) = cu_x + u_t = 0,$$

so u is constant on this line, and only depends on the choice of parameter k. Call this functional dependence $f(k)$ and thus we may set

$$u(x, t) = f(k) = f(x - ct).$$

That is, given any differentiable function f on the real line, we obtain a solution

$$u(x, t) = f(x - ct)$$

and all solutions are of this form. Note this solution represents simply the waveform $f(x)$ moving along to the right at velocity c.

Choosing *which* solution is a question of initial conditions and boundary values. In fact, if we are given the initial values for $u = u(x, 0)$ then this determines f, since $u(x, 0) = f(x - c0) = f(x)$. That is, the initial values for u determine the function f, and the function f determines u everywhere on the plane by following the characteristic lines.

We note in passing that in the usual (two-way) wave equation in three dimensions,

$$u_{tt} = c^2\nabla^2 u,$$

[5] You might ask yourself, why not $D = \nabla$? If you don't know why not, then you are in trouble!

there are characteristic hyperplanes determined by constants (k_x, k_y, k_z, ω) with

$$c^2(k_x^2 + k_y^2 + k_z^2) = \omega^2.$$

It is easy to verify that given any twice-differentiable function $f(x)$, that the functions

$$\begin{aligned} u_1(x, y, z, t) &= f(k_x x + k_y y + k_z z - \omega t) \\ u_2(x, y, z, t) &= f(k_x x + k_y y + k_z z + \omega t) \end{aligned}$$

are solutions to the wave equation. Note that the vector $\mathbf{k} = (k_x, k_y, k_z)$ can be interpreted as a direction of propagation of the traveling wave, and ω is related to temporal frequency. When f is a 1-periodic function, $\mathbf{k}$ is wavenumber and ω is frequency. Wavenumber and frequency are related by the velocity c. Unlike the order one example above, not all solutions are of this form, since we have many characteristic hypersurfaces. More general solutions can be obtained by forming linear combinations of these elementary solutions. Lecture Two will discuss how the elementary solutions can be combined to give the general solution.

1.5 Solution by separation of variables

This is a powerful technique that is applicable to many areas of mathematics. The idea is to look first for solutions of a particularly simple form, then combine to obtain the most general solution. In the PDE case, we look for solutions of the form

$$u(x, y, z, t) = A(x)B(y)C(z)D(t).$$

For instance, consider the equation for vibrations of a thin beam (with constant $k = 1$):

$$u_{tt} = -u_{xxxx}.$$

Plugging in $u = A(x)D(t)$, a separable function of two variables, we obtain

$$A(x)D''(t) = -A''''(x)D(t),$$

which can be rearranged into an equation with all the functions of x on one side, all those of t on the other

$$\frac{A''''(x)}{A(x)} = -\frac{D''(t)}{D(t)} = \lambda$$

which must therefore be a constant (independent of both x and t), which we have denoted by λ. Thus the problem has separated into solving for two ODEs,

$$A''''(x) = \lambda A(x) \text{ and } D''(t) = -\lambda D(t)$$

with a free parameter λ. These are linear, constant coefficient ODEs are easily solved using trig and exponential functions, so for instance we see that some elementary solutions include

$$\begin{aligned} u_1(x, t) &= \sin(\omega x)\sin(\omega^2 t) \\ u_2(x, t) &= \cos(\omega x)\sin(\omega^2 t) \\ u_3(x, t) &= \sinh(\omega x)\sin(\omega^2 t) \\ u_4(x, t) &= \cosh(\omega x)\sin(\omega^2 t) \end{aligned}$$

where we have chosen $\omega > 0$ with $\omega^4 = \lambda > 0$. The fact that the temporary frequency is the square of the spatial frequency in the oscillating solutions tells us this physical situation is very different than the wave equation. The interested reader might consider finding all possible separable solutions.

Notice once again we have many solutions, and more general solutions are obtained by taking linear combinations of these solutions. We defer to Lecture Two the question of how boundary conditions, or initial conditions, restrict this smorgasbord to a unique solution.

1.6 The Helmholtz equation

The Helmhotz equation is a spatial PDE involving the Laplacian, and usually appears in the form

$$\nabla^2 u + k^2 u = 0,$$

on some domain of interest, with given boundary conditions. It arises quite naturally in the separation of variables technique applied to solving the wave equation

$$u_{tt} - c^2 \nabla^2 u = 0.$$

If we look for solutions of the form $u(\mathbf{x}, t) = A(\mathbf{x})B(t)$, then we obtain the two separate systems

$$\nabla^2 A + \lambda A = 0, \qquad B'' + c^2 \lambda B = 0,$$

where λ is the separation constant. There is no apriori reason to assume this constant λ is positive; however in most problems with physical boundary conditions, the only solutions have $\lambda > 0$. One usually sets $k = \sqrt{\lambda}$, and $\omega = ck$, to connect the temporal and spatial frequencies, and the k^2 factor appears as the parameter in the Helmhotz equation..

The Helmhotz equation is also obtained by Fourier transforming the wave equation with respect to the time variable alone. The u_{tt} term becomes $-4\pi^2\omega^2$ times the transformed function. We thus have a new equation, for the transformed function $v = v(x, y, z, \omega)$ with

$$\nabla^2 v + 4\pi^2 \omega^2 v = 0,$$

which is again the Helmholz equation, with $k^2 = 4\pi^2\omega^2$.

It is worth noting the Helmholtz equation is an eigenvalue problem for the Laplacian, which is an elliptic operator. This makes the problem quite elegant, from a mathematical point of view. Including suitable boundary conditions (eg. Dirichlet conditions) turns this into a well-posed problem. For instance, if we solve the 1D wave equation via separation of variables, for a vibrating string with boundary conditions $u(0, t) = u(L, t) = 0$, we obtain the Helmholtz problem

$$A'' + k^2 A = 0 \text{ with boundary conditions } A(0) = A(L) = 0.$$

The general solution to the ODE will be sums of complex exponentials; the boundary conditions restrict this to solutions $\sin(kx)$, with kL a multiple of π. Thus the eigenfunctions for this problem are the functions

$$A_n(x) = \sin \pi n x/L.$$

Continuing with the separation of variables, the corresponding frequency in time is $\omega_n = \pi n c/L$ with eigensolutions $\sin \pi n c t/L$ and $\cos \pi n c t/L$. The general solution to the wave equation on an interval is thus the linear combinations

$$u(x,t) = \sum [a_n \sin(n\pi c t/L) + b_n \cos(n\pi c t/L)] \sin \pi n x/L.$$

Initial conditions will fix the coefficients a_n, b_n uniquely.

Solving the wave equation by this technique, on different domains (disks, balls, etc) lead to different eigenvalue/eigenfunction problems for the Laplace operator. The famous problem "Can you hear the shape of a drum," posed by Marc Kac in 1966, asks whether the eigenvalues of the solution uniquely characterizes a planar domain, since we can hear the corresponding frequency in the time domain. The answer was known to be negative in dimension 16 (!), but it took until 1991 to find counterexamples in the plane. The figure gives an example of two different regions in the plane with the same eigenvalues for the Laplacian.

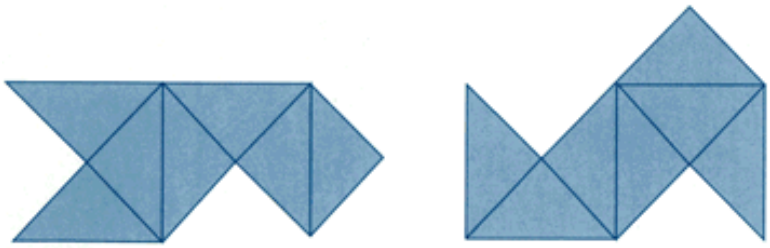

Figure 2: Two regions with different eigenvalues for the Laplacian.

1.7 Classification of second order, linear PDEs

A second order linear PDE in two variables x, t is an equation of the form

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, E, F, G are constants, or specified functions of the variables x, t. The equation is classified into one of three types, based on the coefficients A, B, C, as

- *Elliptic:* if $B^2 - 4AC < 0$;

- *Parabolic:* if $B^2 - 4AC = 0$;
- *Hyperbolic:* if $B^2 - 4AC > 0$.

So for instance, Laplace's equation is elliptic, the heat equation is parabolic, and the wave equation is hyperbolic. It is useful to classify equations because the solution techniques, and properties of the solutions are different, depending on whether the equation is elliptic, parabolic, or hyperbolic. Also, the physical nature of the corresponding problems are different. For instance, elliptic equations often arise in steady-state and equilibrium problems; parabolic equations arise in diffusion problems; hyperbolic problems arise in wave motion and vibrational problems.

An equation can be of *mixed type* if it changes from one type to another, depending on the value of the functions A, B, C. For instance, the equation

$$tu_{xx} + u_{tt} = 0$$

is of mixed type, for $B^2 - 4AC = -4t$ is zero along the line $t = 0$ (parabolic), is positive for $t < 0$ (hyperbolic), and negative for $t > 0$ (elliptic).

When A, B, C are constant, it is always possible to make a linear change of variables to put the equation in a canonical form. This is result is as simple as diagonalizing a 2 by 2 symmetric matrix. The canonical forms as

- *Elliptic:* $u_{xx} + u_{tt} = G(x, y, u, u_t, u_x)$;
- *Parabolic:* $u_{xx} = G(x, y, u, u_t, u_x)$;;
- *Hyperbolic:* $u_{xx} - u_{tt} = G(x, y, u, u_t, u_x)$ or $u_{xt} = G(x, y, u, u_t, u_x)$.

The form $B^2 - 4AC$ is reminiscent of the quadratic formula, but it really should make you think of the determinant of the matrix

$$\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

where the sign of the determinant tells you whether there are two non-zero eigenvalues of the same sign (elliptic), opposite sign (hyperbolic), or one zero eigenvalue (parabolic). This is the key to understanding the classification for linear PDEs with more variables.

For a function $u = u(x_1, x_2, x_3, \ldots, x_n)$ of n independent variables, the general linear second order PDE will be of the form

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=i}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = D$$

where the coefficients A_{ij}, B_i, C, D are constants or functions only of the independent variables. The matrix

$$\mathbf{A} = [A_{ij}]$$

can be chosen symmetric. The equation is then classified into four types, as

- *Elliptic:* if all the eigenvalues of $\mathbf{A}$ are nonzero, and of the same sign;
- *Parabolic:* if exactly one of the eigenvalues is zero, and the rest have the same sign;
- *Hyperbolic:* if $n-1$ of the eigenvalues are of the same sign, the other of opposite sign;
- *Ultrahyperbolic:* If at least two eigenvalues are positve, at least two negative, and none are zero.

This doesn't cover all cases, but it does cover most of the interesting ones. The first three are the typical ones that appear in physics.

1.8 Hyperbolic equations and the wave equation

The wave equation

$$u_{tt} = c^2 \nabla^2 u$$

is an example of a hyperbolic second order linear PDE for a function $u = u(x, y, z, t)$ of four independent variables. By a change of variables, any hyperbolic equation

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=i}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = G$$

can be put into a canonical form that looks a lot like the standard wave equation

$$u_{tt} = \nabla^2 u + G(x, y, z, t, u, u_x, u_y, u_z, u_t),$$

at least locally. This is why we spend so much effort understanding the standard wave equation. Eventually we would like to examine techniques to solve the general hyperbolic equation, which will allow us to consider more realistic physical situations like nonhomogeneous media, anisotropy, and so on. But that will not happen in this series of lectures.

2 Lecture Two: Solutions with boundary conditions and initial conditions

In the first lecture, we saw several examples of partial differential equations that arise in physics, and ways to find many possible solutions. In most physical applications, we are looking for "the solution;" that is, extra conditions are imposed on the equation to pick out a unique, useful solution For PDEs, these are usually boundary conditions and/or initial conditions. We will look at some typical boundary and initial conditions that might ensure a unique solution, and consider what it means to properly pose a mathematical question that has a good solution. We then look at some theorems on existence and uniqueness.

2.1 Boundary and initial conditions

Usually we think of satisfying a PDE only in a particular region in xyz space, for instance in a ball of some radius R. If we denote the region by Ω, typically it is assumed to be an open, connected set with some piecewise smooth boundary $\partial\Omega$. A *boundary condition* is then an additional equation that specifies the value of u and some of its derivatives on the set $\partial\Omega$. For instance,

$$u = f(x, y, z) \text{ on } \partial\Omega$$

or

$$u_x = g(x, y, z) \text{ on } \partial\Omega$$

are boundary conditions.

An *initial* condition, on the other hand, specifies the value of u and some of its derivatives at some initial time t_0 (often $t_0 = 0$). So the following are examples of initial conditions:

$$u(x, y, z, t_0) = f(x, y, z) \text{ on } \Omega$$

or

$$u_t(x, y, z, t_0) = f(x, y, z) \text{ on } \Omega.$$

As an example, consider the 1D wave equation restricted to the interval $[0, L]$. The region of interest is the open interval $\Omega = (0, L)$ with boundary points $x = 0, L$. A typical physical problem is to solve (for $u = u(x, t)$) the equation

$$\begin{aligned}
u_{tt} &= c^2 u_{xx}, \text{ on the region } 0 < x < L, 0 < t \\
u(0, t) &= 0, \quad \text{a boundary condition} \\
u(L, t) &= 0, \quad \text{a boundary condition} \\
u(x, 0) &= f(x), \quad \text{an initial condition, at } t_0 = 0 \\
u_t(x, 0) &= g(x), \quad \text{an initial condition.}
\end{aligned}$$

It is routine now to solve this by separation of variables. The boundary conditions force terms like $\sin(\lambda x), \cos(\lambda x)$ to vanish at the endpoints; this kills the cosine, and puts λ to be integer multiples of π/L. The (unique) solution is of the form of an infinite series

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi ct/L) + b_n \sin(n\pi ct/L)) \sin(n\pi x/L),$$

where the coefficients a_n, b_n are now determined by the initial conditions. In fact, they are determined by the sine expansion of the functions $f(x), g(x)$ on the interval $[0, L]$.[6] Provided that f, g are reasonably smooth, there always is a unique solution to this problem.

The boundary condition

$$u(0,t) = 0$$

is a reasonable assumption for a vibrating string where the string is fixed at the endpoint $x = 0$. If, on the other hand, we have a free end to the string, the physical constraint could be expressed by the boundary condition

$$u_x(0,t) = 0.$$

A combination of these conditions,

$$au(0,t) + bu_x(0,t) = 0$$

for given constants a, b leads to the classical Sturm-Liouville problems.

In higher dimensions, a boundary condition

$$u(x,y,z,t) = 0 \text{ on } \partial\Omega$$

corresponds to a vibrating system that is fixed on the boundary. The condition that the normal component of the gradient vanish on the boundary, say,

$$\eta \cdot \nabla u(x,y,z,t) = 0 \text{ on } \partial\Omega,$$

where η denotes the normal to the surface, gives the physical restriction for a vibrating system that slides freely along a fixed surface.[7]

2.2 Cauchy, Dirichlet, and Neumann conditions

We will often hear reference to these three types of boundary/initial conditions. So let's make it clear what it is.

[6] Notice the constants. The problem is simpler if we set $c = 1$ and $L = \pi$, but this is just a bit extra algebra.

[7] In seismic, the displacements u are vector-valued functions; the boundary conditions are determined by forcing continuity of the stress tensor. The condition for a free surface are different. But for pressure waves, as measured by hydrophones, say, then one has the vanishing condition, $u = 0$ on the free boundary.

The *Cauchy* condition specifies the values of u and several of its normal derivatives, along some given smooth surface in the coordinate space of all the independent variables (including time). To have any hope of getting a well-posed problem, it is important to get the dimensions right. So, if u is a function of n variables, the surface S should have dimension $n-1$ (it is a hypersurface), and if the PDE is order k, the Cauchy data must specify the values of u and its first $k-1$ derivatives along the normal to S:

$$u = f_0, u_\eta = f_1, u_{\eta\eta} = f_2, \ldots u_{\eta\ldots\eta} = f_{k-1} \text{ on } S,$$

where $f_0, \ldots, f_{k-1}$ are given functions. Here u_η means the derivative along the normal to the surface. If u is an analytic function, you can consider doing a power series expansion at points along S, using the Cauchy data and PDE to solve for the coefficients in series expansion.[8]

The initial value problem

$$\begin{aligned} u(x,y,z,0) &= f(x,y,z) \text{ for all } x,y,z \\ u_t(x,y,z,0) &= g(x,y,z) \text{ for all } x,y,z \end{aligned}$$

is an example of a Cauchy problem for any second order ODE, with hypersurface $S = \{(x,y,z,t) : t = 0\}$.

It is important that the hypersurface not be a characteristic surface for the Cauchy problem to be solvable. We won't define characteristic surface here; they come from the coefficients of the PDE, and you would notice if you were on one!

The *Dirichlet* condition specifies the value of u on the boundary $\partial\Omega$ of the region of interest. Think Dirichlet = Data on boundary.

The *Neumann* condition specifies the value of the normal derivative, u_η, of the boundary $\partial\Omega$. Think Neumann = Normal derivative on boundary.

Note that $\partial\Omega$ is a hypersurface, and so the Dirichlet and Neumann conditions each specify less information than the Cauchy condition for second order and higher PDEs. It is rather remarkable that for certain elliptic problems, merely Dirichlet or Neumann data alone suffices to solve the problem.

The point of including boundary and initial problems is to force our solutions to be unique, and hopefully well-behaved. Lets look at what it means to pose a good mathematical problem.

[8] If you are wondering why only the normal derivatives are specified, note that the tangential derivatives along S are already determined by forcing $u = f_0$ along S.

2.3 Well-posed problems

We say a mathematical problem is *well-posed* [9] if it has the following three properties:

1. *Existence* There exists at least one solution to the problem;
2. *Uniqueness:* There is at most one solution;
3. *Stability* The unique solution depends in a continuous manner on the data of the problem. A small change in the data leads to only a small changes in the solution.

It is easy enough to illustrate these ideas with the example of solving for $\mathbf{x}$ a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{y},$$

for given matrix $\mathbf{A}$ and vector $\mathbf{y}$. If the matrix $\mathbf{A}$ is singular, for some inputs y, no solution may exist; for others inputs y there may be multiple solutions. And if $\mathbf{A}$ is close to singular, a small change in $\mathbf{y}$ can lead to a large change in solution $\mathbf{x}$.

To see this in a PDE context, consider the following problem of solving the 1D heat equation in the positive quadrant $x, t > 0$. We add some reasonable boundary and initial conditions to try to force a unique solution:

$$\begin{aligned} u_t &= u_{xx}, \qquad x > 0, t > 0 \\ u(x,0) &= 0, \qquad x > 0, \text{ a boundary condition} \\ u(0,t) &= 0, \qquad t > 0, \text{ an initial condition.} \end{aligned}$$

The boundary and initial conditions strongly suggest "the solution" is

$$u(x,t) \equiv 0,$$

which is indeed a solution satisfying the BC and IC. But it is not the only solution; for instance, another solution satisfiying that BC and IC is the function

$$u(x,t) = \frac{x}{t^{3/2}} e^{-x^2/4t}.$$

It is easy to check that this function satisfies the PDE in the open quadrant $x, t > 0$ and extends to be zero on both the positive x axis $x > 0$, and the positive t-axis $t > 0$. It is curious that by ignoring the behaviour of the function at the origin $(x, t) = (0, 0)$ somehow allows for more than one solution.[10]

One might suppose this is only a mathematical oddity; perhaps one would reject the second solution based on physical grounds. However, keep in mind that many PDE problems

[9] Hadamard came up with this definition.

[10] In fact the function has an interesting singularity at the origin, it blows up along certain curves approaching the origin.

may be solved numerically: it is unlikely that your numerical method will be smart enough to reject non-physical solutions, without you considering these possibilities.[11]

The heat equation can also be used to illustrate instability in solutions by observing that diffusion processes, when run backwards, tend to be chaotic. But instability can also come up in elliptic equations as well (which we often think of as "nice"). For instance, fix $\epsilon > 0$ a small parameter and consider Laplace's equation on the upper half plane, with

$$\begin{aligned} u_{xx} + u_{tt} &= 0, \qquad -\infty < x < \infty, t > 0 \\ u(x,0) &= 0 \text{ all } x, \text{ a boundary condition} \\ u_t(x,0) &= \epsilon \sin \frac{x}{\epsilon} \text{ all } x, \text{ a boundary condition.} \end{aligned}$$

This has solution $u(x,t) = \epsilon^2 \sin(x/\epsilon) \sinh(t/\epsilon)$, which gets very large as $\epsilon \to 0$. Compare this with the zero solution $u_0(x,t) \equiv 0$, which is the solution to the problem for $\epsilon = 0$. Thus we have an instability: the input $u_t(x,0) = \epsilon \sin(x/\epsilon)$ goes to zero as $\epsilon \to 0$ but the output does not converge to the zero solution.

2.4 Existence and uniqueness theorems

The first result, the Cauchy-Kowalevski Theorem, tells us that the Cauchy problem is always locally solvable, if all the functions that appear are analytic [12]. The result is usually stated in terms of an initial value problem; the general result follows by transforming the general Cauchy problem, locally, to an initial value problem.

Theorem 1 (Cauchy-Kowalevski) *If the functions $F, f_0, f_1, \ldots f_{k-1}$ are analytic near the origin, then there is a neighbourhood of the origin where the following Cauchy problem (initial value problem) has a unique analytic solution $u = u(x,y,z,t)$:*

$$\begin{aligned} \frac{\partial^k u}{\partial t^k}(x,y,z,t) &= F(x,y,z,t,u,u_x,u_y,u_z,...) \qquad \textit{a k-th order PDE} \\ \frac{\partial^j u}{\partial t^j}(x,y,z,0) &= f_j(x,y,z) \qquad \textit{for all } 0 \le j < k. \end{aligned}$$

The statement means to indicate that the function F depends on the independent variables x, y, z, t as well as u and all its partial derivatives up to order k, except for the "distinguished" one $\frac{\partial^k u}{\partial t^k}$. The proof amounts to chasing down some formulas with power series. We've stated the case for (3+1) dimensons, but it is true in other dimensions as well.

[11] I like to remember that the existence of the positron was predicted from some "unphysical" solution to certain equations in quantum mechanics. So you never know when a non-physical solution might actually be real.

[12] They have power series expansions that converge in some neighbourhood.

It is a rather odd theorem, though, because it assumes a lot of analyticity. We might expect, in geophysics, certain coefficients in our wave equation to be only piecewise continuous; Cauchy-Kowalevski does not guarantee that we have a solution then.[13]

The next result concerns the Laplacian, which is an elliptic operator. You can consider it in any dimension you like.

Theorem 2 (Dirichlet and Neumann problems) *Suppose Ω is an open, bounded, connected region with smooth boundary $\partial\Omega$. Then the Dirichlet problem*

$$\begin{aligned} \nabla^2 u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega \end{aligned}$$

has a unique solution for each continuous function f on $\partial\Omega$.

The Neumann problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ in } \Omega, \\ u_\eta &= f \text{ on } \partial\Omega \end{aligned}$$

has a solution for continuous function f if and only if $\int_{\partial\Omega} f = 0$. In this case, the solution is unique up to an additive constant.

For simple regions (half plane, open ball), the existence is proved by constructing Greens functions for the Laplacian operator. For more complicated regions, the method of layer potentials is used. The types of regions Ω this is valid for can be expanded, to include finite unions of the given type, or their unbounded complements. There are similar results for the more general Laplace equation, $\nabla^2 u = g$. Notice that both the Dirichlet and Neumann problems specify only one half of the data we expect for Cauchy problems: this is a special feature of elliptic problems.

It is worthwhile to note that for more general elliptic PDEs with non-constant coefficients, there are similar existence and uniqueness results, provided the coefficients are smooth, and the operator is coercive (a measure of just how elliptic it is). In general, for a k-th order elliptic PDE, only $k/2$ normal derivatives need to be specified.[14]

Stability is shown from explicit formulas connecting the forcing terms and boundary conditions to the solution. Unique is often proved using an energy formula.

[13] And there are some strange counterexamples to any "extensions" of Cauchy-Kowaleski that would drop analyticity. Eg. the Lewy example.

[14] For k odd, this will seem like a strange requirement, but there is a way to make precise what half a data set means.

2.5 D'Alembert's solution to the 1D wave equation

Separately, we look at the existence of solutions to the initial value problem (Cauchy problem) for the wave equation. In one dimension, this is called D'Alembert's solution.

From the Cauchy-Kowaleski theorem, we expect to specify u and its first time derivative along the initial curve $t = 0$. Thus, we look for a solution $u = u(x,t)$ for the 1D wave equation with initial conditions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= f(x) \text{ for } -\infty < x < \infty \\ u_t(x,0) &= g(x) \text{ for } -\infty < x < \infty \end{aligned}$$

From the method of characteristic curves, we know to look for a solution as a sum of left and right going waves,

$$u(x,t) = u_l(x+ct) + u_r(x-ct).$$

From the initial equations, we have

$$u_l(x) + u_r(x) = f(x) \text{ and } c(u_l'(x) - u_r'(x)) = g(x).$$

Differentiating the first equation, we solve the 2 by 2 system to obtain $u_l' = \frac{1}{2}(f' + g/c)$, $u_r' = \frac{1}{2}(f' - g/c)$, and integrate to find

$$u_l(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + C_1 \qquad u_r(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + C_2,$$

where $C_1 + C_2 = 0$, as $u_l + u_r = f$. Now we have

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

It is easy to see this solution satisfies the initial value problem, and the solution$u$ is smooth when f, g are both smooth. What's more, if f, g are discontinuous, or even generalized functions, this solution still makes sense in a distributional sense.

2.6 Solution to the n-dimensional wave equation

The n-dimensional initial value problem (Cauchy problem)

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u \text{ for } \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x},0) &= f(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^n \\ u_t(\mathbf{x},0) &= g(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

can be reduced to a 1D wave equation by integrating the wavefield $u(\mathbf{x}, t)$ over spheres of radius r in the spatial variables, at least when n is odd. To summarize the derivation, we note if $\phi = \phi(\mathbf{x})$ is a smooth function in the spatial variables $\mathbf{x}$, then the spatial average

$$M_\phi(\mathbf{x}, r) = \int_{|\mathbf{y}|=1} \phi(\mathbf{x} + r\mathbf{y})\, d\sigma(\mathbf{y})$$

satisfies the the PDE

$$\nabla^2 M_\phi(\mathbf{x}, r) = \left[\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right] M_\phi(\mathbf{x}, r).$$

Taking averages over a wavefield $u(\mathbf{x}, t)$, which depends on time t, we find that u satisfies the wave equation if and only if

$$c^2 \left[\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right] M_u(\mathbf{x}, r, t) = \frac{\partial^2}{\partial t^2} M_u(x, r, t).$$

This PDE for M_u can be converted to the 1D wave equation and solved using d'Almerbert's approach.

Skipping a few details, we present the solution for the 3-dimensional wave equation, with c constant, which is given as

$$u(\mathbf{x}, t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|\mathbf{y}|=1} f(\mathbf{x} + ct\mathbf{y})\, d\sigma(\mathbf{y})\right) + \frac{t}{4\pi}\int_{|\mathbf{y}|=1} g(\mathbf{x} + ct\mathbf{y})\, d\sigma(\mathbf{y}).$$

The integrals are simply averages over sphere of fixed radius, using the usual surface area integral; This has the effect of computing the value of $u(\mathbf{x}, t)$ in terms of data on a ball (in space) centered at $\mathbf{x}$, with radius ct. That is, the information propagates from initial data to solutions at exactly the speed c. This integral over the sphere corresponds to a distribution in $\mathbb{R}^3$ with support on the set $\{\mathbf{z} \in \mathbb{R}^3 : |\mathbf{z} - \mathbf{x}|^2 = c^2t^2\}$. This distribution is denoted $\delta(|\mathbf{z}-\mathbf{x}|^2 - c^2t^2)/|\mathbf{z}-\mathbf{x}|$, where δ is the Dirac delta distribution on the real line. The integral formulation is an equivalent formulation.

For $n = 2$, the solution is obtained from the previous formula by the method of descent; we just solve with initial conditions that depend only on two spatial variables. Again, skipping the details, we write down the solution to Cauchy problem for the 2-dimensional wave equation as

$$u(\mathbf{x}, t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\left(t \int_{|\mathbf{y}|\le 1} \frac{f(\mathbf{x} + ct\mathbf{y})}{\sqrt{1 - |\mathbf{y}|^2}}\, d\mathbf{y}\right) + \frac{t}{2\pi}\int_{|\mathbf{y}|\le 1} \frac{g(\mathbf{x} + ct\mathbf{y})}{\sqrt{1 - |\mathbf{y}|^2}}\, d\mathbf{y}.$$

In contrast to the 3-dimensional case, here the integral is over the unit disk $|\mathbf{y}| \le 1$, so the solution depends on data points that are within a distance ct or less. So propagation of information in 2D is quite different, in particular Huygens principle is not true here.

The solutions to the Cauchy problem in higher dimensions can be given; they are remarkably similar to the above forms. In fact, the formulas just involve some extra time derivatives in front of the integrals. (See Folland's book for details.)

2.7 Huygens principle

The above integrals show that in dimension 3 (and in fact, for all odd dimensions), the solution of the wave equation at point $\mathbf{x}_0$, at time t_0, depends only on the initial data in an infinitesimal neighbourhood of the sphere $|\mathbf{x} - \mathbf{x}_0| = ct_0$. This is a statement of Huygens principle. In particular, it says that information from a point source travels in the form of a sphere. The wavefront is thus sharp, with a sudden onset at the start, and sudden cutoff at the end.

In dimension 2 (and all even dimensions), the behaviour is different. Wavefronts do have a sharp onset, but they decay with a long tail. This we see because the solution at $\mathbf{x}_0, t_0$ depends on the initial conditions on the entire disk $|\mathbf{x}_-\mathbf{x}_0| \leq ct_0$. This behaviour can be observed in the ripples of a pond that are formed around a pebble that falls into the pond.

However, it is worth pointing out that although Huygens principle is not true in even dimensions, it is still approximately true! The integrand in solution to the Cauchy problem has a factor of $\sqrt{1 - |\mathbf{y}|^2}$ in the denominator, which gives a hefty singularity at the surface of the sphere. Not as much as a delta function, but still significant.

2.8 Energy and uniqueness of solutions

The energy of a wavefield $u(\mathbf{x}, t)$ in some region of space Ω is simply defined as the integral

$$E = \frac{1}{2} \int_\Omega (u_t)^2 + c^2 \left(\nabla_{\mathbf{x}} u\right)^2 d\mathbf{x}.$$

Why this is an energy is an interesting question. The u_t term is related to a velocity, so this a kinetic energy term; the spatial gradient $\nabla_{\mathbf{x}} u$ is related to displacement, and thus to the stored energy in a stretched elastic (Hooke's law), so this is a potential energy term.[15]

This energy definition can be used to prove the uniqueness of solutions to the wave equation. For instance, if we fix a point $\mathbf{x}_0$ in space, and some time $t_0 \geq 0$, consider the ball of points close enough to communicate with $\mathbf{x}_0$ using a travel time of $t_0 - t \geq 0$. We define this ball as

$$B_t = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| \leq c(t_0 - t)\}.$$

The energy on this shrinking ball is then defined as

$$E(t) = \frac{1}{2} \int_{B_t} (u_t)^2 + c^2 \left(\nabla_{\mathbf{x}} u\right)^2 d\mathbf{x}.$$

It is an easy exercise in the divergence theorem to show that if u satisfies the wave equation, and twice continuously differentiable, then

$$\frac{dE}{dt} \leq 0.$$

[15] We are being a bit casual about the choice of units here.

If we know from initial conditions that $E(0) = 0$, then it follows that energy $E(t) = 0$ is constant on the interval $[0, t_0]$, since by definition, the energy is a nonegative function.

So, we get the following uniqueness theorem:

Theorem 3 *Suppose u_1, u_2 are C^2 solutions to the wave equation $u_{tt} = c^2\nabla^2 u$ whose initial data (Cauchy data) agree on the ball*

$$B = \{(\mathbf{x}, 0) : |\mathbf{x} - \mathbf{x}_0| \leq ct_0\}.$$

Then the two solutions u_1, u_2 agree on the cone

$$C = \{(\mathbf{x}, t) : |\mathbf{x} - \mathbf{x}_0| \leq c(t_0 - t)\}.$$

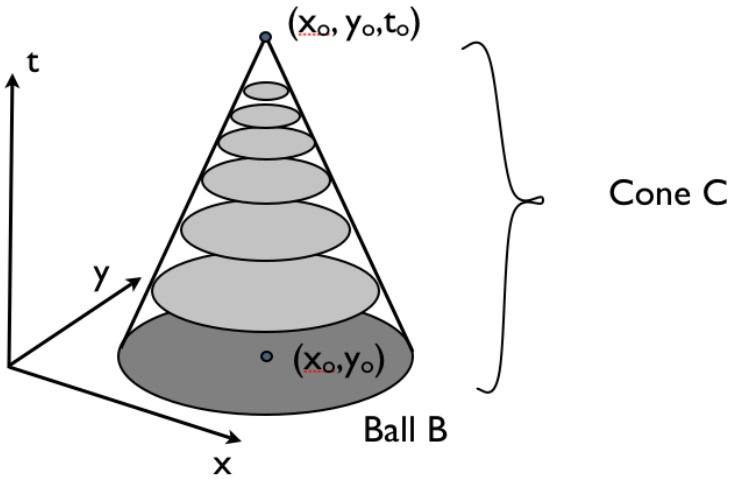

Figure 3: The cone of influence on solutions to the wave equation.

Roughly speaking, this says that if two solutions agree on a given ball, then any disturbances that come in to make them different must creep into the ball at no faster than the propagation speed c.

The proof proceeds by considering the function $u = u_1 - u_2$ which is another solution to the wave equation, but with zero initial values. The corresponding energy defined above (on the shrinking balls) is thus zero, and so $u_t = 0, \nabla_x u = 0$ on the cone. Thus u is a constant on the cone, and by the initial conditions, this constant is zero. Thus $u_1 = u_2$.

Note the same analogous uniqueness theorem is true for the inhomogeneous wave equation

$$u_{tt} - c^2\nabla^2 u = h$$

since via the same proof, the difference function $u = u_1 - u_2$ is a solution to the homogeneous wave equation. Even the forcing term h for the two solutions u_1, u_2 could be different functions, that happen to agree on the cone.

This will be useful to remember for the next lecture, where we consider the inhomogeneous wave equation.

3 Lecture Three: Inhomogeneous solutions - source terms

In this lecture we solve the inhomogeneous wave equation

$$u_{tt} - c^2\nabla^2 u = h$$

where $h = h(x, y, z, t)$ is a forcing term that drives the vibrational system. One might find it a little odd that this is so important in seismic imaging, since in typical seismic experiments, the seismic source (dynamite, Vibroseis) is on the earth's surface and could be treated as a boundary condition. There are not really seismic sources inside the earth, except for earthquakes. However, the reflectors within the earth (geological structures, reservoirs, etc) can be treated as a source of energy, that initiates a wave traveling back to the surface the the earth, where the resulting wave is measured by geophones.[16] For this reason we must consider solving the wave equation with a forcing term.

3.1 Particular solutions and boundary, initial conditions

Given any particular solution to the inhomogeneous equation

$$u_{tt} - c^2\nabla u = h$$

another solution can be obtained by adding any nonzero solution of the homogeneous equation

$$u_{tt} - c^2\nabla u = 0.$$

Thus, the most general solution to the inhomogeneous equation can be written as a sum of a homogeneous solution u^H, and a particular solution u^P, with solution

$$u = u^H + u^P$$

the general solution. As we saw earlier, there are many solutions to the homogeneous equation, and thus again we have a situation with many different solutions.

So, how to we set up a well-posed problem with a unique solution? Again, the approach is to specify boundary conditions and/or initial conditions that will pick out a unique solution. However, we have done all the necessary work in the lectures above; we don't have to do any addition work for the inhomogeneous case. The procedure is as follows:

1. Fix a forcing term h and, say, Cauchy conditons $u = f, u_t = g$ at $t = 0$;

[16] Perhaps this is called the exploding reflector model.

2. Find any particular solution that satisfies $u_{tt} - c^2\nabla u = h$, call. it u^P. Note that u^P need not satisfy the Cauchy conditions;

3. Solve the homogeneous equation $u_{tt} - c^2\nabla^2 u = 0$ with Cauchy condtions $u = f - u^P, u_t = g - u_t^P$ at $t = 0$. Call the solution to this homogeneous problem u^H. It is unique.

4. The unique solution to the inhomogeneous equation satisfying the Caucy conditions, is $u = u^H + u^P$.

So, the uniqueness is assured by solving the homogeneous problem with modified Cauchy conditions.

I'm taking some pains to point out here that the particular solution u^P is not unique, and so we should not really call it "the" solution. It is "a" solution to the inhomogeneous equation, and is just one of many, unless we add boundary or initial conditions or something. In this lecture, we will only be looking for "a" solution to the inhomogeneous problem. It would be a mistake, then, to talk about "the" solution obtained by Green's functions, by Fourier transform, by variation of parameters, because these solutions are not unique and can even be different particular solutions.[17]

3.2 Solution via variation of parameters

Since we've solved homogeneous wave equation with Cauchy data, it is useful to know that this solution can be used to solve the inhomogeneous equation using the technique of variation of parameters.

Theorem 4 (Variation of parameters) *Fix forcing term $h = h(\mathbf{x}, t)$. For each real number s, let $v = v(\mathbf{x}, t; s)$ be the solution to the homogeneous wave equation*

$$\begin{aligned} v_{tt} - c^2\nabla^2 v &= 0 \\ v(\mathbf{x}, 0; s) &= 0 \\ v_t(\mathbf{x}, 0; s) &= h(\mathbf{x}, s). \end{aligned}$$

Then $u(\mathbf{x}, t) = \int_0^t v(x, t - s; s)\, ds$ is a solution to the inhomogeneous wave equation

$$\begin{aligned} u_{tt} - c^2\nabla^2 u &= h \\ u(\mathbf{x}, 0) &= 0 \\ u_t(\mathbf{x}, 0; s) &= 0. \end{aligned}$$

[17] Personally, I have found this very confusing. So for instance, there is not such thing as "the" Green's function, or "the" fundamental solution the wave equation. There is "a" Green's function; in fact there are infinitely many if we don't specify initial or boundary conditions. Personal note: try not to worry when physicists throw a Green's function at you and call it "the" Green's function.

For this theorem to hold, one must assume some continuity on the first few derivatives of h. The proof proceeds simply by verifying that the function $u(\mathbf{x}, t)$ so defined is a solution as stated. So,

$$u(\mathbf{x}, 0) = \int_0^0 v(\mathbf{x}, 0 - s; s)\, ds = 0.$$

Also, by the fundamental theorem of calculus (remember to differentiate the endpoints too)

$$u_t(\mathbf{x}, t) = v(\mathbf{x}, 0; t) + \int_0^t v_t(\mathbf{x}, t - s; s)\, ds = \int_0^t v_t(\mathbf{x}, t - s; s)\, ds,$$

and thus $u_t(\mathbf{x}, 0) = 0$. Differentiating again, we have

$$\begin{aligned} u_{tt}(\mathbf{x}, t) &= v_t(\mathbf{x}, 0; t) + \int_0^t v_{tt}(\mathbf{x}, t - s; s)\, ds \\ &= h(\mathbf{x}, t) + \int_0^t c^2 \nabla^2 v(\mathbf{x}, t - s; s)\, ds = h(\mathbf{x}, t) + c^2 \nabla^2 u(\mathbf{x}, t). \end{aligned}$$

So, u satisfies the required inhomogeneous wave equation. This completes the proof.

It is sometime convenient to restate the theorem while hiding the initial time $t = t_0$. So if the forcing term $h = h(\mathbf{x}, t)$ is zero for $t \ll 0$, just let $v(\mathbf{x}, t; s)$ be the solution to the homogeneous problem

$$\begin{aligned} v_{tt} - c^2 \nabla^2 v &= 0 \\ v(\mathbf{x}, 0; s) &= 0 \\ v_t(\mathbf{x}, 0; s) &= h(\mathbf{x}, s). \end{aligned}$$

and set $u = u(\mathbf{x}, t) = \int_{-\infty}^t v(\mathbf{x}, t - s; s)\, ds$ for the solution to the inhomogeneous problem, which is zero at $t \ll 0$.

3.3 Fundamental solutions

A *fundamental solution* to a linear, constant coefficient partial differential operator L is a distribution Φ which satisfies the distributional equation

$$L\Phi = \delta_0,$$

where δ_0 is the Dirac delta function at the origin (eg. a unit point mass). Note that if Φ is a fundamental solution to the wave equation, then for any (smooth, compactly supported) function $h = h(\mathbf{x}, t)$ we have

$$(\partial_{tt} - c^2 \nabla)(h * \Phi) = h * (\partial_{tt} - c^2 \nabla)\Phi = h * \delta_0 = h.$$

Thus the function $u = h * \Phi$ is a solution to the inhomogeneous wave equation

$$u_{tt} - c^2 \nabla u = h.$$

The converse is true as well; if Φ is a distribution with the property that $u = h * \Phi$ is a solution to the inhomogeneous wave equation, for each function h, then Φ is a fundamental solution.

We already have examples of fundamental solutions for the wave equation: we simply reinterpret the integral solutions of the last section as convolutions, and identify the distribution.

In dimension $n = 1$, the d'Alembert solution for the variation of parameters problem gives

$$v(x,t;s) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(r,s)\,dr = \frac{1}{2c}\int_{-ct}^{ct} h(x+r,s)\,dr$$

and thus the particular solution to the homogeneous equation is

$$u(x,t) = \frac{1}{2c}\int_{-\infty}^{t} v(x,t-s;s)\,ds = \frac{1}{2c}\int_{-\infty}^{t}\int_{-c(t-s)}^{c(t-s)} h(x+r,s)\,dr\,ds.$$

The inner integral looks like a convolution with a boxcar function, with support of length $2c$. The outer integral looks like a convolution with a boxcar function with half-infinite support.

With this hint, we can guess the form of the convolution, and verify that this integral is simply the convolution of h with the function

$$\Phi_+(x,t) = \begin{cases} 0 & t \le 0; \\ \frac{1}{2c} & t > 0, -ct \le x \le ct. \end{cases}$$

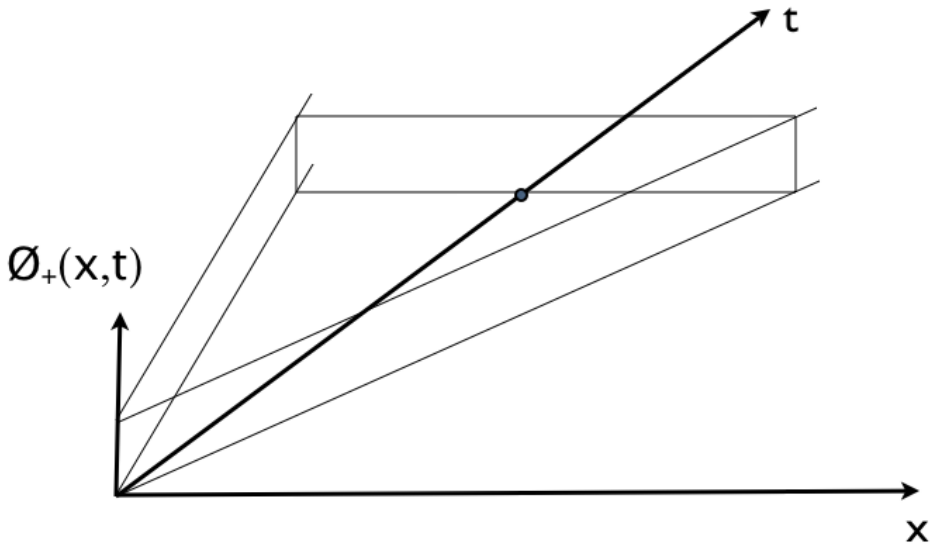

Figure 4: The fundamental solution to the 1D wave equation.

That is, for this piecewise constant function Φ_+, we can express the solution u to the inhomogeneous wave equation as

$$u = h * \Phi_+.$$

This function Φ_+ is an example of a fundamental solution to the wave equation in 1D.

Note the cross-sections $\Phi(\cdot, t)$ at fixed times are the boxcar functions described in Ursenbach's lectures.

Similarly, in dimensions 2 and 3, for each time $t > 0$ we define spatial distributions Φ_t^2, Φ_t^3 by the integrals

$$\begin{aligned} \langle \Phi_t^2, g \rangle &= \frac{t}{2\pi} \int_{|\mathbf{y}|\leq 1} \frac{g(ct\mathbf{y})}{\sqrt{1-|\mathbf{y}|^2}}\, d\mathbf{y} \\ \langle \Phi_t^3, g \rangle &= \frac{t}{4\pi} \int_{|\mathbf{y}|=1} g(ct\mathbf{y})\, d\sigma(\mathbf{y}), \end{aligned}$$

which are simply the integrals that appeared in the solution to the Cauchy problem, centered at $\mathbf{x} = 0$. The fundamental solutions to the wave equation, in dimensions two an three, are given by defining the space-time distribution Φ_+ as

$$\Phi_+(x,t) = \begin{cases} 0 & t \leq 0; \\ \Phi_t^n & t > 0, -ct \leq x \leq ct. \end{cases} \quad \text{for } n = 2, 3.$$

An interesting exercise is to verify that the convolution $u = h * \Phi_+$ is exactly the solution to the inhomogeneous wave equation we derived earlier.

Notice the values of the solution $u = h * \Phi_+$ at point t_0 depends only on the values of h at times $t \leq t_0$. It is a natural choice for a solution that arises from h driving the system. Another perfectly acceptable fundamental solution Φ_- is obtained by time reversal, with

$$\Phi_-(\mathbf{x}, t) = \Phi_+(\mathbf{x}, -t).$$

In physics, these are usually called the retarded and advanced Green's functions for the wave equation.[18]

3.4 Green's functions, Green's theorem

Mr. Green was apparently a very popular man, and many things got named after him, which can be confusing for the rest of us. A fundamental solution to a (constant coefficient linear) PDE is often called a Green's function; thus the fundamental solutions of the previous section are called the Green's function for the wave equation. And again, there is nothing

[18] In dimension three, the distribution can be expressed as $\delta(|\mathbf{x}|^2 - c^2t^2)/|\mathbf{x}|$, scaled by 4π.

unique about them, so they shouldn't really be called "the" Green's function. Also note that these "functions" are actually distributions.

It is somewhat confusing that when we talk about the wave equation, another Green's function for a different PDE is often mentioned. Within the wave equation, there is the Laplacian operator ∇^2. The fundamental solution for the Laplacian $\nabla^2 u = \delta_0$ is often called "the" Green's function; in dimension $n = 2$ it is given by

$$\Phi(\mathbf{x}) = \frac{1}{2\pi} \log |\mathbf{x}|,$$

while in dimension $n > 2$ it is given by

$$\Phi(\mathbf{x}) = \frac{|\mathbf{x}|^{2-n}}{(2-n)\mu_n},$$

where μ_n is a normalizing constant (the volume of the unit ball in dimension n.) This is easily verified by confirming that $\nabla^2(g * \Phi) = g$ for any smooth function g.

And, to be really precise, *the* Green's function for the Laplacian, applied to a specific bounded domain Ω with smooth boundary $\partial\Omega$, is the unique function $G(\mathbf{x}, \mathbf{y})$ such that for each point $\mathbf{x}$ in Ω,

1. $\nabla^2_{\mathbf{y}}[G(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y})] = 0$;
2. $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y})$ is continuous on $\overline{\Omega}$;
3. $G(\mathbf{x}, \mathbf{y}) = 0$ for each $\mathbf{x} \in \Omega, \mathbf{y} \in \partial\Omega$,

where Φ is the fundamental solution mentioned in the last paragraph. That is, the difference between the Green's function and a fundamental solution is the unique solution to the Dirichlet problem on a specific domain. Note for certain domains, the solution to this problem could be very complicated.

Similarly, to specify a unique Green's function, or fundamental solution, for the wave equation, one must pose some boundary and initial conditions. The answer always will involve a distribution, in dimensions $n > 1$. The examples in the last section are precise formulations of particular distributional solutions.

Green's theorem, on the other hand, consists of some identites Laplacian, namely:

Theorem 5 (Green's Identities) *Suppose Ω is a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is smooth. If u, v are functions with continuous first derivatives, then*

$$\begin{aligned} \int_{\partial\Omega} v\partial_\eta u \, d\sigma &= \int_\Omega (v\nabla^2 u + \Delta v \cdot \Delta u) \, d\mathbf{x} \\ \int_{\partial\Omega} (v\partial_\eta u - u\partial_\eta v) \, d\sigma &= \int_\Omega (v\nabla^2 u - u\nabla^2 v) \, d\mathbf{x} \end{aligned}$$

where ∂_η indicates the outward normal derivative, and $d\sigma$ the surface area.

The first identity is a consequence of the divergence theorem[19] with vector field $\mathbf{F} = v\nabla u$, and $\int_{\partial\Omega}(\mathbf{F}\cdot\eta)d\sigma = \int_\Omega(\nabla\cdot\mathbf{F})d\mathbf{x}$. The second identity follows by subtracting the symmetric result from the first identity. They have obvious uses when computing with the Laplacian.

3.5 Why the convolution with the fundamental solutions?

Let's point this out specifically. There is something magical in the fact that we can solve the inhomogeneous wave equation in the form of convolution $u = h * \Phi$. The only reason this works is because we have been working with a constant coefficient PDE. Thus the fundamental solution at the origin (the one that gives the Dirac delta function at the origin) can be transported around to other points $(\mathbf{x}_0, t_0)$ simply by translation. Summing these up with weights $h(\mathbf{x}_0, t_0)$ results in a convolution.

For non-constant coefficient equations, we expect something much more complicated to happen. For instance, you might need a different fundamental solution at each point, and then sum them up with the weighting h. This will *not* be a convolution.

In general, one might hope to solve a non-constant coefficient PDE $Lu = h$ with an integral solution

$$u(\mathbf{x}) = \int \Phi(\mathbf{x},\mathbf{y})h(\mathbf{y})\,d\mathbf{y}$$

where, for each $\mathbf{x}$, the distributional map $\mathbf{y}\mapsto\Phi(\mathbf{x},\mathbf{y})$ satisfies $L\Phi(\mathbf{x},\cdot) = \delta_{\mathbf{x}}$. Looking for a fundamental solution at each $\mathbf{x}$ could be very challenging.

3.6 The Fourier transform and solutions

Constant coefficient linear PDEs can also be effectively solved using the Fourier transform. The Fourier transform of a function $f = f(t)$ of a single variable is defined as

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi it\cdot\omega}\,dt;$$

the function is recovered from its transform using the inverse transformation,

$$f(t) = \int_{\mathbb{R}} \hat{f}(\omega)e^{+2\pi it\cdot\omega}\,d\omega.$$

[19] And, as I like to explain to my kids, the Divergence Theorem simply says you can measure how much methane gas a cow produces by either measuring how much is produced in each cubic centimeter of the cow, or simply by measuring how much leaks out its mouth and rear end, and other places on the surface of the cow.

The factor of 2π in the exponent is particularly convenient for normalizing this operator; it also gives useful physical units for the dual variable ω[20]

One should worry a bit about what kinds of function the Fourier transform is defined for. However, it is easy to note it is well-defined on smooth functions which decay rapidly at infinity, faster than the reciprocal of any polynomial (Schwartz class functions). By duality, the Fourier transform is extended to all tempered distributions, and thus it behaves nicely on a wide class of (generalized) functions.

The Fourier transform preserves L^2 norm; maps convolutions to pointwise products, and vice versa; it converts derivatives to mutliplications by polynomials in the dual variable, and vice versa.

In higher dimensions, the Fourier transform is defined using a higher dimensional integral; similar properties hold. Since we are working with the wave equation, it is convenient to indicate the space and time variables separately. Thus, for $u = u(\mathbf{x}, t)$ we define its transform as

$$\hat{u}(\mathbf{k}, \omega) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} u(\mathbf{x}, t) e^{-2\pi i(\mathbf{x}\cdot\mathbf{k} + t\cdot\omega)} \, d\mathbf{x} dt,$$

and of course the inverse transform gives

$$u(\mathbf{x}, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \hat{u}(\mathbf{k}, \omega) e^{+2\pi i(\mathbf{x}\cdot\mathbf{k} + t\cdot\omega)} \, d\mathbf{k} d\omega.$$

To solve the inhomogeneous wave equation $u_{tt} - c^2\nabla^2 u = h$, we simply apply the Fourier transform to the equation to obtain

$$-4\pi^2\omega^2\hat{u} + c^2 4\pi^2 |\mathbf{k}|^2 \hat{u} = \hat{h}$$

and thus

$$\hat{u} = \frac{1}{4\pi^2(c^2|\mathbf{k}|^2 - \omega^2)} \hat{h}.$$

We would then recover u via the inverse Fourier transform.

However, there are some problems with this approach, namely because the factor $4\pi^2(c^2|\mathbf{k}|^2 - \omega^2)$ is zero along the light cone, and dividing by zero is not well-defined. In particular, we can say the reciprocal $1/4\pi^2(c^2|\mathbf{k}|^2 - \omega^2)$ is not the Fourier transform of a tempered distribution.

To fix things, one can define distributions Φ^ϵ_+ and Φ^ϵ_- as those distributions with Fourier transform

$$\begin{aligned} \widehat{\Phi^\epsilon_+}(\mathbf{k}, \omega) &= \frac{1}{4\pi^2(c^2|\mathbf{k}|^2 - (\omega - i\epsilon)^2)} \\ \widehat{\Phi^\epsilon_-}(\mathbf{k}, \omega) &= \frac{1}{4\pi^2(c^2|\mathbf{k}|^2 - (\omega + i\epsilon)^2)} \end{aligned}$$

[20]Eg. if t is time measured in seconds, then ω is frequency measured in Hertz. What could be more natural?

and take the limits of these distributions as $\epsilon \to 0^+$. It can be verified that in fact, the retarded and advanced Green's functions (fundamental solutions) derived in the earlier section are these limits.[21]

It is curious that the addition of an infinitesimally small, imaginary part to the temporary frequency frequency switches solutions from the retarded to the advanced. This gives an idea of how bad the behaviour could be if one is not careful about those zeros.

A closely related solution is to the the limit of distributions Φ_f^ϵ, whose Fourier transform is

$$\widehat{\Phi_f^\epsilon}(\mathbf{k}, \omega) = \frac{1}{4\pi^2(c^2|\mathbf{k}|^2 - \omega^2 - i\epsilon)}.$$

The limiting distribution is called the causal Green's function, or the Feynmann propagator Φ_f.

3.7 Analyticity and avoiding zeros

There is another way to avoid the zeros that comes up in the polynomial $c^2|\mathbf{k}|^2 - \omega^2$; we mention this because it works well for all constant coefficient PDEs. The key idea is that when a function $f(t)$ has compact support, then its Fourier transform

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \cdot \omega} \, dt$$

extends to an analytic function for complex values of ω. This is also true in higher dimensions. Thus, by moving off the real line, we can avoid the zero set of any polynomials that arise.

For the wave equation, this is easy to do. In the dual variable for time, we look at values $\omega + i\epsilon$. These will never give a zero in the characteristic polynomial, and in fact $c^2|\mathbf{k}|^2 - (\omega + i\epsilon)^2$ is bounded away from zero. So, rather than integrating over the real line for ω, we integrate on a line in the complex plane, where $\mathrm{Im}(\omega) = \epsilon \neq 0$.

Thus, for any compactly supported forcing term h, we may define

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \int_{\mathrm{Im}(\omega)=\epsilon} \frac{\hat{h}(\mathbf{k}, \omega)}{4\pi^2(c^2|\mathbf{k}|^2 - \omega^2)} e^{2\pi(\mathbf{x}\cdot\mathbf{k} + t\cdot\omega)} \, d\omega d\mathbf{k},$$

where the inner integral is along a line in the complex plane which thus avoids zeros in the denominator. Applying the wave operator, we pick up the polynomial term that cancels the denominator, so

$$u_{tt} - c^2\nabla^2 u = \int_{\mathbb{R}^n} \int_{\mathrm{Im}(\omega)=\epsilon} \hat{h}(\mathbf{k}, \omega) e^{2\pi(\mathbf{x}\cdot\mathbf{k} + t\cdot\omega)} \, d\omega d\mathbf{k},$$

[21] This would be a good exercise.

By analyticity of the integrand, we can deform the integral along the line $\mathrm{Im}(\omega) = \epsilon$ to the real line (Cauchy's theorem in complex variables), and obtain the desired result, that $u_{tt} - c^2\nabla^2 u = h$.

This does not reveal the fundamental solution[22], but it is an effective solution technique for the inhomogenous wave equation. That is, it is mathematically exact, there are not approximations or wishful ignoring of singularities. It would seem these solutions are related to the distributions Φ_+^ϵ defined in the last section. They are close, but not the same thing.

It is certainly worth pointing out that the factors $e^{2\pi(\mathbf{x}\cdot\mathbf{k}+t\cdot\omega)}$ appear to be plane waves in variables $\mathbf{x}, t$, for any fixed $\mathbf{k}, \omega$. Thus this solution looks like a sum of plane waves. However, we really do have some exponential behaviour happening, since the ω is actually a complex number in this integral formulation.

3.8 Spatial Fourier transforms

In this section and the next, we let $\hat{u}$ denote the Fourier transform in the spatial variables alone, that is,

$$\hat{u}(\mathbf{k}, t) = \int_{\mathbb{R}^n} u(\mathbf{x}, t) e^{-2\pi i \mathbf{x}\cdot\mathbf{k}}\, d\mathbf{x}.$$

This transforms the initial value problem

$$u_{tt} - c^2\nabla^2 u = 0, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u_t(\mathbf{x}, 0) = g(\mathbf{x})$$

to a second order ODE in variable t, with

$$\hat{u}_{tt} + 4\pi^2 c^2 |\mathbf{k}|^2 \hat{u} = 0, \qquad \hat{u}(\mathbf{k}, 0) = \hat{f}(\mathbf{k}), \qquad \hat{u}_t(\mathbf{k}, t) = \hat{g}(\mathbf{k}).$$

Taking into account the initial conditions, we solve this ODE as a linear combination of cosines and sines, so

$$\hat{u}(\mathbf{k}, t) = \hat{f}(\mathbf{k}) \cos 2\pi c|\mathbf{k}|t + \hat{g}(\mathbf{k}) \frac{\sin 2\pi c|\mathbf{k}|t}{2\pi c|\mathbf{k}|}.$$

Thus, taking the inverse Fourier transforms, these products turn into convolutions, so we can write the solution as

$$u(\cdot, t) = f * \Psi_t + g * \Phi_t,$$

where we use a spatial convolution. Of course, we know what these distributions Ψ_t, Φ_t are, since we know their Fourier transforms[23] But, it is simpler to note that Φ_t is the same distribution that appeared in Section 3.3 on fundamental solutions, and

$$\Psi_t = \frac{\partial \Phi}{\partial t}$$

[22] the one that gives our convolutional results

[23] A good exercise would be to compute these distributions by finding the inverse Fourier transform of the functions $c(\mathbf{k}) = \cos 2\pi c|\mathbf{k}|t$ and $s(\mathbf{k}) = \dfrac{\sin 2\pi c|\mathbf{k}|t}{2\pi c|\mathbf{k}|}$.

is its time derivative.

3.9 Local Fourier theory: pseudodifferential operators

We saw that the spatial Fourier transform represents the Laplacian as multiplication by a polynomial in the Fourier dual variables $\mathbf{k} = (k_x, k_y, k_z)$, with,

$$\nabla^2 u(\mathbf{x}) = \int_{\mathbb{R}^3} -4\pi^2 \left(k_x^2 + k_y^2 + k_z^2\right) \hat{u}(\mathbf{k}) e^{2\pi i \mathbf{x}\cdot\mathbf{k}} \, d\mathbf{k}.$$

By the Fourier inversion formula, a linear differential operator with non-constant coefficients can be represented by a polynomial in $\mathbf{k}$ with coefficients that depend on the spatial variable $\mathbf{x}$. So, for instance, an inhomogeneous Laplacian can be calculated as

$$a(\mathbf{x})u_{xx} + b(\mathbf{x})u_{yy} + c(\mathbf{x})u_{zz} = \int_{\mathbb{R}^3} -4\pi^2 \left(a(\mathbf{x})k_x^2 + b(\mathbf{x})k_y^2 + c(\mathbf{x})k_z^2\right) \hat{u}(\mathbf{k}) e^{2\pi i \mathbf{x}\cdot\mathbf{k}} \, d\mathbf{k}.$$

A *pseudodifferential operator* is a linear operator given by an Fourier integral formula as above, but replacing a polynomial such as $p(\mathbf{x}, \mathbf{k}) = -4\pi^2 \left(a(\mathbf{x})k_x^2 + b(\mathbf{x})k_y^2 + c(\mathbf{x})k_z^2\right)$ with an arbitrary (but given) function $\sigma(\mathbf{x}, \mathbf{k})$. We can define the corresponding operator K_σ as

$$K_\sigma u(\mathbf{x}) = \int_{\mathbb{R}^3} \sigma(\mathbf{x}, \mathbf{k}) \hat{u}(\mathbf{k}) e^{2\pi i \mathbf{x}\cdot\mathbf{k}} \, d\mathbf{k}.$$

With reasonable restrictions on the smoothness and growth of the function σ, the corresponding operator K_σ retains many of the important properties of a real differential operators. The function σ is called the symbol of the operator. There is a well-developed mathematical theory which provides a functional calculus for combining symbols to create useful pseudodifferential operators. So for instance, the square root of the Laplacian can be approximated by a pseudodifferential operator whose symbol approximates the square root of the symbol for the Laplacian, such as

$$\sigma(\mathbf{x}, \mathbf{k}) = 2\pi i \sqrt{k_x^2 + k_y^2 + k_z^2 + \epsilon}.$$

In other words, this provides an approximation to the Dirac operator.

Pseudodifferential operators provide a useful tool for working with PDEs with non-constant coefficients, such as those describing an inhomogeneous earth. These will be described in other lectures of this workshop.

3.10 Radon transform

There is a very elegant way of solving the Cauchy problem for the wave equation using the Radon transform. The basic idea is to consider the simple case where the initial data f, g are

constant along (hyper)planes, with the same normal vector $\mathbf{k}$. That is, we have two function F, G of one variable and wish to solve the Cauchy problem

$$u_{tt} - c^2\nabla^2 u = 0, \qquad u(\mathbf{x}, 0) = F(\mathbf{x}\cdot\mathbf{k}), \qquad u_t(\mathbf{x}, 0) = G(\mathbf{x}\cdot\mathbf{k}).$$

We expect the solution to have the same symmetry, and in fact the solution is obtained as in the d'Alembert 1D solution, so we find

$$u(\mathbf{x}, t) = \frac{1}{2}[F(\mathbf{x}\cdot\mathbf{k} + t) + F(\mathbf{x}\cdot\mathbf{k} - ct)] + \frac{1}{2c}\int_{-ct}^{ct} G(\mathbf{x}\cdot\mathbf{k} + s)\, ds.$$

The more general solution, for more general Cauchy conditions, is to decompose the initial condition into a sum (or integral) of functions which are constant along various hyperplanes. Thus, suppose we can write the functions f, g as integrals

$$f(\mathbf{x}) = \int_{S_n} F(\mathbf{x}\cdot\mathbf{k}, \mathbf{k})\, d\sigma(\mathbf{k}), \qquad g(\mathbf{x}) = \int_{S_n} G(\mathbf{x}\cdot\mathbf{k}, \mathbf{k})\, d\sigma(\mathbf{k}).$$

Then the solution to the wave equation would be

$$u(\mathbf{x}, t) = \frac{1}{2}\int_{S_n} \left[F(\mathbf{x}\cdot\mathbf{k} + t, \mathbf{k}) + F(\mathbf{x}\cdot\mathbf{k} - ct, \mathbf{k}) + \frac{1}{c}\int_{-ct}^{ct} G(\mathbf{x}\cdot\mathbf{k} + s, \mathbf{k})\, ds\right] d\sigma(\mathbf{k}),$$

where we integrate over the unit sphere of all possible direction vectors $\mathbf{k}$.

That is all there is to it: once we know how to decompose the functions $f(\mathbf{x}), g(\mathbf{x})$ in terms of integrals of some functions $F(s, \mathbf{k}), G(s, \mathbf{k})$, we have the formula for the solution to the wave equation. So we present here how to obtain this decomposition using the Radon transform.

This decomposition is obtained by the modified Radon transform. First, the Radon transform is defined on spatial functions $f = f(\mathbf{x})$ by integrating the function over all possible (hyper)planes $\mathbf{x}\cdot\mathbf{k} = s$. Thus, for each direction $\mathbf{k}$ and each displacement s, we obtain a value for the transform, with

$$(Rf)(s, \mathbf{k}) = \int_{\mathbf{x}\cdot\mathbf{k}=s} f(\mathbf{x})\, d\mathbf{x},$$

where the integral is the usual $(n-1)$ dimensional area integral. It is interesting to note that the Radon transform is closely related to the spatial Fourier transform; indeed, it is easy to check that

$$Rf(s, \mathbf{k}) = \int_{\mathbb{R}} \hat{f}(\omega\mathbf{k}) e^{2\pi i s\cdot\omega}\, d\omega.$$

As you might expect, this transform is invertible, and the inverse can be written as an integral over points $\mathbf{k}$ in the sphere S_n and displacements $s \geq 0$. Integrating over the displacements, and taking care of some symmetries, gives the modified Radon transform $\tilde{R}$ which satisfies the decomposition

$$f(\mathbf{x}) = \int_{S_n} \tilde{R}f(\mathbf{x}\cdot\mathbf{k}, \mathbf{k})\, d\sigma(\mathbf{k}).$$

Thus it is the modified Radon transform that should be used in computing the solution to the wave equation.

In dimension 2, the modified Radon transform involves a Hilbert transform, so it should be computed with some care[24]. In dimension 3, we can just write it down

$$\tilde{R}(s, \mathbf{k}) = \frac{1}{2} \int_{-\infty}^{\infty} \widehat{Rf}(\omega, \mathbf{k}) \omega^2 e^{2\pi i s \omega} \, d\omega,$$

where $\widehat{Rf}$ here means the Fourier transform of the Radon transform $Rf(s, \mathbf{k})$ with respect to the first variable s.

3.11 Things we haven't covered

Given more time, it would have been useful for cover topics such as smoothness results, ray theory, and numerical solutions.

For a brief summary of each:

Smoothness: we can ask, given certain smoothness conditions on the input to a PDE problem, how smooth will the solution be? For elliptic operators, the answer is very nice: there is no loss of smoothness. That is, if the Cauchy data is $f \in C^k, g \in C^{k-1}$, then the solution u is also C^k. For the wave equation, there is a loss of smoothness, of about order $n/2$. So, for Cauchy data $f \in C^{k+n/2}$, and $g \in C^{k-1+n/2}$, then the solution is u is in C^k. This can be thought of as the result of weak singularities colliding into other weak singularities, to produce stronger singularities. For constant coefficient wave equations, this can only happen in a limited sense, hence the loss of only $n/2$ orders of derivatives. Moreover, this loss can only happen in startup of the solution: we can loss $n/2$ orders of smoothness in going from $u(\mathbf{x}, 0)$ to $u(\mathbf{x}, \epsilon)$, but after that there is no further loss moving on to any point $u(\mathbf{x}, t)$. And if one uses L^2 derivatives, there is no loss at all! A complete discussion of this would require the introduction of Sobolev spaces. And it is not clear (to me) that analogous results hold for non-constant coefficient wave equation, because the creation of caustics can produce very bad singularities.

Ray theory: Light travels in rays. Well, not really, but this is a useful approximation in geometric optics. A similar approximation can be made in seismic imaging, where one assumes most of the energy in a seismic experiment has travelled along a ray that follows the path of least time.[25] These paths can be computed by looking at the coefficients in the wave equation, in particular in the velocity term $c = c(\mathbf{x})$. In the constant velocity case, the ray paths are straight lines; for non-constant velocities, they are other curves. The problem

[24] Huygens principle fails in dimension 2, which is related to this problem.

[25] Or possibly an extremal for time.

of computing paths of fastest travel time is a standard problem in the calculus of variations, and not so much a problem in PDEs.

Numerical methods: There are many techniques for approximating the solution to a PDE using numerical methods. For non-constant coefficient equations, sometimes these are the only known methods to finding accurate approximations to the solutions. Finite differences, finite elements, matrix solvers, filtering methods, and so on, are all useful techniques. The seismic problem is difficult enough that the tradeoff between accuracy and speed becomes one of the most important issues in choosing a solution method.

4 Summary

Here are the key points to remember from these lectures.

First, there are many solutions to the wave equation. It is only by adding boundary and initial conditions that we turn a question about the wave equation into a well-posed mathematical problem.

Second, for the constant coefficient wave equation, there are many techniques for computing the solutions to (well-posed) problems. These included separation of variables, characteristic curves, variations of parameters, one-way wave equations, d'Alembert and higher dimensional explicit solutions, Green's functions, Fourier and Radon transforms. A technical facility with these techniques will help you in solving real problems about waves.

Finaly, for non-constant coefficient problems, we still expect many solutions to exist for the wave equation. Well-posedness will come from imposing boundary and initial conditions. Local solutions can always be found. We can optimistically expect that the many solutions to the constant coefficient case can be applied, at least locally, to give methods of solution to the more general, non-constant coefficient case.

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