Wave-equation migration: Theory

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SISS

Symbols

Symbol	Name	Units
$\mathbf{x} \Leftrightarrow (x_1, x_2, x_3)$	position vector (x_3 is depth)	m
$\mathbf{y} \Leftrightarrow (y_1, y_2, y_3)$	position vector (y_3 is depth)	m
t	time	s
ω	circular frequency	s^{-1}
$\mathbf{p} \Leftrightarrow (p_1, p_2, p_3)$	slowness vector (p_3 is vertical slowness)	s m^{-1}
ψ	P-wave scalar potential	
φ	spectrum of ψ	
A	amplitude	
$\mathbf{C} \Leftrightarrow C_{ijkl}$	elastic coefficients	N m^{-3}
$\sigma \Leftrightarrow \sigma_{ij}$	stress	N m^{-2}
$\mathbf{u} \Leftrightarrow (u_1, u_2, u_3)$	displacement vector	m
λ	Lamé parameter	N m^{-3}
v	P-wave velocity	m s^{-1}
ρ	density	kg m^{-3}
$\mathbf{W}$	extrapolation operator	
$\mathbf{R}$	reflection operator	
r	a single element of $\mathbf{R}$	
ϕ	angle measured from the normal to a reflector	rad / deg

Introduction

- The path between a source at depth $-x_3$, a boundary at depth 0, and a receiver at depth $-x_3$ may be represented as follows

Source → Down → Reflect → Up → Receive

- Symbolically, for each ω, the path can be written

$$\boxed{\psi_S|_{-x_3}} \longrightarrow \boxed{\mathbf{W}_{x_3}} \longrightarrow \boxed{\mathbf{R}_0} \longrightarrow \boxed{\mathbf{W}_{-x_3}} \longrightarrow \boxed{\psi_R|_{-x_3}}$$

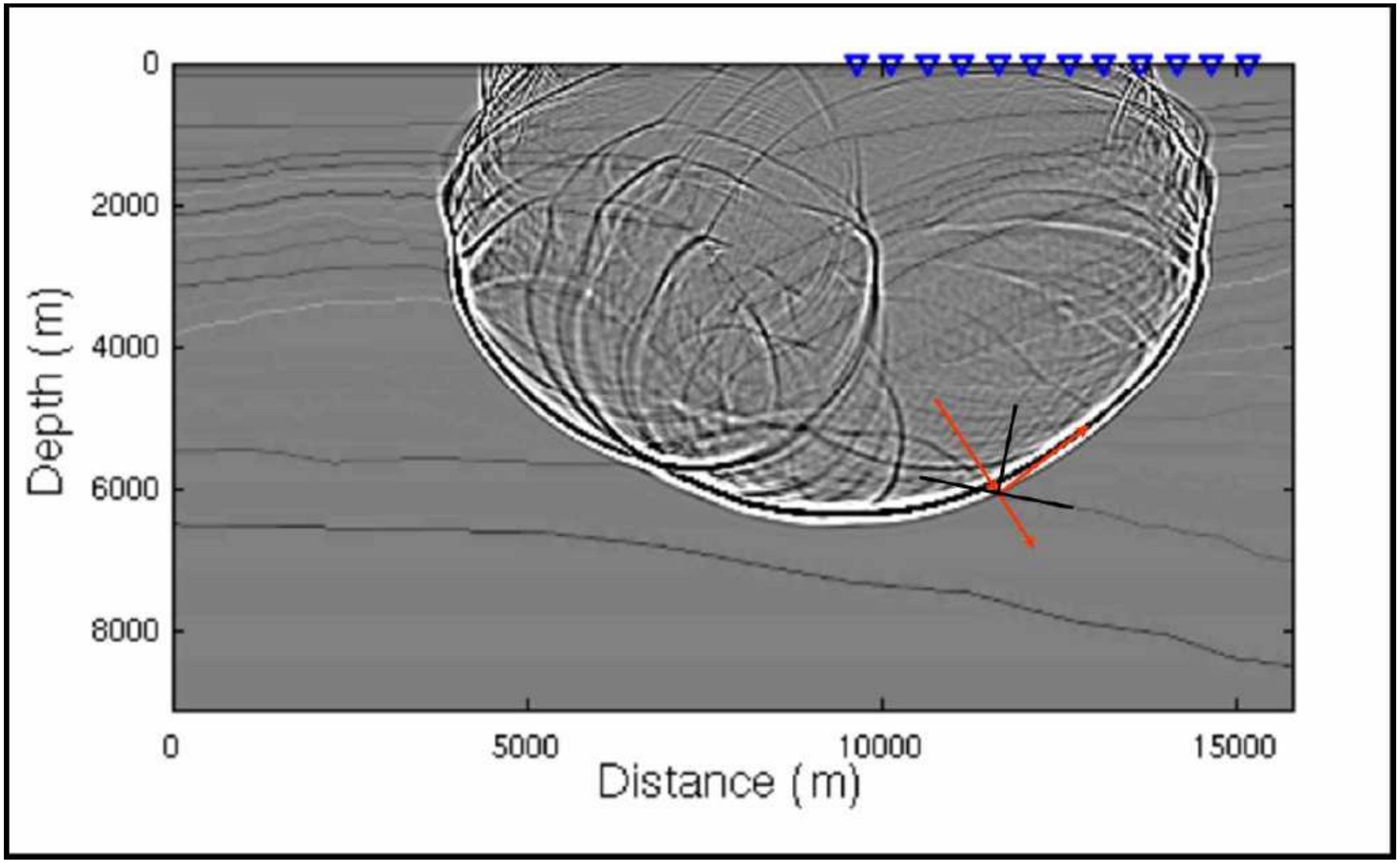


Figure 1: Snapshot of a propagating wavefield in an elastic medium. (Courtesy of L. Fishman)

- An elementary equation for modeling is

$$\psi_R|_{-x_3} = [\mathbf{W}_{-x_3}\,\mathbf{R}_0\,\mathbf{W}_{x_3}\,\psi_S|_{-x_3}]_{-x_3}$$

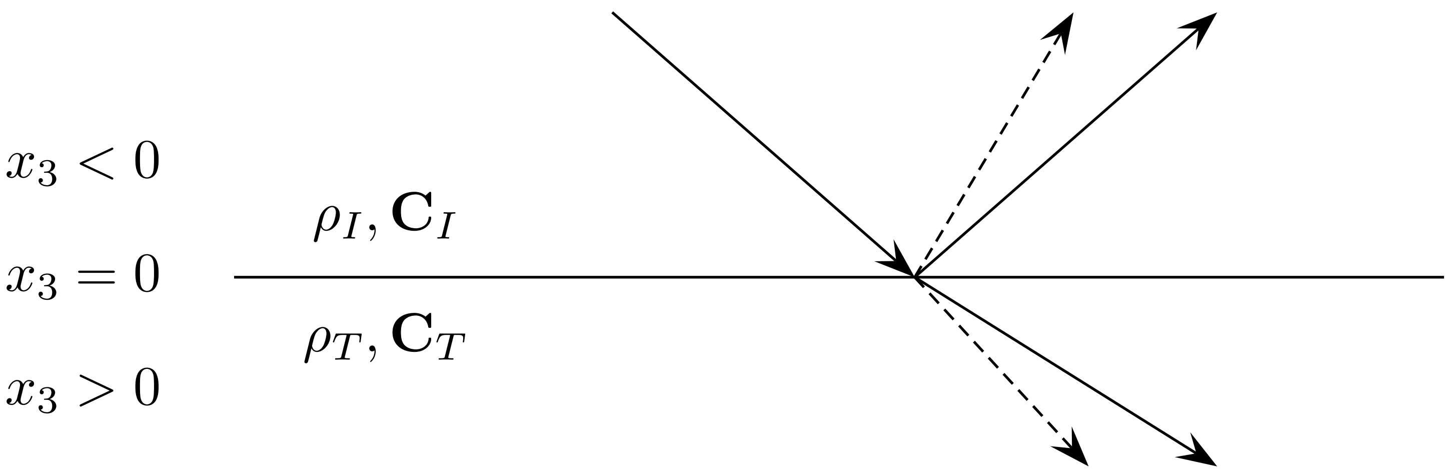
- An elementary equation for imaging is

$$\left[\mathbf{W}_{-x_3}^{-1}\,\psi_R|_{-x_3}\right]\,\left[\mathbf{W}_{x_3}\,\psi_S|_{-x_3}\right]^{-1} = \mathbf{R}_0$$

- In general, $\mathbf{R}$ and $\mathbf{W}$ are heterogeneous and anisotropic

Reflection operator $\mathbf{R_0}$

- At a boundary between elastic media, ...



... we have continuity equations:

1. Continuity of displacement $\mathbf{u} = (u_1, u_2, u_3)$

$$\left[\mathbf{u}^+ + \mathbf{u}^-\right]_I = \mathbf{u}_T^+$$

2. Continuity of stress σ

$$\left[\sigma^+ + \sigma^-\right]_I = \sigma_T^+$$

- For small deformations, $\mathbf{C}$ relates σ and $\mathbf{u}$ through

$$\sigma_{ij} = C_{ijkl}\,\frac{1}{2}\,\left[u_{k,l} + u_{l,k}\right]$$

- In terms of $\mathbf{u}$, continuity of σ becomes

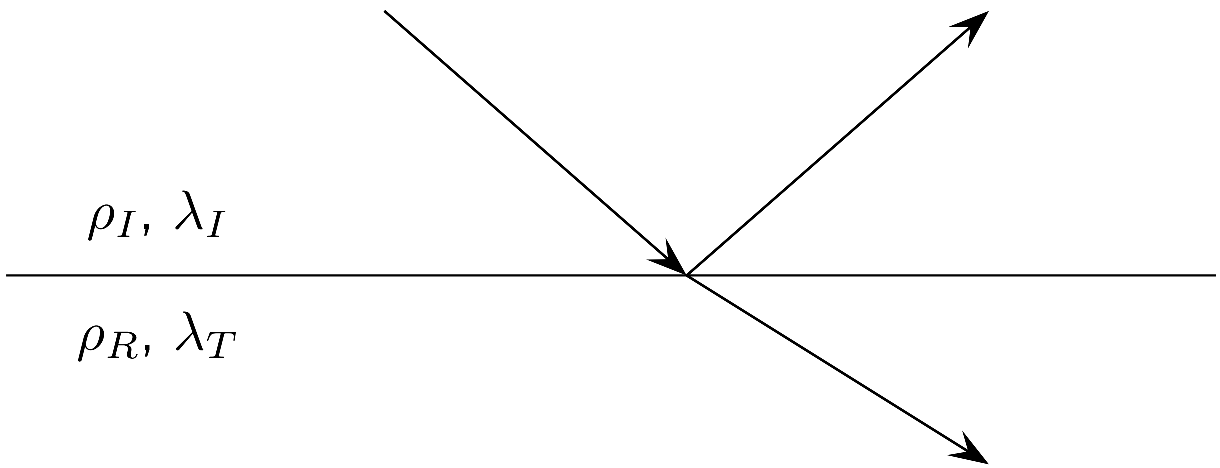
$$\left[C_{ijkl}\ \left[u^{+}_{k,l} + u^{+}_{l,k} + u^{-}_{k,l} + u^{-}_{l,k}\right]\right]_I$$
$$= \left[C_{ijkl}\ \left[u^{+}_{k,l} + u^{+}_{l,k}\right]\right]_T$$

- Given $\mathbf{u}^{+}_I$, $\mathbf{u}^{-}_I$, and $\mathbf{C}_I$ we can compute $\mathbf{C}_T$

- Practical realities make $\mathbf{C}_T$ estimation difficult
  - only $\left[u^{-}_3\right]_I$ (land), or pressure (sea) are recorded
  - measurements of $\mathbf{u}^{+}_I$ in the far field are rare
  - only a scalar estimate of $\mathbf{C}_I$ is obtained

- To gain insight, try a simpler model of the medium

- Consider, then, a boundary between fluid media



- Fluids don't support shear, so the continuity equations simplify

1. Continuity of displacement

$$\left[u_3^+ + u_3^-\right]_I = \left[u_3^+\right]_T$$

2. Continuity of stress

$$\left[\lambda\ \left[u^{+}_{3,3} + u^{-}_{3,3}\right]\right]_{I} = \left[\lambda\, u^{+}_{3,3}\right]_{T}$$

- In the Fourier domain

$$u^{\pm}_{3}\left(x_3, \omega\right) = \frac{1}{2\pi}\int \omega\, A\left(p_3, \omega\right) e^{\pm\, i\, \omega p_3\, x_3} dp_3$$

and

$$u^{\pm}_{3,3}\left(x_3, \omega\right) = \pm\frac{1}{2\pi}\int i\,\omega^{2}\, p_3\, A\left(p_3, \omega\right)\ e^{\pm\, i\, \omega p_3\, x_3} dp_3$$

- Then, for a boundary at $x_3 = 0$, the continuity equations become

1. Continuity of displacement

$$\left[A^{+} + A^{-}\right]_{I} = \left[A^{+}\right]_{T}$$

2. Continuity of stress

$$\left[\lambda\, p_3\ \left[A^+ - A^-\right]\right]_I = \left[-\lambda\, p_3\, A^+\right]_T$$

- Define $r = [A^-/A^+]_I$, and use the continuity equations to get

$$r = \frac{[\lambda\, p_3]_I - [\lambda\, p_3]_T}{[\lambda\, p_3]_I + [\lambda\, p_3]_T}$$

- For reflection of the plane wave defined by $p_2 = 0$, we have from the scalar wave-equation

$$\lambda\, p_3 = \frac{\lambda}{v}\sqrt{1 - \left(v\, p_1\right)^2} = \rho\, v\sqrt{1 - \left(v\, p_1\right)^2}$$

- So r in a fluid is depends on p according to

$$r = \frac{Z_T - Z_I}{Z_I + Z_T},$$

where

$$Z\left(p_1\right) = \rho\, v \sqrt{1 - \left(v\, p_1\right)^2}$$

- Reflectivity r is $angle\ dependent$

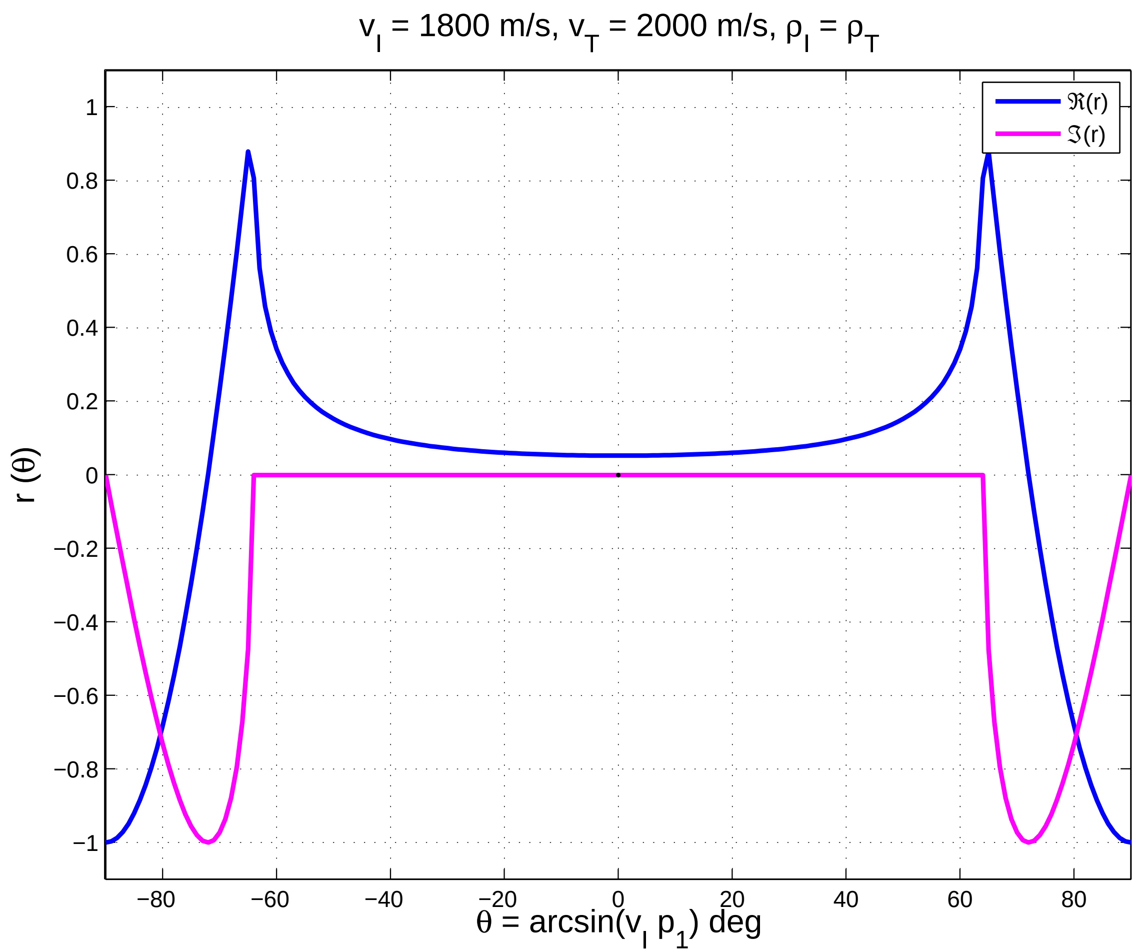


Figure 2: Acoustic reflectivity in angle coordinates.

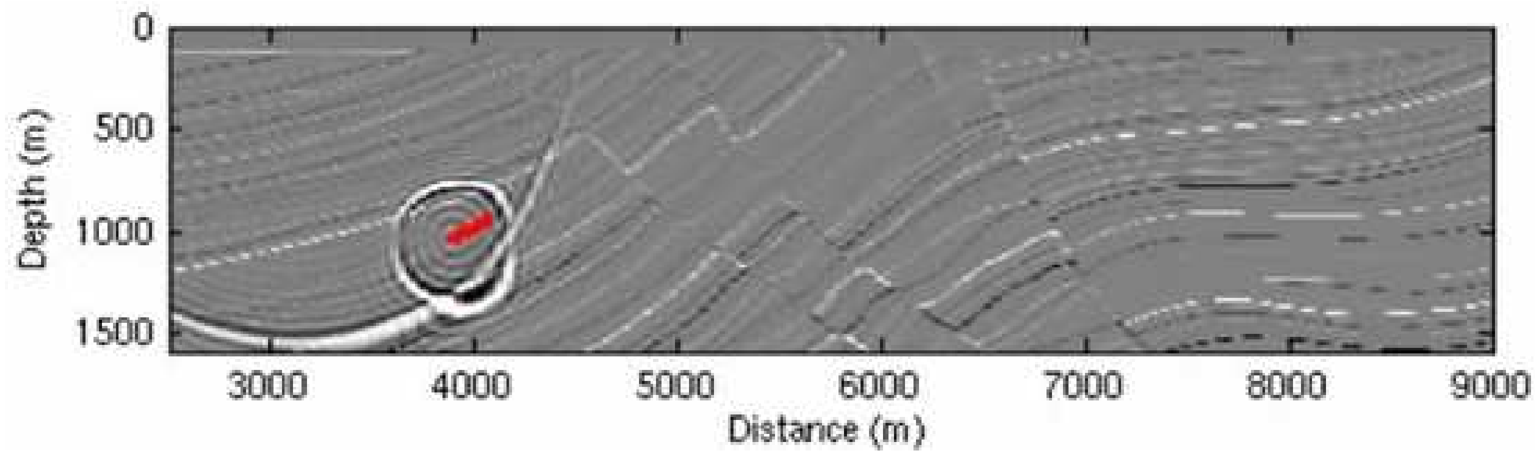


Figure 3: Close up of seismic reflection.

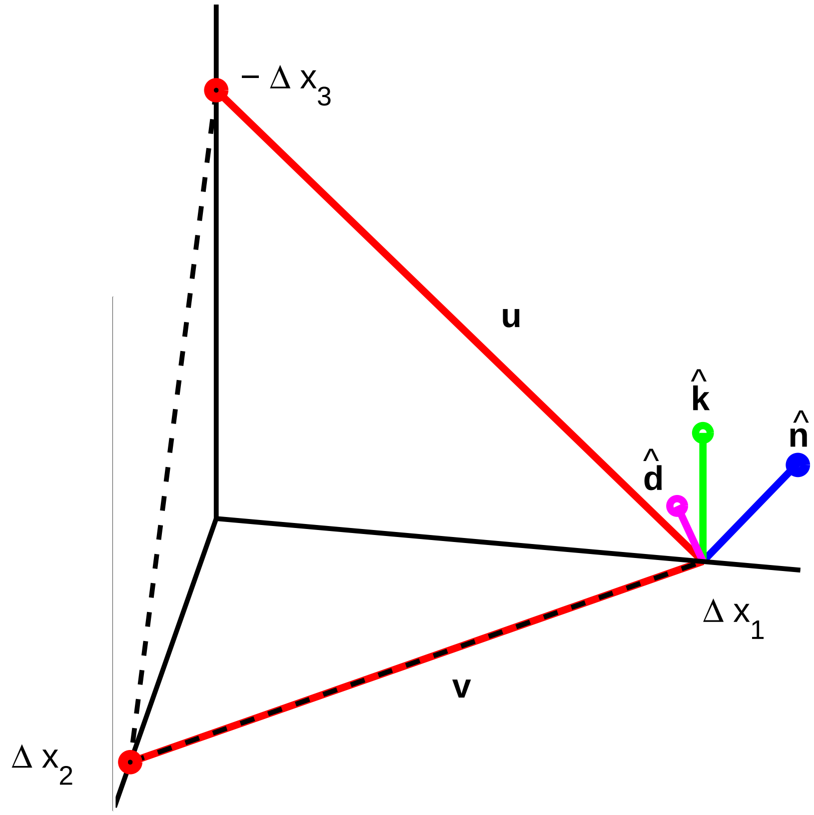


Figure 4: Unit vector $\hat{\mathbf{n}}$ is normal to a plane wave, $\hat{\mathbf{d}}$ is normal to a reflecting boundary, and $\hat{\mathbf{k}}$ is normal to the recording surface. Vectors $\mathbf{u}$ and $\mathbf{v}$ are in-the-plane of the plane wave.

Plane waves

- Ray parameters p_1, p_2, and p_3 define a plane wave in (x_1, x_2, x_3) where

$$p_3 = \frac{1}{v}\sqrt{1 - (v\,p_1)^2 - (v\,p_2)^2}$$

  - The equation for p_3 comes from $FT\left\{\nabla^2\psi + \left(\frac{\omega}{v}\right)^2\psi = 0\right\}$, where $\psi(\mathbf{x},\omega) = \frac{1}{(2\pi)^3}\int \omega\,\varphi(\mathbf{p},\omega)\,e^{i\,\omega\,[\mathbf{p}\cdot\mathbf{x}-t]}\,d\mathbf{p}$, and v is constant

- Given vectors $\mathbf{u}$ and $\mathbf{v}$ *in the plane* of the plane wave, normal $\hat{\mathbf{n}}_I$ to the plane wave is computed

$$\hat{\mathbf{n}}_I = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$$

- $\hat{\mathbf{n}}_I$ points in the direction of propagation of the incident plane-wave

- At a boundary, angle ϕ between $\hat{\mathbf{n}}_I$ and normal $\hat{\mathbf{d}}$ to the boundary provides wavenumber $p_{\hat{\mathbf{n}}_I}$ from which to compute $r\left(p_{\hat{\mathbf{n}}_I}\right)$ according to

$$p_{\hat{\mathbf{n}}_I} = \frac{\sin\phi}{v} = \frac{1}{v}\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|$$

- Given, $\mathbf{u} = \left(\Delta x_1\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} - \Delta x_3\,\hat{\mathbf{k}}\right)$, and $\mathbf{v} = \left(\Delta x_1\,\hat{\mathbf{i}} + \Delta x_2\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}\right)$ for example, $\mathbf{u} \times \mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \Delta x_3\,\Delta x_2\,\hat{\mathbf{i}} + \Delta x_3\,\Delta x_1\,\hat{\mathbf{j}} + \Delta x_1\,\Delta x_2\,\hat{\mathbf{k}}$$

- For plane waves, write travel time in terms of p_3

$$\Delta x_j = \frac{\Delta t}{p_j} = \frac{\Delta x_3\, p_3}{p_j},$$

and $\hat{\mathbf{u}} \times \hat{\mathbf{v}}$ becomes

$$\mathbf{u} \times \mathbf{v} = \Delta x_3^2\, p_3 \left[\frac{1}{p_2} \hat{\mathbf{i}} + \frac{1}{p_1} \hat{\mathbf{j}} + \frac{p_3}{p_1\, p_2} \hat{\mathbf{k}} \right] = \frac{\Delta x_3^2\, p_3}{p_1\, p_2} \left[p_1 \hat{\mathbf{i}} + p_2 \hat{\mathbf{j}} + p_3 \hat{\mathbf{k}} \right]$$

- Normal $\hat{\mathbf{n}}_I = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$ to the incident plane-wave is then computed as

$$\hat{\mathbf{n}}_I = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{p_1\, \hat{\mathbf{i}} + p_2\, \hat{\mathbf{j}} + p_3\, \hat{\mathbf{k}}}{\sqrt{p_1^2 + p_2^2 + p_3^2}}$$

– recall, $p_3 = \frac{1}{v}\sqrt{1 - (v\,p_1)^2 - (v\,p_2)^2}$, so $\hat{\mathbf{n}}_I \Rightarrow \hat{\mathbf{n}}_I\,(p_1, p_2)$

- Given incident unit-vector $\hat{\mathbf{n}}_I$ and normal to the boundary $\hat{\mathbf{d}}$, reflection coefficient $r\left(p_{\hat{\mathbf{n}}_I} = \frac{1}{v}\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|\right)$ may now be computed

- As an example, for a horizontal boundary, $\hat{\mathbf{d}} = \left(0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\right)$, and $\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}$ is computed as

$$\hat{\mathbf{n}}_I \times \hat{\mathbf{d}} = \hat{\mathbf{n}}_I \times \hat{\mathbf{k}} = \frac{p_2\,\hat{\mathbf{i}} + p_1\,\hat{\mathbf{j}}}{\sqrt{p_1^2 + p_2^2 + p_3^2}},$$

and effective wavenumber $p_{\hat{\mathbf{n}}_I}$ is

$$p_{\hat{\mathbf{n}}_I} = \frac{1}{v}\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{k}}\right| = \frac{1}{v}\sqrt{\frac{p_1^2 + p_2^2}{p_1^2 + p_2^2 + p_3^2}}$$

- Then, for a horizontal boundary in 2D, $p_2 = 0$, $p_3 = \frac{1}{v}\sqrt{1 - (v\,p_1)^2}$, and $p_{\hat{\mathbf{n}}_I} \Rightarrow p_1$

$$p_{\hat{\mathbf{n}}_I}\Big|_{p_2=0} = \frac{1}{v}\,\frac{p_1}{\sqrt{p_1^2 + p_3^2}} = \frac{1}{v}\,\frac{p_1}{\sqrt{1/v^2}} = p_1,$$

as expected

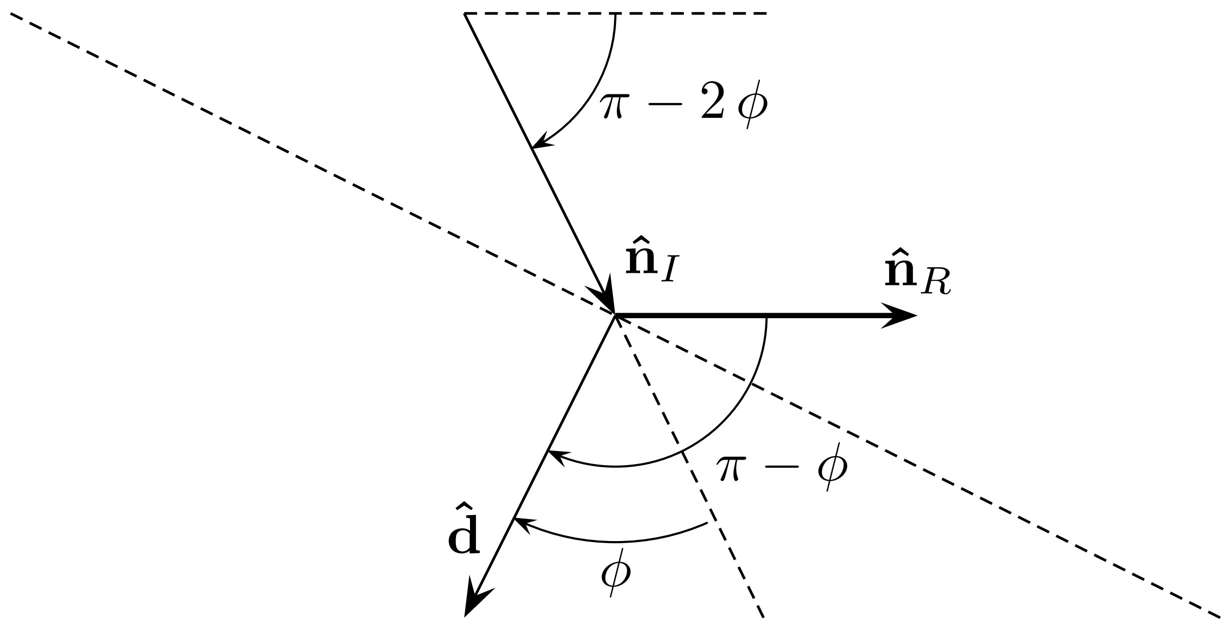


Figure 5: A model of reflection from a dipping boundary.

Plane-wave reflection

- Following reflection, $\hat{\mathbf{n}}_I$ and $\hat{\mathbf{d}}$ are related to reflected plane-wave $\hat{\mathbf{n}}_R = \left(n_{R1}\,\hat{\mathbf{i}} + n_{R2}\,\hat{\mathbf{j}} + n_{R3}\,\hat{\mathbf{k}}\right)$ through a unit-vector $\hat{\mathbf{a}}$
  - $\hat{\mathbf{a}}$ is normal to the plane containing $\hat{\mathbf{n}}_I$, $\hat{\mathbf{d}}$, and $\hat{\mathbf{n}}_R$
- From $\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}$ and $\sin\phi = \left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|$ we have

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}}{\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|}$$

- Trig' identity $\sin(\pi - \phi) = \sin\phi = \left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|$, and $\hat{\mathbf{n}}_R \times \hat{\mathbf{d}}$ give

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{n}}_R \times \hat{\mathbf{d}}}{\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|}$$

- From $\sin(\pi - 2\,\phi) = \sin(2\,\phi) = 2\,\sin\phi\,\cos\phi = 2\,\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|\,\hat{\mathbf{n}}_I \cdot \hat{\mathbf{d}}$, and $\hat{\mathbf{n}}_R \times \hat{\mathbf{n}}_I$ we have

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{n}}_R \times \hat{\mathbf{n}}_{\mathbf{I}}}{2\,\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|\,\hat{\mathbf{n}}_I \cdot \hat{\mathbf{d}}}$$

- Three equations for $\hat{\mathbf{a}}$ allow computation of (n_{R1}, n_{R2}, n_{R3})
  - we must solve a system of equations

- Once, (n_{R1}, n_{R2}, n_{R3}) are known, ray parameters (p_{R1}, p_{R2}) of the reflected wavefield are then calculated according to

$$\hat{\mathbf{n}}_R = n_{R1}\hat{\mathbf{i}} + n_{R2}\hat{\mathbf{j}} + n_{R3}\hat{\mathbf{k}} = \frac{p_{R1}\,\hat{\mathbf{i}} + p_{R2}\,\hat{\mathbf{j}} + p_{R3}\,\hat{\mathbf{k}}}{\sqrt{p_{R1}^2 + p_{R2}^2 + p_{R3}^2}}$$

where $p_{R3} = \frac{1}{v}\sqrt{1 - (v\,p_{R1})^2 - (v\,p_{R2})^2}$

- For example, when $\hat{\mathbf{d}} = \hat{\mathbf{k}}$, we have

$$\frac{\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}}{\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|} = \frac{n_{I2}\,\hat{\mathbf{i}} + n_{I1}\,\hat{\mathbf{j}}}{\sqrt{n_{I1}^2 + n_{I2}^2}},$$

and

$$\frac{\hat{\mathbf{n}}_R \times \hat{\mathbf{d}}}{\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|} = \frac{n_{R2}\,\hat{\mathbf{i}} + n_{R1}\,\hat{\mathbf{j}}}{\sqrt{n_{I1}^2 + n_{I2}^2}},$$

so that $n_{R1} = n_{I1}$ and $n_{R2} = n_{I2}$

- Further, to compute n_{R3}, we have

$$\frac{\hat{\mathbf{n}}_R \times \hat{\mathbf{n}}_I}{2\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right| \; \hat{\mathbf{n}}_I \cdot \hat{\mathbf{d}}} = \frac{1}{2\,n_{I3}\sqrt{n_{I1}^2 + n_{I2}^2}} \left[\begin{array}{c} (n_{R2}\,n_{I3} - n_{R3}\,n_{I2})\;\hat{\mathbf{i}} \\ (n_{R1}\,n_{I3} - n_{R3}\,n_{I1})\;\hat{\mathbf{j}} \\ (n_{R1}\,n_{I2} - n_{R2}\,n_{I1})\;\hat{\mathbf{k}} \end{array} \right]^T,$$

where, from the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components we have

$$n_{R3} = -n_{I3},$$

and so,

$$\hat{\mathbf{n}}_R = n_{I1}\hat{\mathbf{i}} + n_{I2}\hat{\mathbf{j}} - n_{I3}\hat{\mathbf{k}} = \frac{p_1\,\hat{\mathbf{i}} + p_2\,\hat{\mathbf{j}} - p_3\,\hat{\mathbf{k}}}{\sqrt{p_1^2 + p_2^2 + p_3^2}},$$

where $p_3 = \frac{1}{v}\sqrt{1 - (v\,p_1)^2 - (v\,p_2)^2}$

- As a check, for $\hat{\mathbf{d}} = \hat{\mathbf{k}}$, $\hat{\mathbf{n}}_R \cdot \hat{\mathbf{k}} = \cos\theta_R = -p_3/\sqrt{p_1^2 + p_2^2 + p_3^2}$, and $\hat{\mathbf{n}}_I \cdot \hat{\mathbf{k}} = \cos\theta_I = p_3/\sqrt{p_1^2 + p_2^2 + p_3^2}$, and $|\theta_R| = |\theta_I|$ as expected

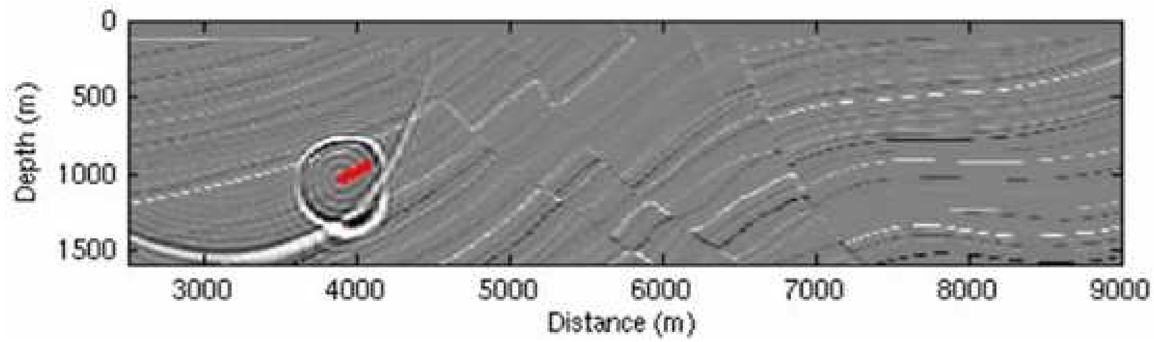


Figure 6: Non-specular reflection.

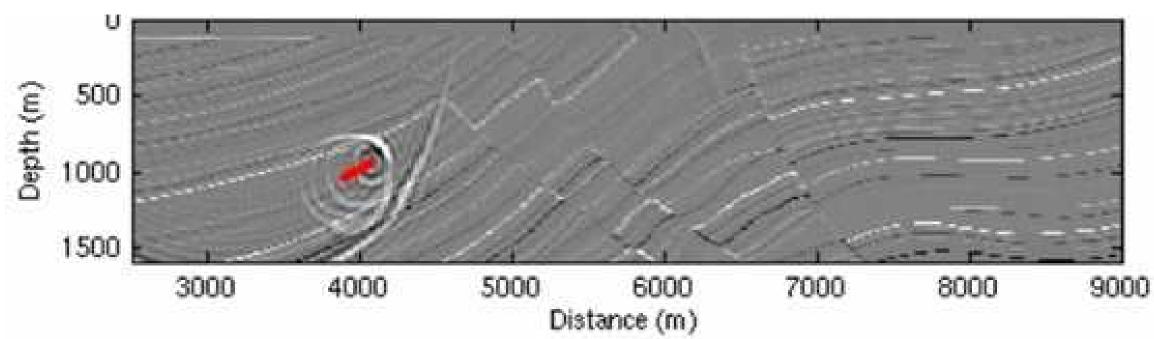


Figure 7: Specular(ish) reflection.

A model of the reflected wavefield

- A $model$ of reflected wavefield φ_R is computed as

$$\varphi_R\left(\mathbf{p}_R\right) = r\left(\mathbf{p}_R, \mathbf{p}\right)\, \varphi_I\left(\mathbf{p}\right),$$

or, with coordinates $\mathbf{p}_R = (p_{R1}, p_{R2})$, and $\mathbf{p} = (p_1, p_2)$ written explicitly

$$\varphi_R\left(p_{R1}, p_{R2}\right) = r\left(p_{R1}, p_{R2}, p_1, p_2\right)\, \varphi_I\left(p_1, p_2\right)$$

- If $\hat{\mathbf{d}}$ is unknown, we allow the possibility thea incident plane-wave

$\varphi_I\left(p_1, p_2\right)$ reflects in all directions (scatters)

$$\left[\begin{array}{c} \varphi_R\left(-p_N\right) \\ \vdots \\ \varphi_R\left(0\right) \\ \vdots \\ \varphi_R\left(p_N\right) \end{array}\right]_{p_{R2}} = \left[\begin{array}{c} r\left(-p_N, p_1, p_2\right) \\ \vdots \\ r\left(0, p_1, p_2\right) \\ \vdots \\ r\left(p_N, p_1, p_2\right) \end{array}\right]_{p_{R2}} \varphi_I\left(p_1, p_2\right)$$

where $\left(-p_N \leq p_{R1} \leq p_N\right)$, $p_N = \frac{\pi}{\Delta x\, \omega}$ (Nyquist ray-parameter), and we consider a single p_{R2} for simplicity

- Recognize that, for specular reflection, only one (unknown) combination of $\mathbf{p}_R$ and $\mathbf{p}$ results in non-zero r

- For each $\mathbf{p}_R$, then, sum up all $r\,\varphi_I$ for all incident $\mathbf{p}$ according to

$$\varphi_R\left(p_{R1},p_{R2}\right)=\begin{bmatrix} r\left(p_{R1},p_{R2},-p_N\right) \\ \vdots \\ r\left(p_{R1},p_{R2},0\right) \\ \vdots \\ r\left(p_{R1},p_{R2},p_N\right) \end{bmatrix}_{p_2}^{T} \begin{bmatrix} \varphi_I\left(-p_N\right) \\ \vdots \\ \varphi_I\left(0\right) \\ \vdots \\ \varphi_I\left(p_N\right) \end{bmatrix}_{p_2},$$

where $(-p_N \leq p_1 \leq p_N)$ and we consider a single p_2 for simplicity

- We may consider, then, all combinations of φ_I and φ_R according to

$$\vec{\varphi}_R=\mathbf{R}\,\vec{\varphi}_I,$$

where

$$\vec{\varphi}_R=\left[\varphi_R\left(-p_N\right),\cdots,\varphi_R\left(0\right)\cdots,\varphi_R\left(p_N\right)\right]_{p_{R2}}^{T}$$

and

$$\vec{\varphi}_I = \left[\varphi_I\left(-p_N\right), \cdots, \varphi_I\left(0\right) \cdots, \varphi_I\left(p_N\right)\right]_{p2}^T$$

- Reflectivity $r \rightarrow \mathbf{R}$ is now a matrix

$$\mathbf{R} = \left[\begin{array}{ccccc} r\left(-p_N, -p_N\right) & \cdots & r\left(-p_N, 0\right) & \cdots & r\left(-p_N, p_N\right) \\ \vdots & \ddots & \vdots & & \vdots \\ r\left(0, -p_N\right) & \cdots & r\left(0, 0\right) & \cdots & r\left(0, p_N\right) \\ \vdots & & \cdots & \ddots & \vdots \\ r\left(p_N, -p_N\right) & \cdots & r\left(p_N, 0\right) & \cdots & r\left(p_N, p_N\right) \end{array}\right]_{(p_{R2}, p_2)}$$

- Further we may consider M incident plane-waves and M reflected plane-waves simultaneously according to

$$\vec{\vec{\varphi}}_R = \mathbf{R}\,\vec{\vec{\varphi}}_I,$$

where

$$\vec{\vec{\varphi}}_R = [\vec{\varphi}_1, \cdots, \vec{\varphi}_M]_R\,,$$

and

$$\vec{\vec{\varphi}}_I = [\vec{\varphi}_1, \cdots, \vec{\varphi}_M]_I$$

- Then, to determine the complete reflected-wavefield, compute φ_R for all combinations of p_2 and p_{R2}
- Given $\mathbf{R}$, and using the above model, all specular reflections are computed automatically for all incident plane-waves

Extrapolation operator $\mathbf{W}$

- From the phase-shift theorem, spectrum φ_0 of wavefield ψ at boundary $x_3 = 0$ is computed from $\varphi_{\pm\Delta x_3}$ according to

$$\varphi\left(p_1, p_2, \omega\right)\big|_{x_3=0} = \varphi\left(p_1, p_2, \omega\right)\big|_{x_3=\pm\Delta x_3} e^{\mp\, i\omega\, p_3\, \Delta x_3},$$

where

$$\varphi\left(p_1, p_2, \omega\right)\big|_{x_3=u} = \frac{1}{2\pi}\int \psi\left(\mathbf{x}, t\right) e^{i\omega\,[p_1\, x_1 + p_2\, x_2 - t]}\, \delta\left(x_3 - u\right) d\mathbf{x}\, dt$$

- Wavefield $\psi_{x_3=0}$ is then computed from $\varphi_{x_3=0}$ by inverse transform $(p_1 \rightarrow x_1, p_2 \rightarrow x_2, \omega \rightarrow t)$

- For a single frequency ω, then, we have

$$\psi\left(x_1, x_2\right)\Big|_{x_3=0} = \int \psi\left(y_1, y_2\right)\Big|_{x_3=\pm\Delta x_3} W\left(x_1, x_2, y_1, y_2\right)\Big|_{\mp\Delta x_3} dy_1\, dy_2$$

where, (y_1, y_2) are space coordinates of the wavefield at depth $x_3 = \pm\Delta x_3$ and,

$$\begin{aligned} \mathbf{W} \Leftrightarrow\ & W\left(x_1, x_2, y_1, y_2, \omega\right)\Big|_{\mp\Delta x_3} \\ = & \frac{1}{\left(2\pi\right)^2} \int \omega^2 e^{-i\omega p_1[x_1 - y_1]}\, e^{-i\omega p_2[x_2 - y_2]}\, e^{\mp\, i\omega p_3 \Delta x_3} dp_1\, dp_2 \end{aligned}$$

- Of course, $p_3 \Leftrightarrow p_3\left(p_1, p_2, v\right)$

- If $v \Leftrightarrow v(\mathbf{x})$, then $p_3 \Leftrightarrow p_3(\mathbf{x}, p_1, p_2, \omega)$ and

$$\psi(x_1, x_2)|_{x_3=0} \approx \int \psi(y_1, y_2)|_{x_3=\pm\Delta x_3} \; W(x_1, x_2, y_1, y_2)|_{\mp\Delta x_3} \, dy_1 \, dy_2$$

- In matrix-vector format for constant-velocity media

$$\vec{\vec{\psi}}_0 = \mathbf{W}_{\mp\Delta x_3} \vec{\vec{\psi}}_{\pm\Delta x_3},$$

and for variable-velocity media

$$\vec{\vec{\psi}}_0 \approx \mathbf{W}_{\mp\Delta x_3} \vec{\vec{\psi}}_{\pm\Delta x_3}$$

Summary

- Reflectivity $r\left(p\right)$ is derived, commonly, for horizontal reflectors
- Modification $r\left(p\right) \Rightarrow r\left(p_{\hat{\mathbf{n}}}\left(\mathbf{p}\right)\right)$ permits use of derived r for 3D, dipping boundaries
- When dip is known, the direction of reflected plane-waves $\mathbf{p}_R\left(\mathbf{p}\right)$ is deduced
- When dip is not known, $r \Rightarrow \mathbf{R}$, and specular reflection corresponds to non-zero elements
- $\mathbf{W}_{\pm\Delta x_3}$ are related closely to Fourier integrals - exact in constant-velocity media, approximate in variable-velocity media

Wave-equation migration: practice

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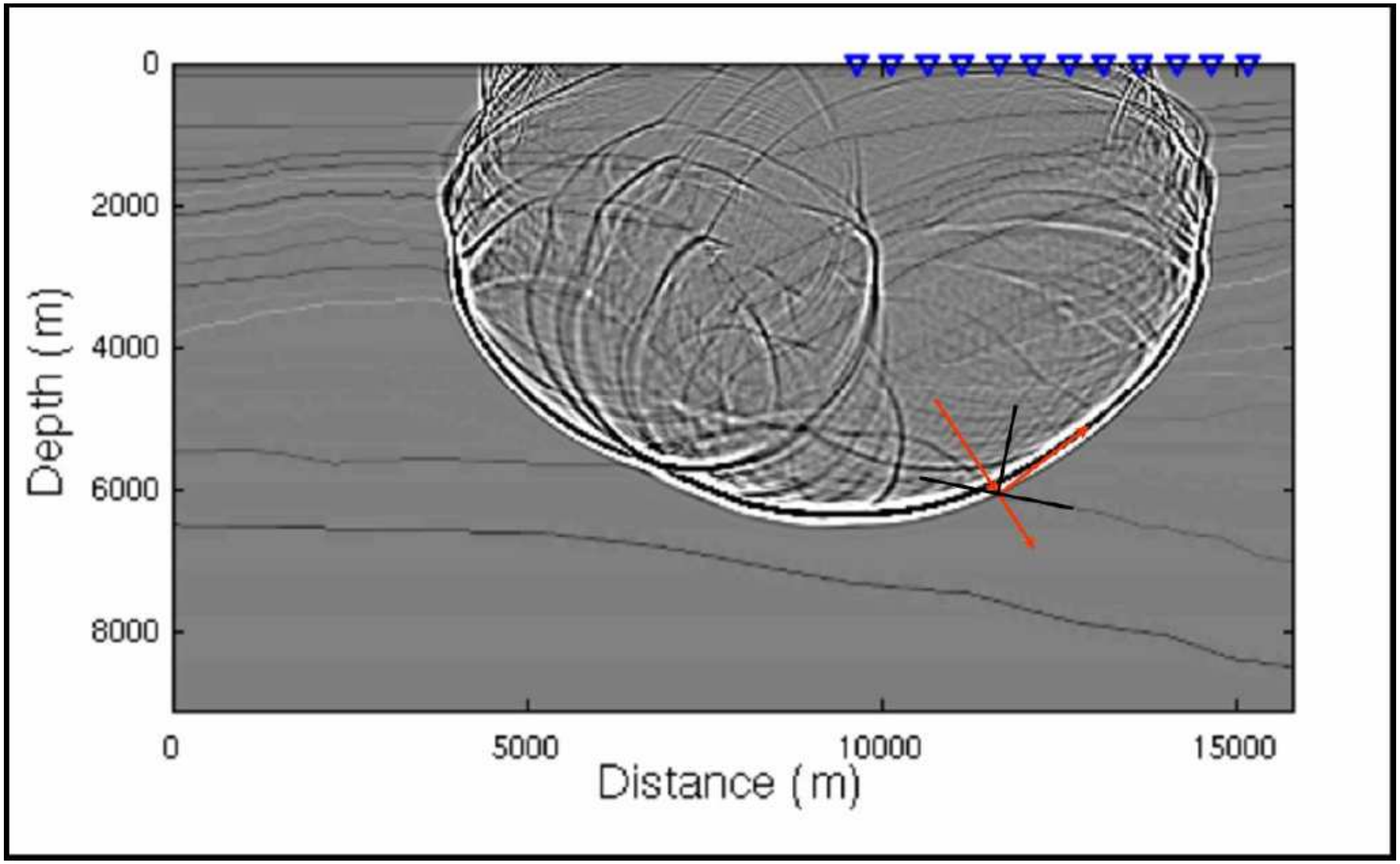

Figure 1: Snapshot of a propagating wavefield in an elastic medium. (Courtesy of L. Fishman)

From last WE class

- From a simple model of reflection

$$\boxed{\psi_I|_{-x_3}} \rightarrow \boxed{\mathbf{W}_{x_3}} \rightarrow \boxed{\mathbf{R}_0} \rightarrow \boxed{\mathbf{W}_{-x_3}} \rightarrow \boxed{\psi_R|_{-x_3}}.$$

  we arrive at a simple model of imaging

$$\left[\mathbf{W}_{-x_3}^{-1}\,\psi_R|_{-x_3}\right]\,\left[\mathbf{W}_{x_3}\,\psi_S|_{x_3}\right]^{-1} = \mathbf{R}_0$$

- What is $\mathbf{R}$?

– Compute acoustic reflectivity $r\left(p_{\hat{\mathbf{n}}_I}\right)$ for dipping boundaries according to incident plane-wave $\left(p_1, p_2, \frac{1}{v}\sqrt{1-(v\,p_1)^2-(v\,p_2)^2}\right)$

$$p_{\hat{\mathbf{n}}_I} = \frac{1}{v}\left|\hat{\mathbf{n}}_I \times \hat{\mathbf{d}}\right|,$$

where $\hat{\mathbf{d}}$ is normal to the boundary, and

$$\hat{\mathbf{n}}_I = \frac{p_1\,\hat{\mathbf{i}} + p_2\,\hat{\mathbf{j}} + p_3\,\hat{\mathbf{k}}}{\sqrt{p_1^2+p_2^2+p_3^2}}$$

– Compute reflected spectrum $\varphi\left(\mathbf{p}_R\right)$ when $\hat{\mathbf{d}}$ is known

$$\varphi_R\left(\mathbf{p}_R\right) = r\left(\mathbf{p}_R, \mathbf{p}\right)\,\varphi_I\left(\mathbf{p}\right),$$

and when $\hat{\mathbf{d}}$ is unknown, compute as

$$\left[\vec{\vec{\varphi}}_R\right]_{p_{R2}} = \mathbf{R}\left[\vec{\vec{\varphi}}_I\right]_{p_2},$$

for all source spectra and reflected spectra

- Matrix $\mathbf{R}$ corresponds to fixed values of p_2 and p_{R2}, and non-zero elements correspond to specular reflection - it converts $\varphi_I(\mathbf{p})$ to $\varphi_R(\mathbf{p_R})$ at the boundary

- All reflected wavefields may then be modeled by looping over p_2 and p_{R2}, followed by inverse transform $\varphi(p_{R1}, p_{R2}, \omega) \Rightarrow \psi(x_1, x_2, t)$

- What is $\mathbf{W}$?

– Extrapolation operator $\mathbf{W}$ works in heterogeneous media according to

$$\vec{\vec{\psi}}_0 \approx \mathbf{W}_{\mp\Delta x_3} \vec{\vec{\psi}}_{\pm\Delta x_3},$$

where

$$\mathbf{W} \Leftrightarrow \left. W\left(x_1, x_2, y_1, y_2, \omega\right)\right|_{\mp\Delta x_3}$$
$$= \frac{1}{\left(2\pi\right)^2} \int \omega^2 e^{-i\omega p_1 [x_1 - y_1]}\, e^{-i\omega p_2 [x_2 - y_2]}\, e^{\mp\, i\omega p_3 \Delta x_3} dp_1\, dp_2,$$

and $p_3 \Rightarrow p_3\left(v\left(\mathbf{x}\right)\right)$

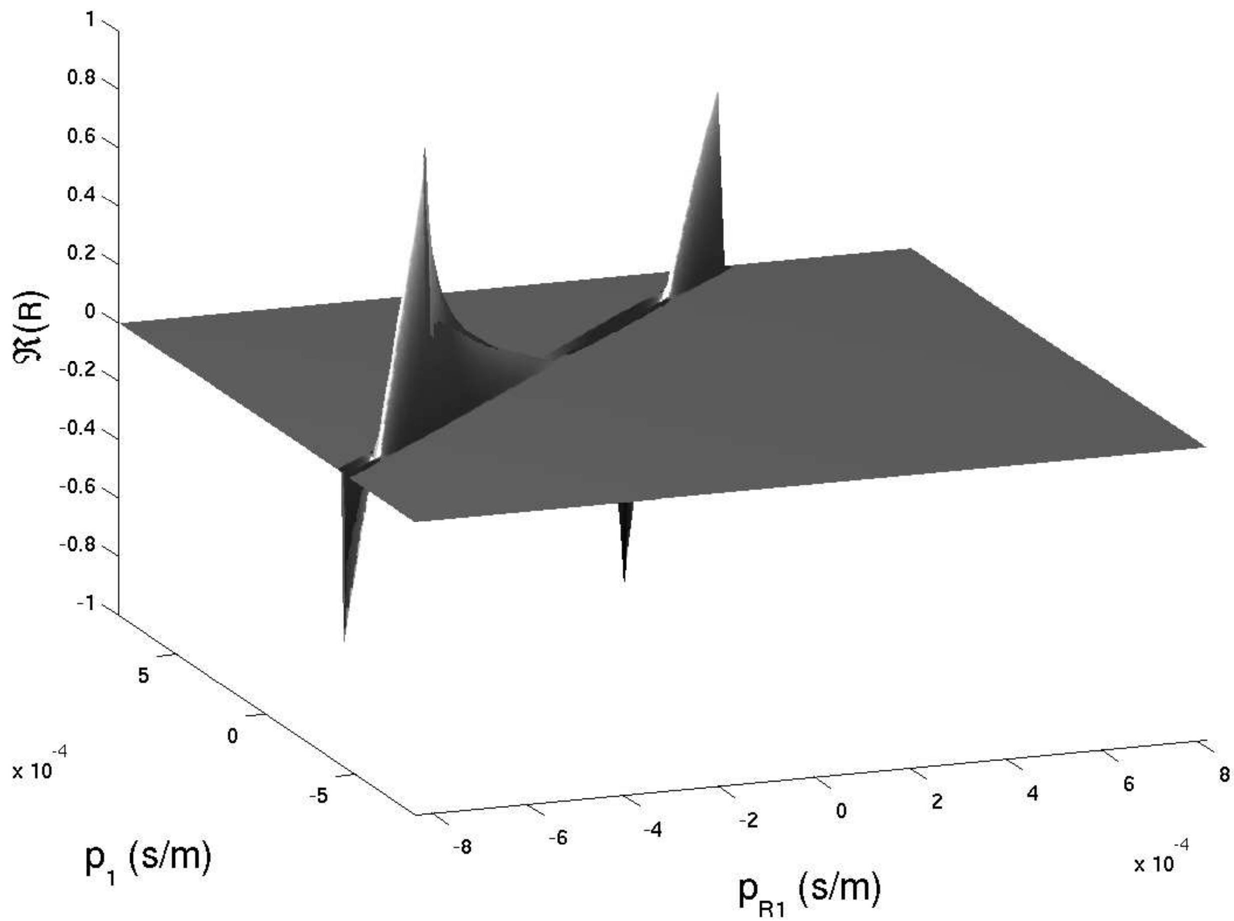

Figure 2: $\mathbf{R}$ for a boundary with dip $\hat{\mathbf{d}}$.

Practical reflection

- When the incident and reflected wavefields $\left[\vec{\vec{\varphi}}_I\right]_{p_2}$ and $\left[\vec{\vec{\varphi}}_R\right]_{p_{R2}}$ are known, $\mathbf{R}$ is estimated by

$$[\mathbf{R}]_{p_{R2},p_2} = \left[\vec{\vec{\varphi}}_R\right]_{p_{R2}} \left[\vec{\vec{\varphi}}_I\right]_{p2}^{-1}$$

- For $\left[\vec{\vec{\varphi}}_I\right]_{p_2}^{-1}$ to exist $\vec{\vec{\varphi}}_I$ must be square and have a non-zero determinant
  - for square $\vec{\vec{\varphi}}_I$, the numbers of shots and receivers is the same, and the spacing is equal - when this is not so, damped least-squares or conjugate gradients can be used

- The result is large matrices and a huge computational cost to resolve each $\mathbf{R}(\mathbf{x})$ in the subsurface
  - for example, inversion of $\vec{\vec{\varphi}}_I$ followed by multiplication by $\vec{\vec{\varphi}}_R$ requires 100's Gflops (estimated for 1000 shots and 1000 receivers) - this is the innermost calculation
  - the innermost calculation lies within three loops: frequency, and the two slownesses p_{R2} and p_2

⋮

```
for w1 to wN
   for p1 to pN
      for pR1 to pRN
          ...
          100's of Gflops calculation
          ...
       end
    end
 end
```

⋮

Figure 3: For each $\mathbf{x}$ in the subsurface, a very expensive calculation lies within 3 loops.

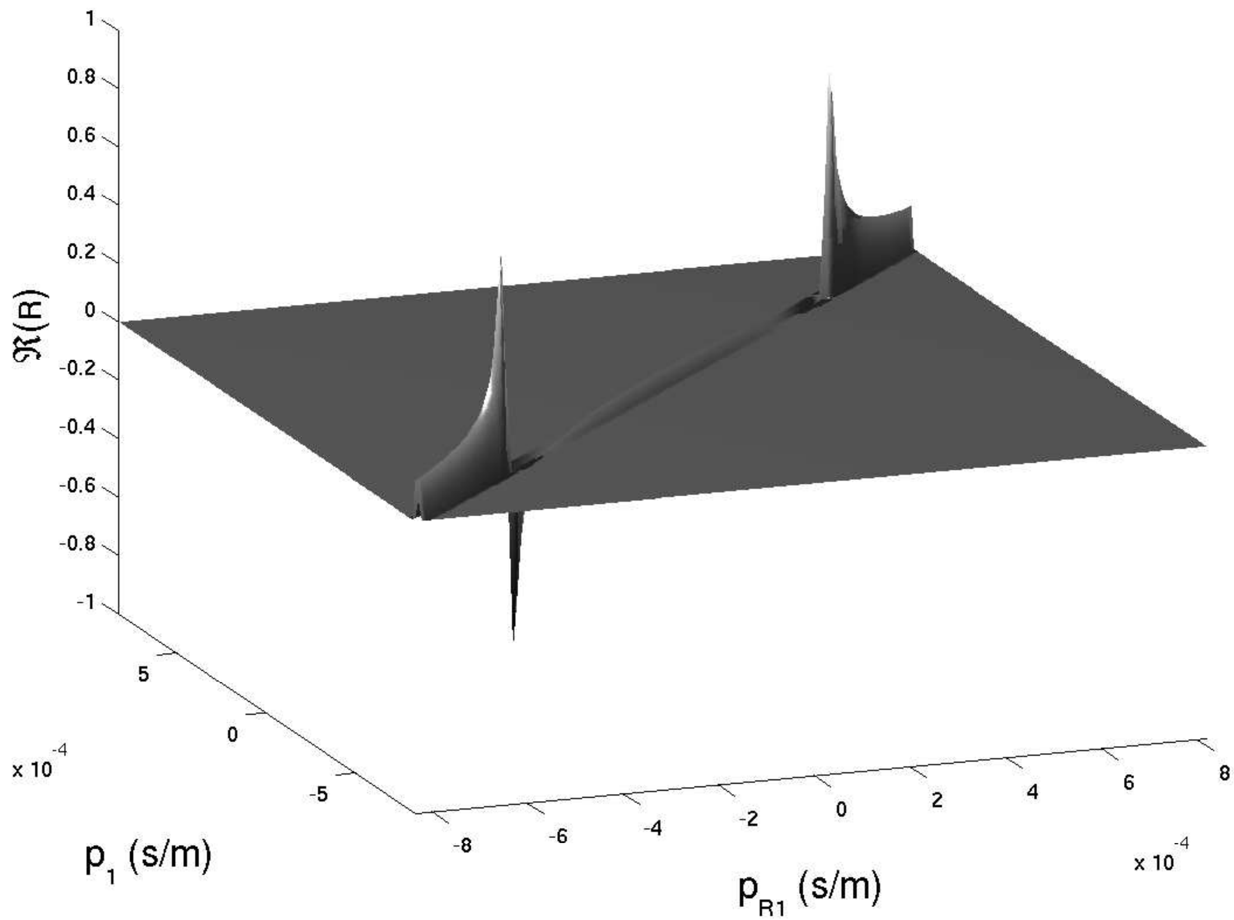

Figure 4: $\mathbf{R}_{p_{R2}=p_2=0}$ for a boundary with dip $\hat{\mathbf{d}} = \hat{\mathbf{k}}$.

- If we know that $\hat{\mathbf{d}}$ for the boundary is the normal $\hat{\mathbf{k}}$, $(p_{R1}, p_{R2}) = (p_1, p_2)$, and $\mathbf{R}$ becomes diagonal

$$\mathbf{R} = \begin{bmatrix} r(-p_N, -p_N) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & r(0,0) & \cdots & 0 \\ \vdots & & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & r(p_N, p_N) \end{bmatrix}_{p_2},$$

- We can then reduce matrix equation for $\mathbf{R}$ to a scalar quotient

$$r(p_1, p_2) = \frac{\varphi_R(p_1, p_2)}{\varphi_I(p_1, p_2)},$$

that may be computed for individual *gathers* of data

- For a *common-shot* gather in 2D $(p_2 = 0)$, for example, r for shot $\tilde{S}$ may be computed as

$$\begin{bmatrix} r\left(p_{-N}\right) \\ \vdots \\ r\left(0\right) \\ \vdots \\ r\left(p_N\right) \end{bmatrix}_{\tilde{S}} = \begin{bmatrix} \frac{\varphi_R\left(p_{-N}\right)}{\varphi_I\left(p_{-N}\right)} \\ \vdots \\ \frac{\varphi_R(0)}{\varphi_I(0)} \\ \vdots \\ \frac{\varphi_R(p_N)}{\varphi_I(p_N)} \end{bmatrix}_{\tilde{S}},$$

where $p_{-N} \leq p \leq p_N$

- For a *common-angle* gather in 3D, let $p_2 = \tilde{p_2}$, and then r may be computed as

$$\begin{bmatrix} r\left(p_{-N}\right) \\ \vdots \\ r\left(0\right) \\ \vdots \\ r\left(p_{N}\right) \end{bmatrix}_{\tilde{p}_2} = \begin{bmatrix} \frac{\varphi_R\left(p_{-N}\right)}{\varphi_I\left(p_{-N}\right)} \\ \vdots \\ \frac{\varphi_R(0)}{\varphi_I(0)} \\ \vdots \\ \frac{\varphi_R(p_N)}{\varphi_I(p_N)} \end{bmatrix}_{\tilde{p}_2},$$

- Computation of r above is done for each ω, so average $\bar{r}$ may be computed by summing them up

$$\bar{r}\left(p_1, p_2\right) = \sum_{\omega} \frac{\varphi_R\left(p_1, p_2, \omega\right)}{\varphi_I\left(p_1, p_2, \omega\right)},$$

- The sum over ω is equivalent to an IFT $\omega \to t$ for $t = 0$

$$\bar{r}\left(p_1, p_2\right) = \frac{1}{2\pi} \int \frac{\varphi_R\left(p_1, p_2, \omega\right)}{\varphi_I\left(p_1, p_2, \omega\right)} e^{i\,\omega\,[t=0]}\, d\omega,$$

  - this is the $t = 0$ *imaging condition*

- Further, if we are not interested in variation of r with (p_1, p_2), we may produce a single $\hat{r}$ at $\mathbf{x}$ by summing over (p_1, p_2)

$$\hat{r} = \sum_{\omega} \sum_{p_1} \sum_{p_2} \frac{\varphi_R\left(p_1, p_2, \omega\right)}{\varphi_I\left(p_1, p_2, \omega\right)},$$

  where we employ the $t = 0$ imaging condition as well

  - this is *stacking* over p_1 and p_2

- Stacking and the $t = 0$ imaging condition help reduce random noise, and they reduce the data volume in a rational way

- In practice, frequently, due probably to early use of W operators cast entirely in $\mathbf{x}$, $\bar{r}$ is computed in $\mathbf{x}$ as

$$\bar{r}\left(x_1, x_2\right)_{x_3=0} = \sum_{\omega} \frac{\psi_R\left(x_1, x_2, \omega\right)_{x_3=0}}{\psi_I\left(x_1, x_2, \omega\right)_{x_3=0}},$$

  where $x_3 = 0$ is the depth to the boundary

  – this implies, however, that r is independent of (p_1, p_2)

- For the example of a common-shot gather in 2D $x_2 = 0$, then, $\bar{r}$ is computed in $\mathbf{x}$ as

$$\begin{bmatrix} \bar{r}\left(x_{-N}\right) \\ \vdots \\ \bar{r}\left(0\right) \\ \vdots \\ \bar{r}\left(x_{N}\right) \end{bmatrix}_{\tilde{S}} = \sum_{\omega} \begin{bmatrix} \frac{\psi_R\left(x_{-N}\right)}{\psi_I\left(x_{-N}\right)} \\ \vdots \\ \frac{\psi_R(0)}{\psi_I(0)} \\ \vdots \\ \frac{\psi_R(x_N)}{\psi_I(x_N)} \end{bmatrix}_{\tilde{S}},$$

returns r that varies with $\mathbf{x}$ (offset) for each shot gather ($x_3 = 0$ is suppressed here for brevity)

- any relationship, however, between $\bar{r}\left(\mathbf{x}\right)$ and $r\left(p\right)$ obtained analytically is broken, and inversion of $\bar{r}\left(\mathbf{x}\right)$ does not have much meaning in an absolute sense
- in a relative sense, inversion of $\bar{r}\left(\mathbf{x}\right)$ has meaning - i.e. basic AVO

- Stacking of common-shot gathers may then be done in an $\mathbf{x}$ consistent way according to

$$\hat{r}(\mathbf{x}) = \sum_{S} \bar{r}(\mathbf{x})_S$$

Practical $\mathbf{W}$

- For simplicity, in 2D media $(\mathbf{x} \Leftrightarrow (x_1, x_3))$ and $(\mathbf{x} \Leftrightarrow (y_1, y_3))$, our expression for $\mathbf{W}$ is

$$\mathbf{W} \Leftrightarrow W(x, y, \omega)|_{\pm \Delta x_3} = \frac{1}{2\pi} \int \omega \, e^{-i\omega p[x-y]} \, \alpha(p_3, \omega)_{\pm \Delta x_3} \, dp,$$

where

$$\alpha(p_3(x, p), \omega)_{\pm \Delta x_3} = e^{\pm i \frac{\omega}{v(x)} \sqrt{1-(v(x)\, p)^2} \Delta x_3}$$

- Because α disrupts the symmetry of the Fourier kernal, computational cost for $\mathbf{W}$ is $\propto$ Cost$\{$FT$\}$ $\propto N^2$ rather than $N \log_2 N$ (N is the number of receivers)

- For 3D, cost $\propto N^4$
  - a Tflop for 1000×1000 receivers

- For efficiency, use a series for α

$$\alpha\,(x,p,\omega)_{\pm\,\Delta x_3} \approx \sum_{j=0}^{n} a_j\,(x,\omega)_{\pm\,\Delta x_3}\; b_j\,(p,\omega)_{\pm\,\Delta x_3}$$

where $0 \leq n < \infty$

- So that

$$W\,(x,y,\omega)_{\pm\Delta x_3} = \sum_{j=0}^{n} a_j\,(x,\omega)_{\pm\,\Delta x_3}\,\frac{1}{2\pi}\int \omega\, e^{-i\omega p[x-y]}\, b_j\,(p,\omega)_{\pm\,\Delta x_3}\, dp,$$

and wavefield ψ_0 is computed

$$\psi\,(x,\omega)_0 = \sum_{j=0}^{n} a_j\,(x,\omega)_{\pm\,\Delta x_3} \int \varphi\,(p,\omega)_{\mp\,\Delta x_3}\; e^{-i\omega p\,x}\, b_j\,(p,\omega)_{\pm\,\Delta x_3}\;\omega\, dp,$$

where $\varphi\,(p,\omega)_{\mp\,\Delta x_3} = \int \psi\,(y)_{\mp\,\Delta x_3}\; e^{i\,\omega\,p\,y}\, dy$

- Now, cost $\propto n \times \text{Cost}\,\{\text{FFT}\} = n\,N\,\log_2 N$ ($\propto 2\,n\,N^2 \log_2 N$ in 3D)
  - $\propto 10n$ Mflops for 1000×1000 receivers

Summary

- $\mathbf{R}$ very expensive to estimate
  - a Gflop computation within 3 loops for every subsurface point $\mathbf{x}$
  - numbers of sources and receivers must be the same and they must have even spacing (or the cost goes up)
- Assume horizontal boundaries
  - a scalar calculation within 2 loops
  - work with individual gathers of data - robust for irregular shots/reviver's
- The sum of r over ω is equivalent to the $t = 0$ imaging condition

- The $\mathbf{x}$ consistent sum of r over gathers (common shot, common receiver, common offset, common p, common mid-point, ...) is stacking

- Estimates of r computed in $\mathbf{x}$ are valid in a relative sense only

- Extrapolation operator $\mathbf{W}$ has a computational cost $\propto N^4$ when applied in 3D

- Factor α into series $\alpha(\mathbf{x}, \mathbf{p}) \approx \sum_j^n a_j(\mathbf{x})\, b_j(\mathbf{p})$ for cost $\propto 2\,n\,N^2 \log_2 N$

Wave-equation migration: examples

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From last WE class

- In 2D space-coordinates, and relative reflection coefficient $\hat{r}$ is given by

$$\hat{r}\left(x_1\right)_{x_3=0} = \sum_{\omega} \sum_{G} \left[\frac{\psi_R\left(x_1, \omega\right)_{x_3=0}}{\psi_I\left(x_1 \omega\right)_{x_3=0}} \right]_G ,$$

where G represents a gather like a shot gather or a CMP

- Using $\mathbf{W}$, wavefields ψ_R and ψ_I on the boundary are computed

$$\psi_R\left(x_1\right)_0 = \int \psi_R\left(y_1\right)_{-\Delta x_3} W\left(x_1, y_1\right)_{-\Delta x_3} dy_1,$$

and

$$\psi_I\,(x_1)_0 = \int \psi_I\,(y_1)_{-\Delta x_3}\, W\,(x_1, y_1)_{-\Delta x_3}\; dy_1,$$

where extrapolator W is given by

$$W\,(x, y, \omega)_{\pm\,\Delta x_3} = \frac{1}{2\pi}\int \omega e^{-i\omega p_1[x_1 - y_1]}\,\alpha\,(x_1, p_3, \omega)_{\pm\,\Delta x_3}\; dp_1$$

and extrapolation-symbol α is

$$\begin{aligned}\alpha\,(x_1, p_3, \omega)_{\pm\,\Delta x_3} &= e^{\pm\,\Delta x_3\, i\,\omega\, p_3(x_1, p_1)} \\ &\approx \sum_{j=0}^{n} a_j\,(x_1, \omega)_{\pm\,\Delta x_3}\; b_j\,(p_1, \omega)_{\pm\,\Delta x_3}\end{aligned}$$

- For $N << \infty$, cost is reduced from N^2 to $n\,N\,\log_2 N$

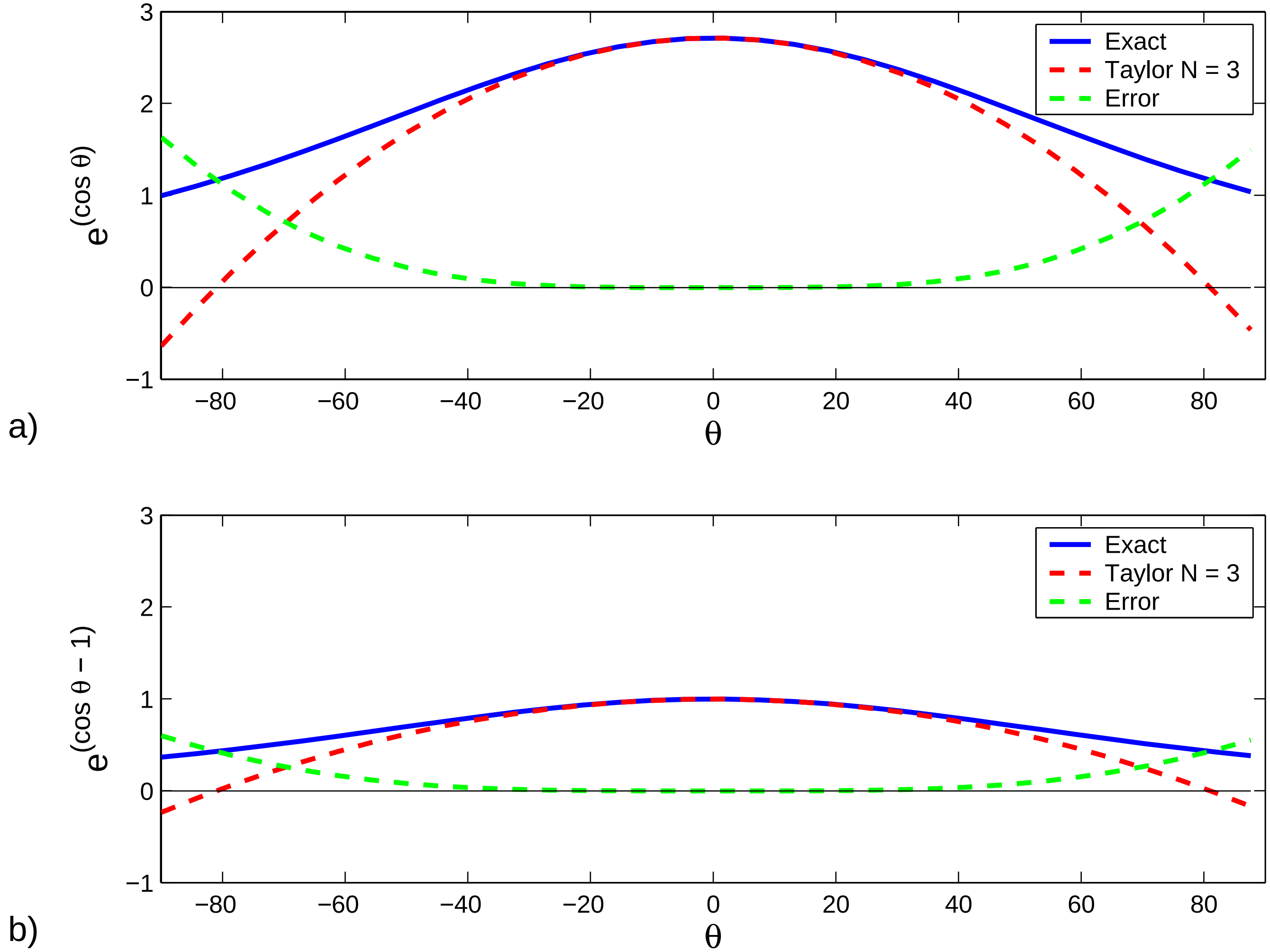

Figure 1: a) Expansion of $e^{\cos\theta}$. b) Expansion of $e^{\cos\theta-1}$.

Fourier finite difference migration

- To ensure stability, calculate vertical slowness p_3 according to

$$p_3\left(x_1, p_1\right) = \frac{\omega}{v\left(x_1\right)}\left[\Re\left\{\sqrt{1-\left(v\left(x_1\right)\, p_1\right)^2}\right\} + \left|\Im\left\{\sqrt{1-\left(v\left(x_1\right)\, p_1\right)^2}\right\}\right|\right]$$

for Δx_3, change the sign in the $\Im$ part for $-\Delta x_3$

  - for horizontal boundaries, we force the evanescent region to decay rapidly, but we must expect leakage for dipping boundaries

- To determine $a_j\left(x_1\right)$ and $b_j\left(p_1\right)$, recall $\cos\theta = v\, p_3$, where θ is phase angle and write α as

$$\alpha = e^{\pm\,\Delta x_3\, i\, \frac{\omega}{v}\, \cos\theta},$$

as part of the expansion of α, we must approximate $\cos\theta$

- For the same number of terms, expansion of $\cos\theta - 1$ has better properties for reflections than does expansion of $\cos\theta$, so a better form for α is

$$\alpha\left(x_1, p_1\right)_{\pm\,\Delta x_3} = e^{\pm\Delta x_3\, i\, k(x_1)\left[\sqrt{1-(v(x_1)\,p_1)^2}-1\right]}\, e^{\pm\Delta x_3\, i\, k(x_1)}$$

where $k\left(x\right) = \frac{\omega}{v(x_1)}$, and p_3 is written in terms of $v\left(x_1\right)$ and p_1 explicitly

- Using

$$\sqrt{1+u} - 1 \sim \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{48} - \cdots\,,$$

and

$$e^u \sim 1 + u + \frac{u^2}{2} + \frac{u^3}{6} - \cdots\,,$$

expand $e^{\pm\Delta x_3\, i\, k(x_1)\left[\sqrt{1-(v(x_1)\,p_1)^2}-1\right]}$ in p_1 and truncate to n terms

- Collect x_1 dependent terms to get

$$a(x_1)_j = e^{\pm\,\Delta x_3\, i\, k(x_1)}\,\gamma_j(x_1)\,,$$

and collect p_1 dependent terms to get

$$b(p_1)_j = (\omega\, p_1)^{2j}$$

where the first five terms $(0 \le j \le n = 4)$ are

$$\begin{aligned}
\gamma_0 &= 1 \\
\gamma_1 &= -\frac{i\,\pi\,\Delta z}{k} \\
\gamma_2 &= \frac{-1/4\, i\,\pi\,\Delta z - 1/2\,\pi^2\,\Delta z^2\, k}{k^3} \\
\gamma_3 &= \frac{-1/8\, i\pi\,\Delta z - 1/4\,\pi^2\,\Delta z^2\, k + 1/6\, i\,\pi^3\,\Delta z^3\, k^2}{k^5} \\
\gamma_4 &= \frac{-5/64\, i\,\pi\,\Delta z - 5/32\,\pi^2\,\Delta z^2\, k + 1/8\, i\,\pi^3\,\Delta z^3\, k^2 + 1/24\,\pi^4\,\Delta z^4 k^3}{k^7}
\end{aligned}$$

- In space coordinates, b_j is applied using finite differences according to

$$\frac{\partial^2}{\partial x^2} f(x) = \int (i\,k_1)^{2j} F(k_1)\, e^{i\,k_1\,x_1}\, dk_x \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2},$$

where the substitution $\omega\, p_1 = k_1$ has been made

- Algorithms based on this factorization are called Fourier finite-difference methods or $FD\ migration$, sometimes $\omega - x$ migration

- FD migration copes with strong $v(x)$ at the expense of steep dips

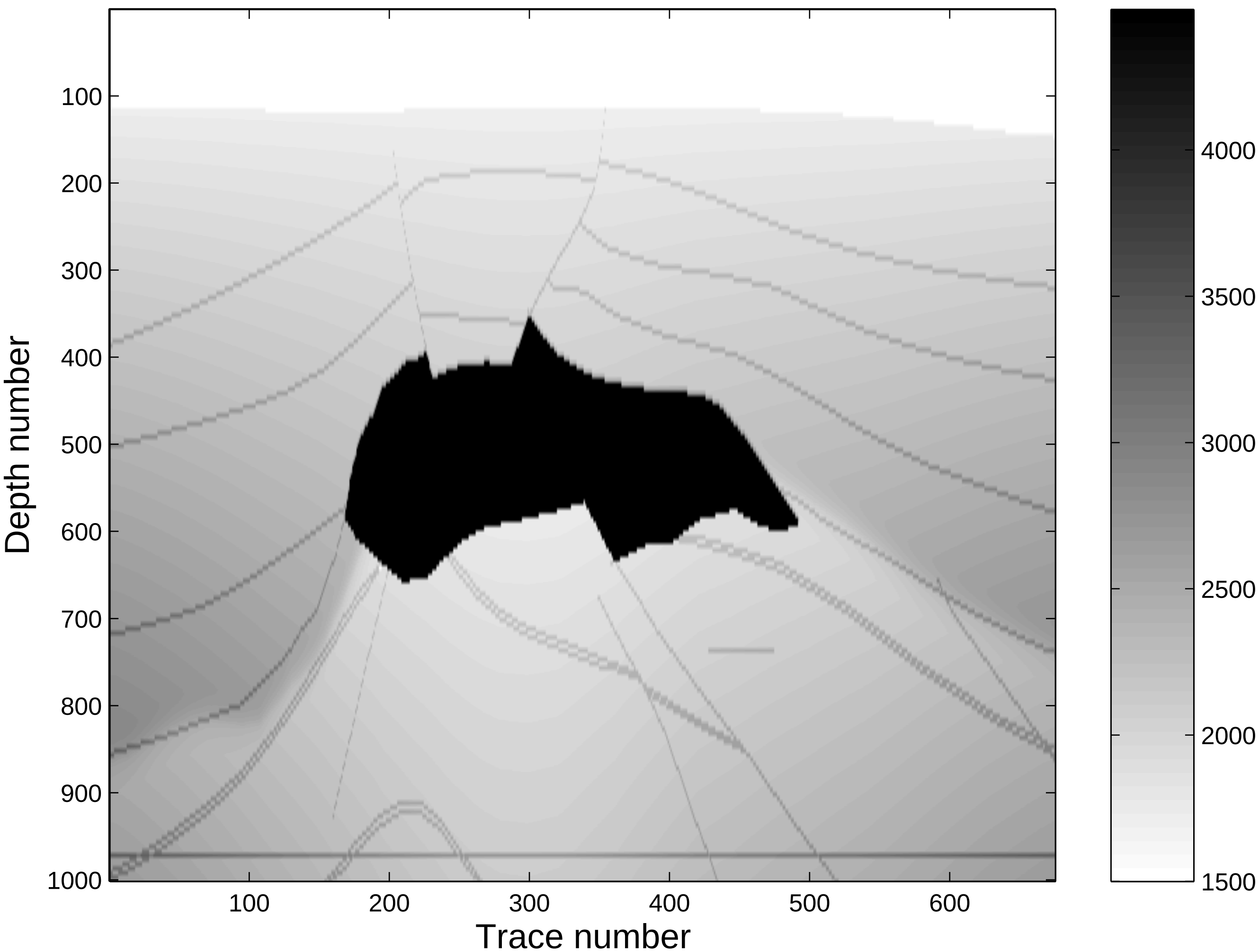

Figure 2: SEG/EAGE salt model.

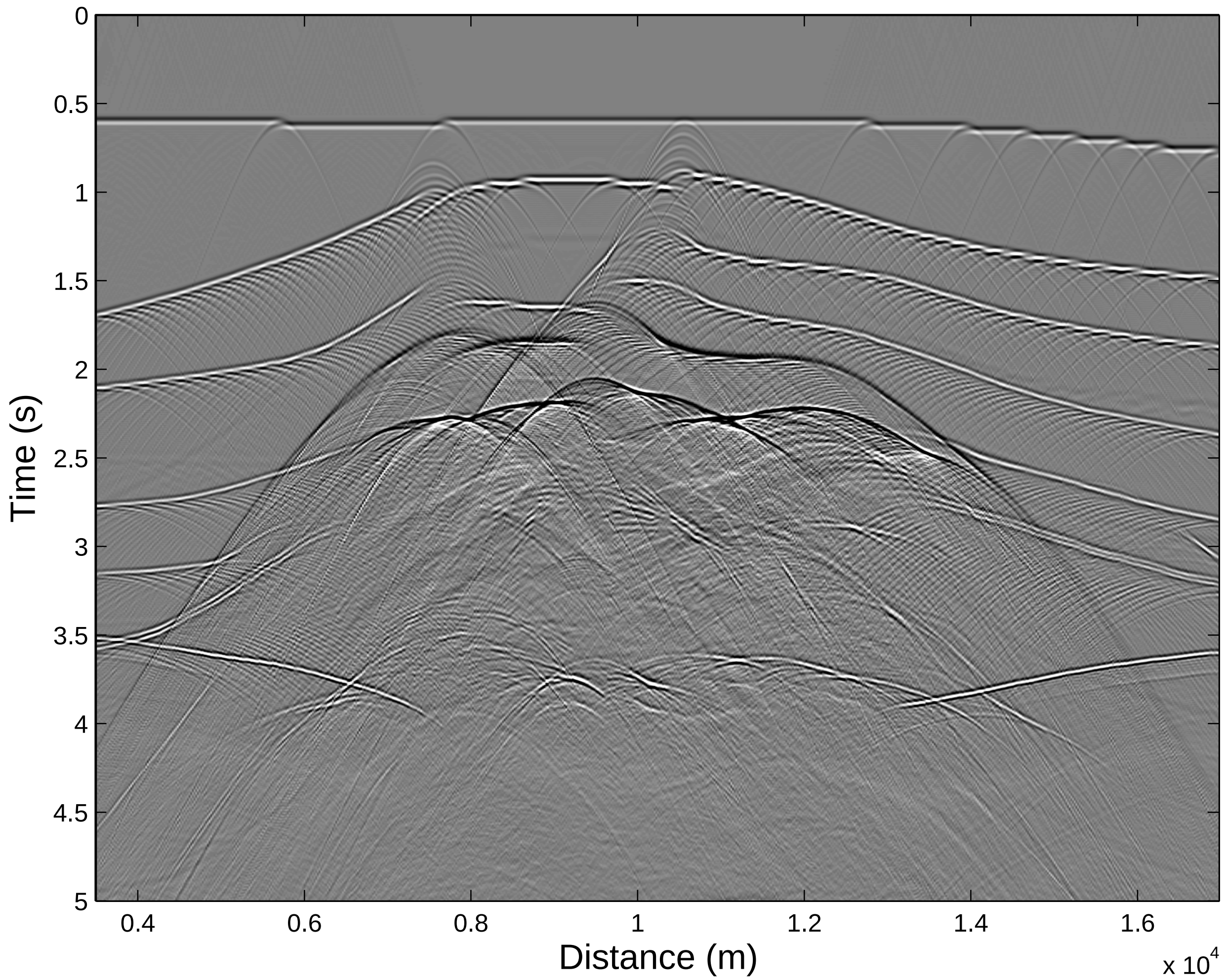

Figure 3: Exploding reflector data.

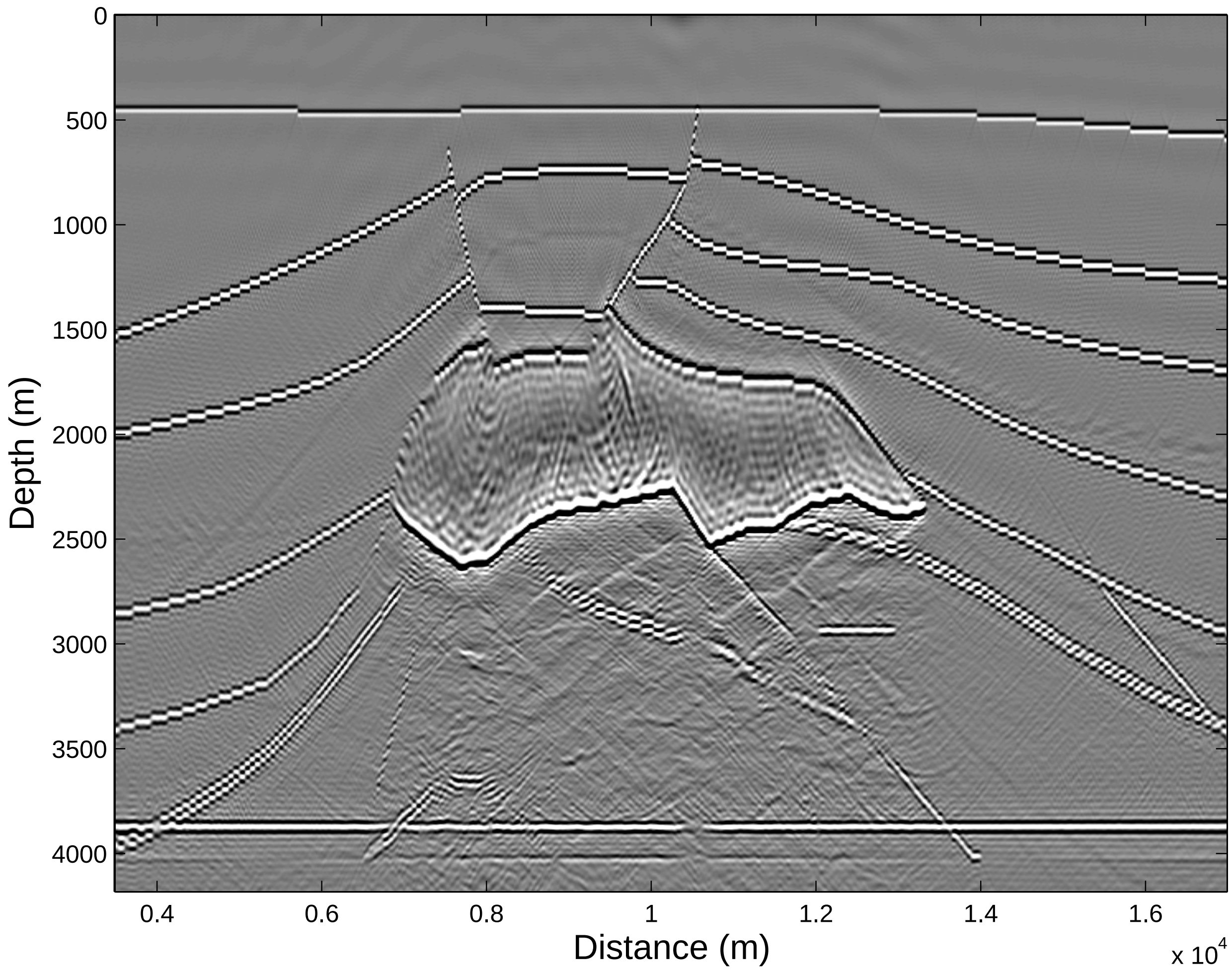

Figure 4: Finite difference migration ($n = 4$, 65 degree).

Split-step Fourier migration

1. Compute $\bar{v} = \mathsf{mean}\,(v)$ and expand $u\,(x_1, p_1) = \sqrt{1 - (v\,(x_1)\;p_1)^2}$ about $\bar{v}$ using

$$u\,(v + \bar{v}) = u\,(\bar{v}) + \frac{\partial u}{\partial v}\,(v - \bar{v}) + \cdots$$

2. Truncate the series at zeroth order, and the resulting approximation for α is given by

$$\alpha_{\mathsf{SS}}\,(x_1, p_1)_{\pm\,\Delta x_3} \approx a_0\,(x_1)_{\pm\,\Delta x_3}\; b_0\,(p_1)_{\pm\,\Delta x_3}$$

where

$$a_0\,(x_1)_{\pm\,\Delta x_3} = e^{\pm\,\Delta x_3\,i\,k(x_1)},$$

and

$$b_0\,(p_1,\omega)_{\pm\,\Delta x_3} = e^{\pm\,\Delta x_3\,i\,\frac{\omega}{\bar{v}}\left[\sqrt{1-(\bar{v}\,p_1)^2}-1\right]}$$

Generalized screen migration

1. Compute $\check{v} = \mathsf{min}\,(v)$

2. Factor $\alpha_{\mathsf{SS}}\,(\check{v})$ from α so that

$$\alpha\,(x_1, p_1)_{\pm\,\Delta x_3} = \alpha_{\mathsf{SS}}\,(x_1, p_1)\, e^{\pm\,\Delta x_3\, i\,\left[\omega\, p_3(x_1,p_1) - \omega\, \check{p}_3(p_1) - k(x_1) + \check{k}\right]}$$

3. Expand the exponential to first order,

$$
\begin{aligned}
& e^{\pm\,\Delta x_3\, i\left[\omega\, p_3(x_1,p_1)-\omega\,\check{p}_3(p_1)-k(x_1)+\check{k}\right]} \\
& = 1 \pm\, \Delta x_3\, i\, \left[\omega\, p_3\left(x_1,p_1\right)-\omega\,\check{p}_3\left(p_1\right)-k\left(x_1\right)+\check{k}\right] \\
= \quad & 1 \pm\, \Delta x_3\, i\, \left\{\omega\,\check{p}_3\left(p_1\right)\left[\sqrt{1-\frac{\check{k}^2-k\left(x_1\right)^2}{\omega\,\check{p}_3^2}}-1\right]-k\left(x_1\right)+\check{k}\right\}
\end{aligned}
$$

4. Expand $\sqrt{1-\frac{\check{k}^2-k^2(x_1)}{\omega\,\check{p}_3^2}}-1$ about $\omega\, p_1$, and truncate to n terms

5. Collect x_1 dependent terms to get

$$a_j\left(x_1\right)=\lambda_j\left(x_1\right)e^{\pm\,\Delta x_3\, i\, k(x_1)}$$

6. Collect p_1 dependent terms to get

$$b_j\left(p_1\right) = \kappa_j\left(p_1\right) e^{\pm\,\Delta x_3\, i\, \frac{\omega}{\check{v}}\left[\sqrt{1-\left(\check{v}\, p_1\right)^2}-1\right]}$$

where

$$\begin{aligned}
\lambda_0 &= 1 \\
\lambda_1 &= \tfrac{1}{2}\left(\check{k}^2 - k\left(x_1\right)^2\right) \\
\lambda_2 &= \tfrac{1}{8}\left(\check{k}^2 - k\left(x_1\right)^2\right)^2 \\
\lambda_3 &= \tfrac{1}{16}\left(\check{k}^2 - k\left(x_1\right)^2\right)^3 \\
\lambda_4 &= \tfrac{5}{128}\left(\check{k}^2 - k\left(x_1\right)^2\right)^4
\end{aligned}$$

and

$$
\begin{aligned}
\kappa_0 &= \omega\, \check{p}_3 \\
\kappa_1 &= (\omega\, \check{p}_3)^{-1} \\
\kappa_2 &= (\omega\, \check{p}_3)^{-3} \\
\kappa_3 &= (\omega\, \check{p}_3)^{-5} \\
\kappa_4 &= (\omega\, \check{p}_3)^{-7}
\end{aligned}
$$

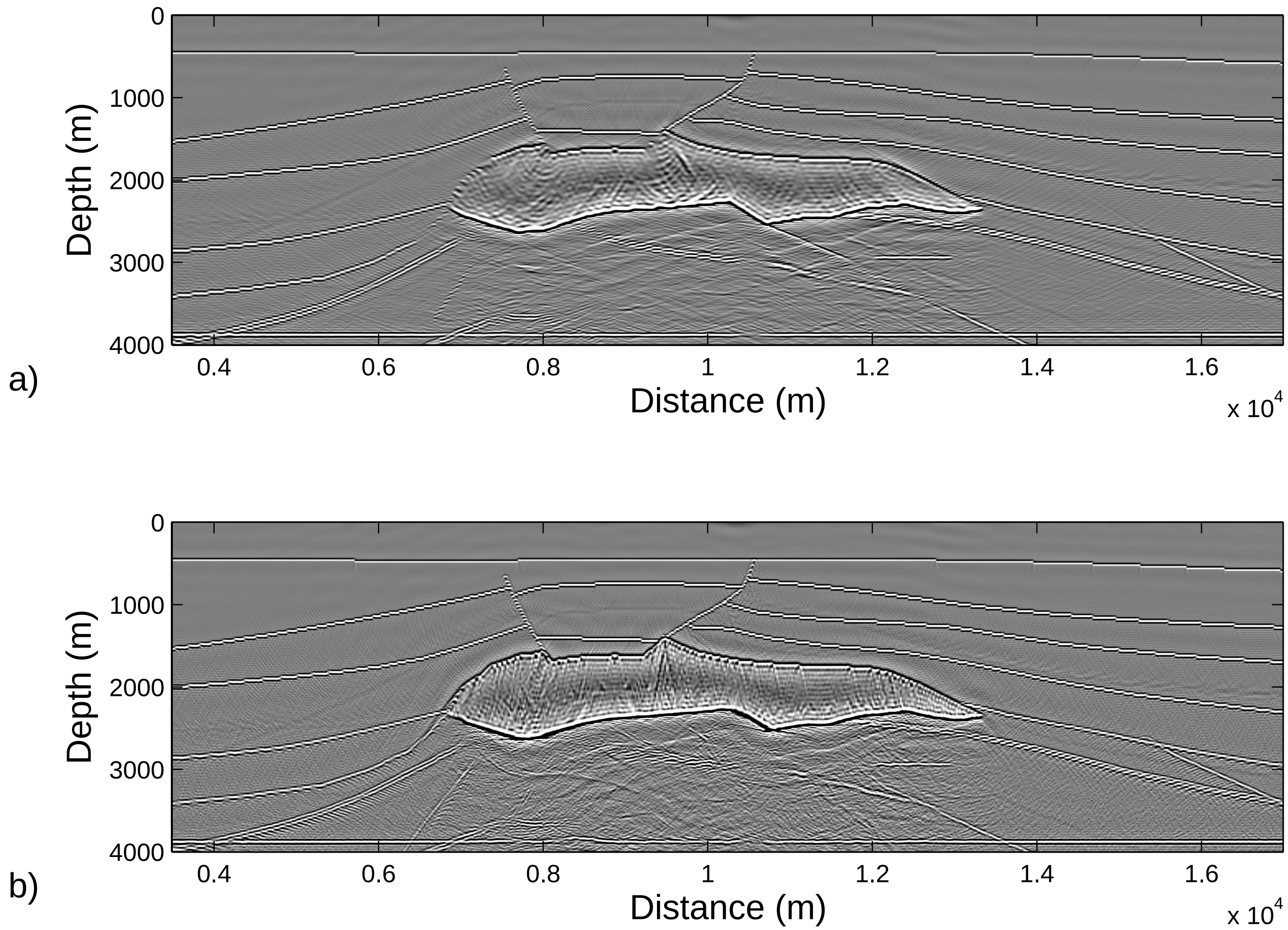

Figure 5: Zero offset migration of the SEG salt model. a) 65° FD. b) GS.

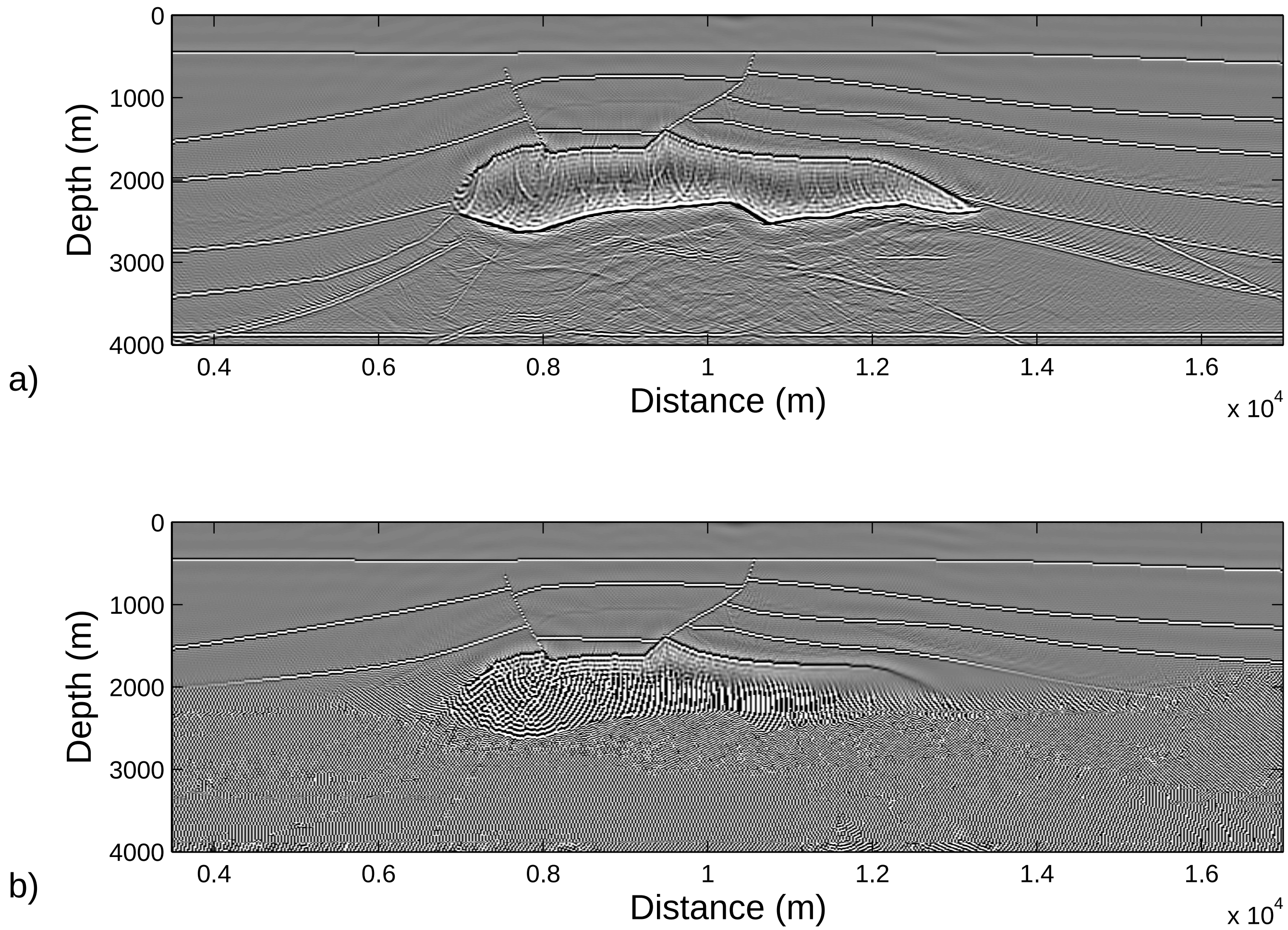

Figure 6: Zero offset migration of the SEG salt model. a) SS. b) BL.

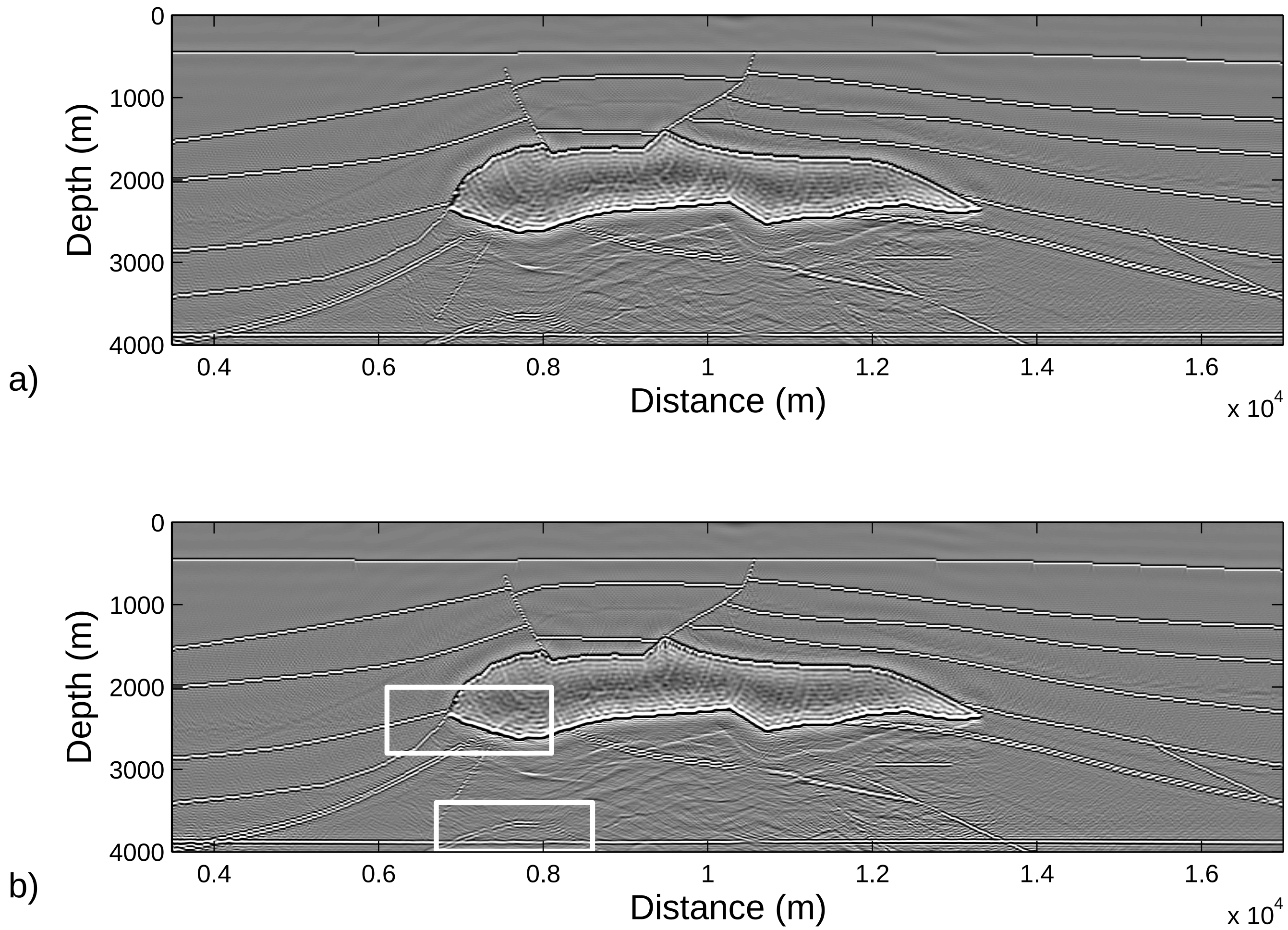

Figure 7: Zero offset migration of the SEG salt model. a) PSPI. b) Hybrid.

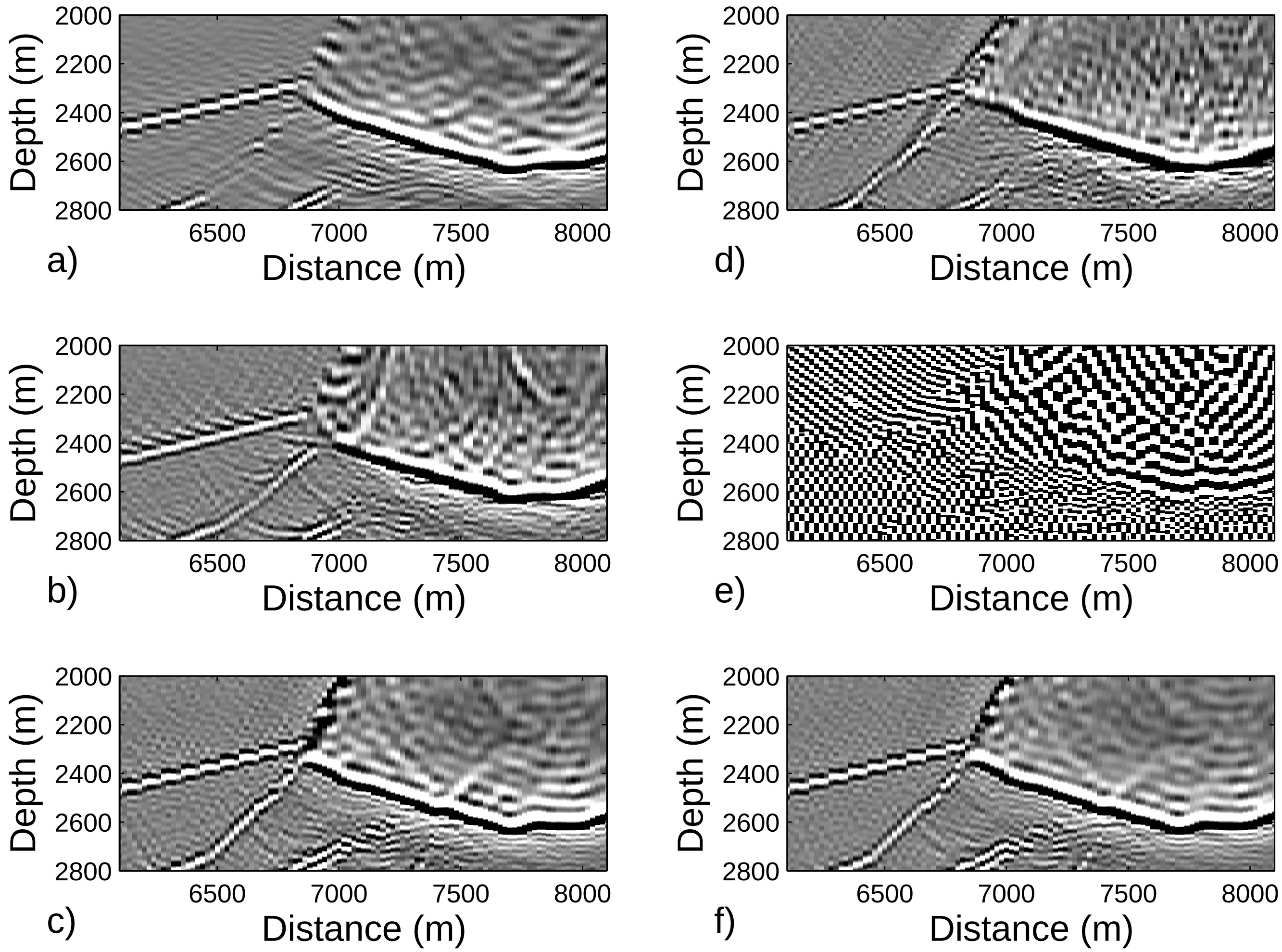

Figure 8: a) $f - x$. b) SS. c) PSPI. D) GS. E). BL. F) Hybrid.

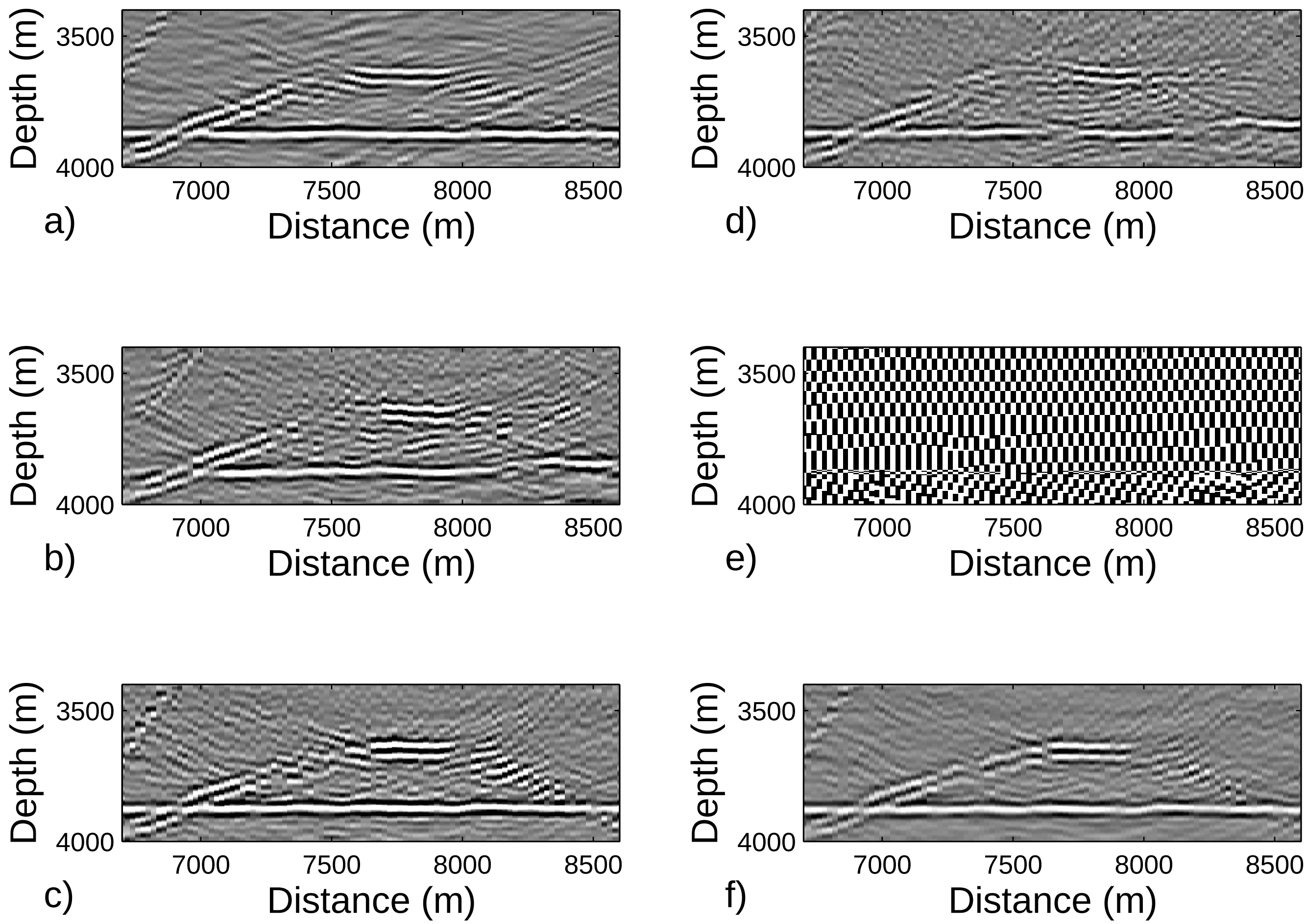

Figure 9: a) $f - x$. b) SS. c) PSPI. D) GS. E). BL. F) Hybrid.

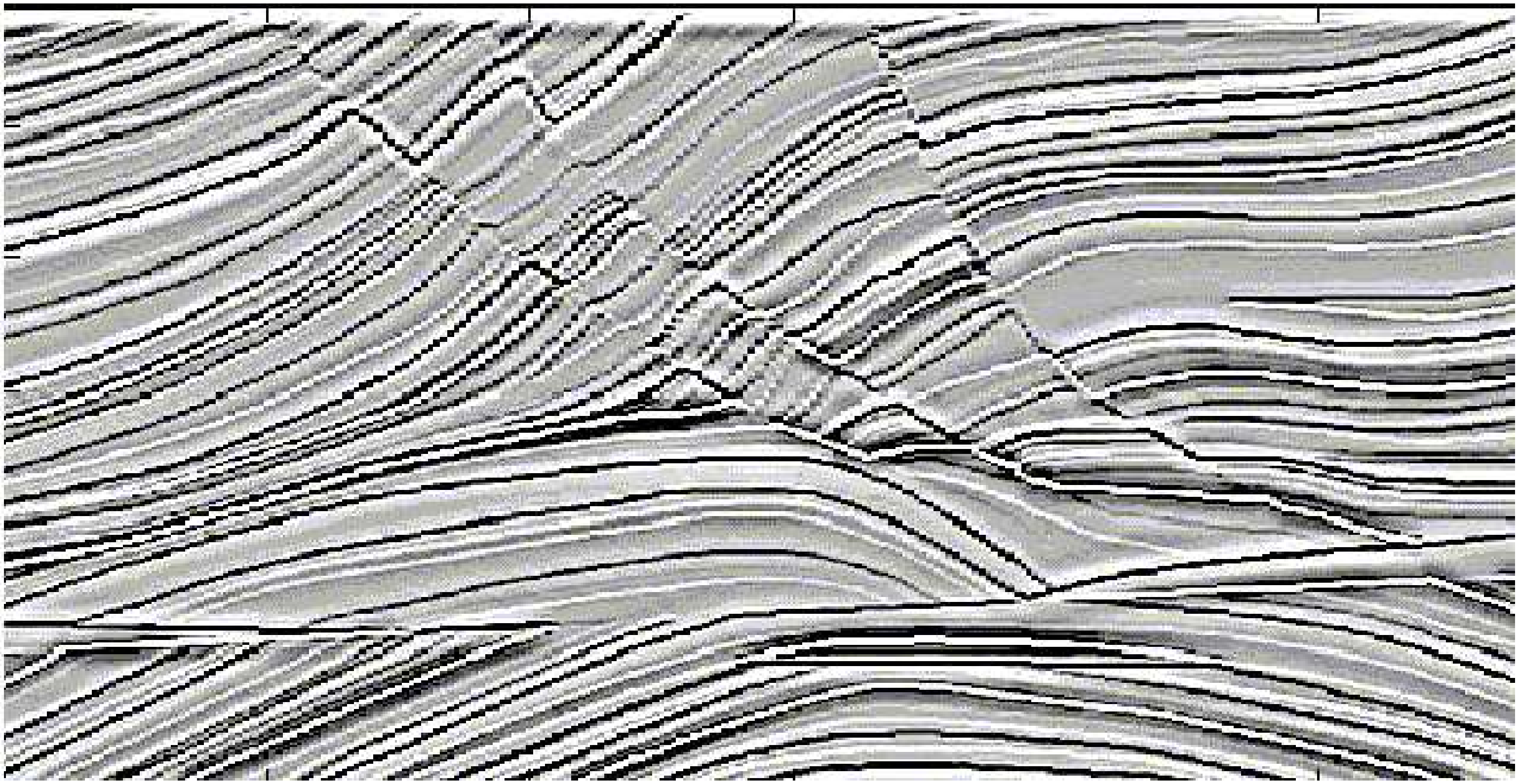

Figure 10: Marmousi reflectivity for zero offset.

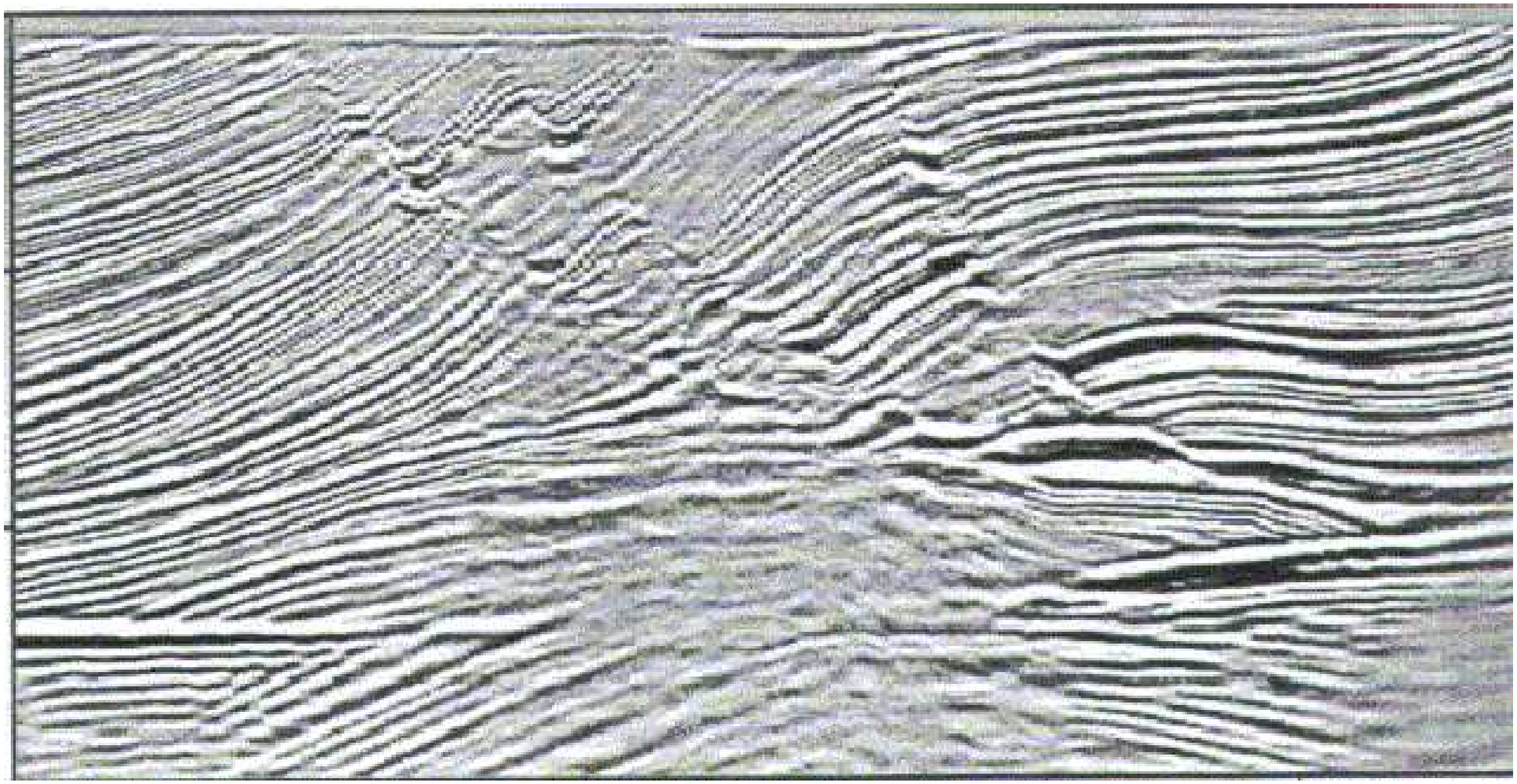

Figure 11: Kirchhoff migration of Marmousi (S. Gray).

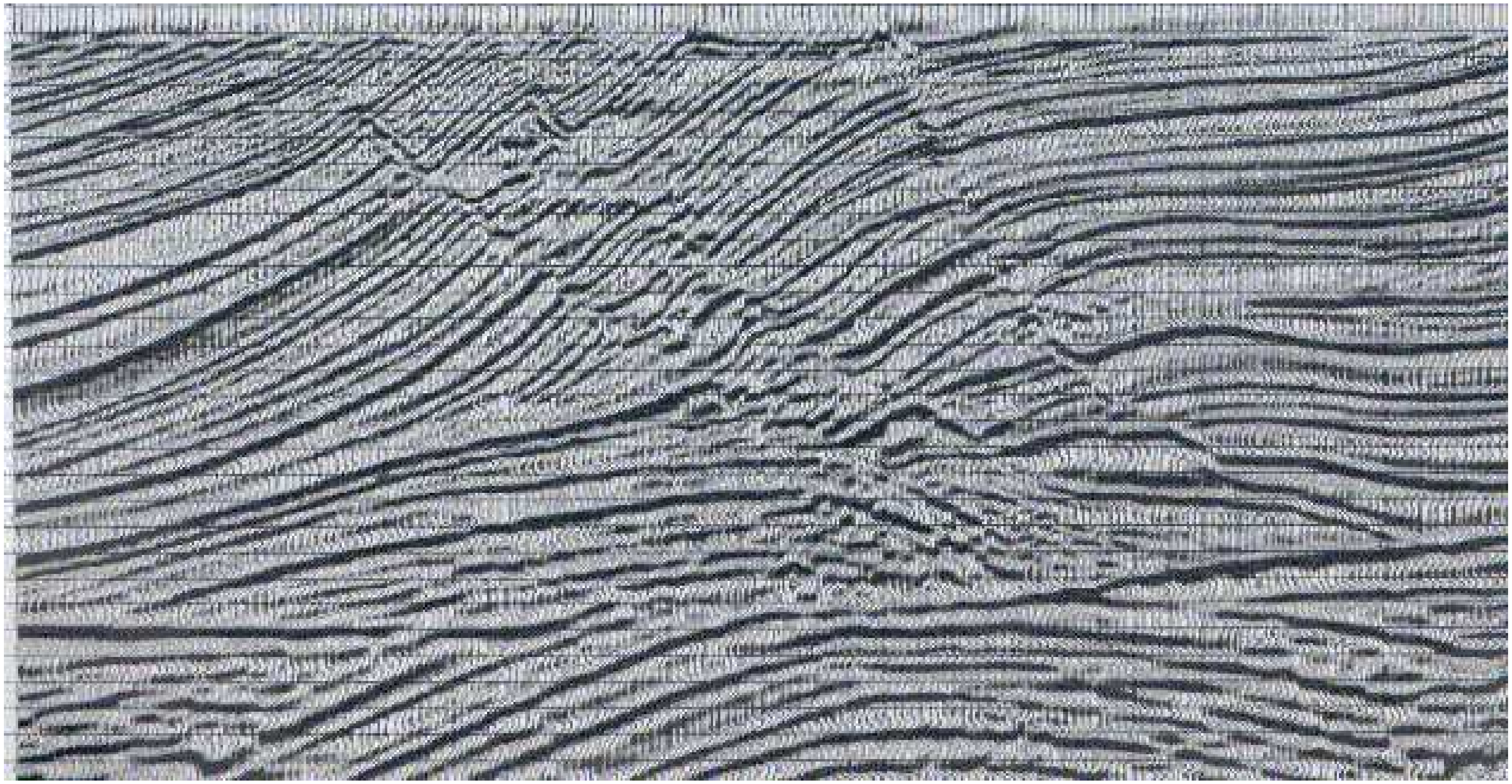

Figure 12: $f - x$ migration of Marmousi (Delft University).

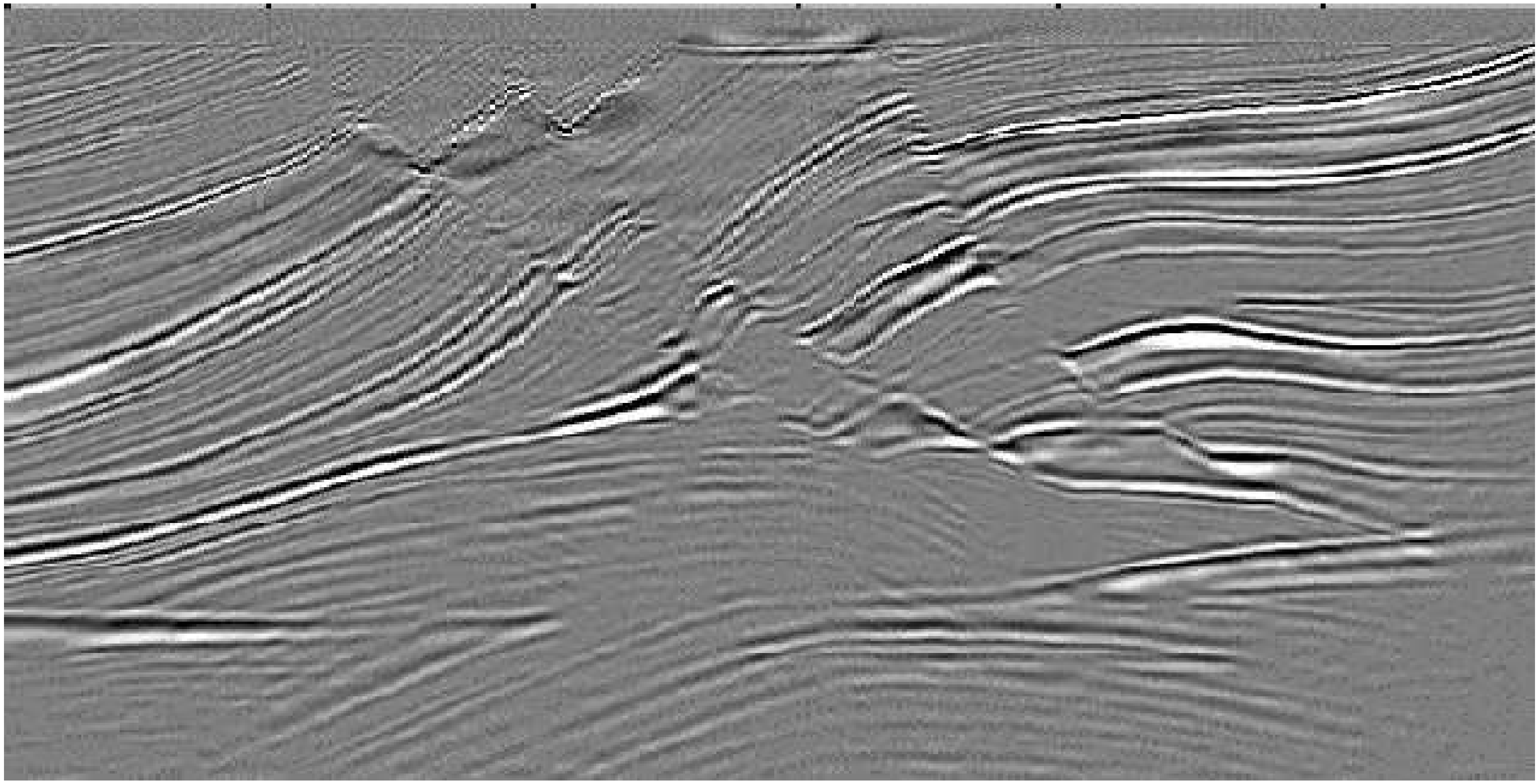

Figure 13: PSPI migration of Marmousi (Nutec).

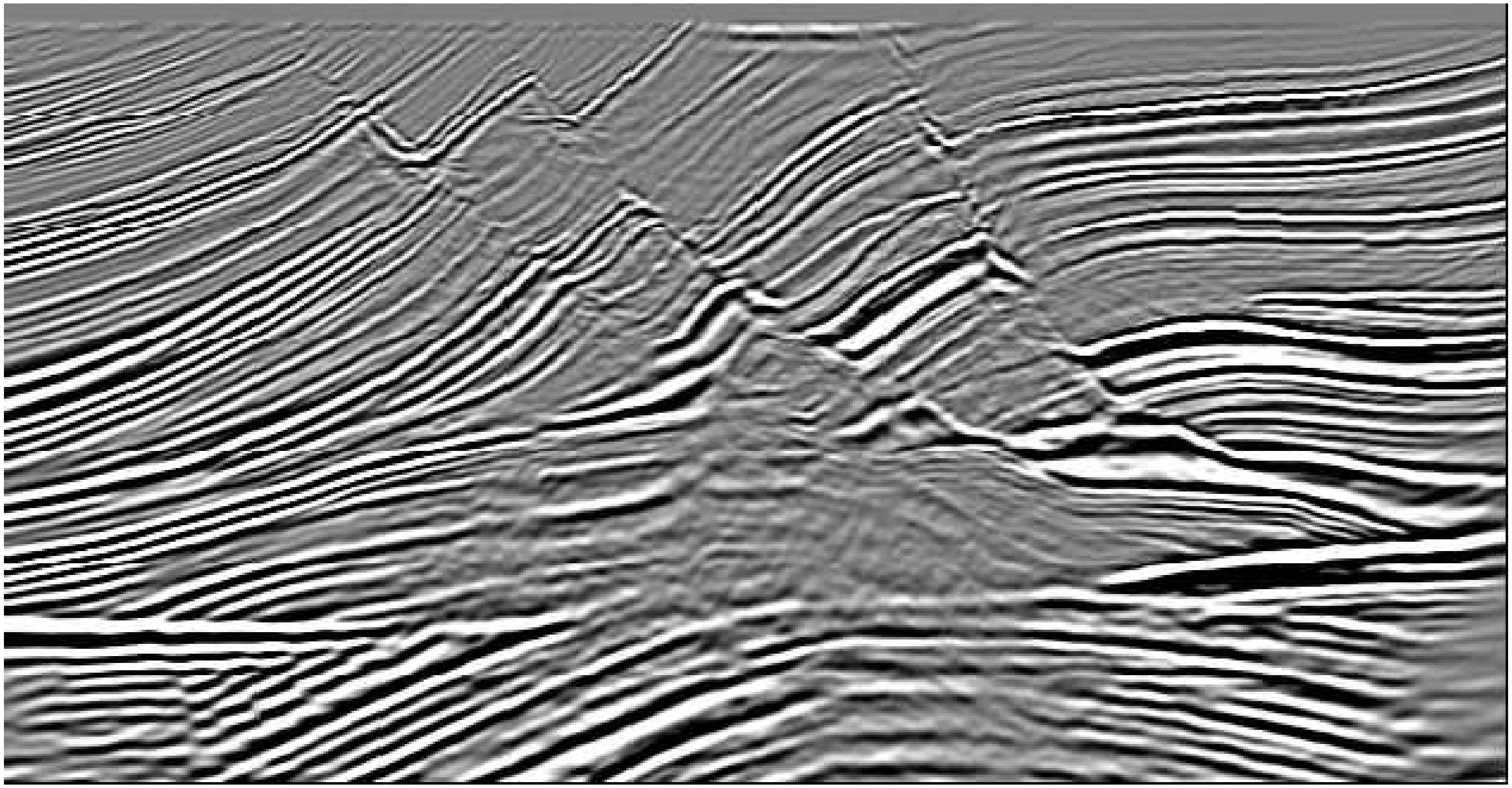

Figure 14: GB migration of Marmousi (R. Hill).