The Direct Scattering Problem

David Colton

University of Delaware

Research supported by AFOSR

Scattering by an Infinite Cylinder

g replacements

E
H
D
ν

PSfrag replacements

E
H
D
ν

1. A Perfect Conductor $(E = (0, 0, u))$

$$\Delta_2 u + k^2 u = 0 \quad \text{in } R^2 \setminus \bar{D}$$

$$\begin{aligned} u &= u^i + u^s \\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

2. An Inhomogeneous Medium

$$\Delta_2 u + k^2 n(x) u = 0 \quad \text{in } R^2$$

$$u = u^i + u^s$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

Assume $u^i(x) = e^{ikx\cdot d}$, $n(x) = 1$ in $R^2 \setminus \bar{D}$, $Im\, n(x) \geq 0$ for $x \in D$, $Re\, n(x) > 0$, $k > 0$, $r = |x|$.

Scattering by an Infinite Cylinder

Let

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|) \quad , x \neq y.$$

Then for $x \in R^2 \setminus \bar{D}$, Green's theorem implies that

$$u^s(x) = \int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y)$$

This is known as Green's representation formula.

Theorem: In $R^2 \setminus \bar{D}$, $u^s(x)$ is a real-analytic function of its independent variables.

Rellich's Lemma: Let $u \in C^2(R^2 \setminus \bar{D})$ be a solution of the Helmholtz equation satisfying

$$\lim_{R \to \infty} \int_{|y|=R} |u(y)|^2 ds(y) = 0.$$

Then $u = 0$ in $R^2 \setminus \bar{D}$.

Definition: The condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

is called the Sommerfeld radiation condition.

The Perfect Conductor: Uniqueness and Existence

Uniqueness Theorem: Let $u^s \in C^2(R^2 \setminus \bar{D}) \cap C(R^2 \setminus D)$ be a solution of the Helmholtz equation in $R^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and $u^s = 0$ on ∂D. Then $u^s = 0$ in $R^2 \setminus D$.

Proof: Let B be a disk centered at the origin such that $B \supset D$. Then by Green's theorem

$$\int_{\partial B} \left(\bar{u}^s \frac{\partial u^s}{\partial r} - u^s \frac{\partial \bar{u}^s}{\partial r} \right) ds = 0. \qquad (1)$$

The Perfect Conductor: Uniqueness and Existence

But for $x \in R^2 \setminus B$,

$$u^s(r,\theta) = \sum_{-\infty}^{\infty} a_n(r) e^{in\theta} \qquad (2)$$

$$a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^s(r,\theta) e^{-in\theta} d\theta$$

$\Rightarrow$

$$a_n(r) = a_n H_n^{(1)}(kr).$$

Using the Wronskian relation for Hankel functions, 1), 2)$\Rightarrow$

$$\sum_{-\infty}^{\infty} |a_n|^2 = 0$$

$\Rightarrow u^s(x) = 0$ for $x \in R^2 \setminus B$ and, by analyticity, $u^s(x) = 0$ for $x \in R^2 \setminus \bar{D}$.

The Perfect Conductor: Uniqueness and Existence

We now try to construct a solution to the direct scattering problem for a perfect conductor. We first look for a solution in the form of a double layer potential

$$u^s(x) = \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) ds(y).$$

However, this approach fails if k^2 is an eigenvalue of the interior Dirchlet problem for the Laplacian in D! Hence, we look for a solution int he form of a modified double layer potential

$$u^s(x) = \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta\Phi(x,y) \right\} ds(y)$$

where $\varphi \in C(\partial D)$ and $\eta \neq 0$. u^s will be a solution of the scattering problem if

$$\varphi + 2\int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta\Phi(x,y) \right\} ds(y) = -2e^{ikx\cdot d}$$

It can be show using the Fredholm alternative that there exists a unique solution to this integral equation. Hence existence of a solution to the scattering problem for a perfect conductor has been established.

The Inhomogeneous Medium: Uniqueness and Existence

In this case uniqueness (and hence, by the Fredholm alternative, existence) is based on the following theorem:

Unique Continuation Principle: Let G be a domain in R^2 and suppose $u \in C^2(G)$ is a solution of

$$\Delta_2 u + k^2 n(x) u = 0$$

in G such that $n \in C(\bar{G})$ and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Uniqueness Theorem: Let $u \in C^2(R^2)$ satisfy

$$\Delta_2 u + k^2 n(x) u = 0 \quad \text{in } R^2$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0$$

Then $u = 0$ in R^2.

Proof: Green's theorem implies that for $D \subset \{x : |x| < a\}$

$$\int\limits_{|x|=a} u \frac{\partial \bar{u}}{\partial \nu} ds = \int\limits_{|x| \leq a} \left\{ |\text{grad } u|^2 - k^2 \bar{n} |u|^2 \right\} dx$$

and hence

$$Im \int\limits_{|x|=a} u \frac{\partial \bar{u}}{\partial r} ds = k^2 \int\limits_{|x|<a} Im\ n |u|^2 dx \geq 0. \qquad (1)$$

The Inhomogeneous Medium: Uniqueness and Existence

But

$$Im \int_{|x|=a} u \frac{\partial \bar{u}}{\partial r} ds = \frac{1}{2i} \int_{|x|=a} \left(u \frac{\partial \bar{u}}{\partial r} - \bar{u} \frac{\partial u}{\partial r} \right) ds$$

and the Wronskian relation for Hankel functions implies that

$$Im \int_{|x|=a} u \frac{\partial \bar{u}}{\partial r} ds < 0$$

unless $u = 0$ for $|x| \geq a$. Hence, from 1) $u = 0$ for $|x| \geq a$ and the theorem follows by the unique continuation principle.

The direct scattering problem for an inhomogeneous medium is easily seen to be equivalent to the problem of solving the Lippmann Schwinger equation

$$u(x) = u^i(x) - k^2 \int_{R^2} \Phi(x,y) m(y) u(y) dy, \; x \in R^2$$

where $m := 1 - n$. The above uniqueness theorem and the Fredholm alternative now imply the existence of a unique solution to the direct scattering problem for an inhomogeneous medium.

Far Field Patterns

Recall that for both a perfect conductor and an inhomogeneous medium,

$$u^s(x) = \int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x, y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x, y) \right) ds(y).$$

Letting $r = |x| \to \infty$ implies that

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + 0(r^{-3/2})$$

where $\hat{x} = x/|x|$ and

$$u_\infty(\hat{x}, d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} (u^s \frac{\partial}{\partial \nu} e^{-ik\hat{x}\cdot y} - \frac{\partial u^s}{\partial \nu} e^{-ik\hat{x}\cdot y}) ds(y).$$

Definition: u_∞ is called the far field pattern corresponding to the specific scattering problem under consideration.

Theorem: Suppose $u_\infty = 0$. Then $u^s = 0$ in $R^2 \setminus D$.

Proof: $\int_{|y|=R} |u^s(y)|^2 ds = \int_{|\hat{x}|=1} |u_\infty(\hat{x}, d)|^2 ds(\hat{x}) + 0(\frac{1}{r})$ as $r \to \infty$. If $u_\infty = 0$ then by Rellich's lemma $u^s = 0$.

Far Field Patterns

Reciprocity Principle: $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$.

It follows from the reciprocity principle that $u_\infty(\hat{x}, d)$ is infinitely differentiable with respect to its independent variables.

Example: Consider the direct scattering problem for a perfect conductor when D is a disk of radius a. Then using the Jacobi-Anger expansion

$$e^{ikr\cos\theta} = \sum_{-\infty}^{\infty} i^n J_n(kr) e^{in\theta}$$

we have that, for $d = (\cos\phi, \sin\phi)$,

$$u^s(r, \theta) = -\sum_{-\infty}^{\infty} i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr) e^{in(\theta-\phi)}$$

and since

$$H_n^{(1)}(kr) = \sqrt{\frac{2}{\pi r}} \exp\left[i(kr - \frac{n\pi}{2} - \frac{\pi}{4})\right] + 0\left(r^{-3/2}\right)$$

we have that

$$u_\infty(\hat{x}, d) = -e^{-i\pi/4}\sqrt{\frac{2}{\pi}} \sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n(ka)} e^{in(\theta-\phi)}.$$

Far Field Operator for a Perfect Conductor

Let $\Omega := \{x : |x| = 1\}$. The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

From the smoothness of u_∞ we see that F is a compact operator. Note that $(Fg)(\hat{x})$ is the far field pattern corresponding to the incident field u^i being a Herglotz wave function $v_g(x)$ defined by

$$v_g(x) := \int_{\Omega} e^{ikx\cdot d} g(d) ds(d).$$

Theorem: If F is the far field operator corresponding to a perfect conductor then F is a normal operator.

Far Field Operator for a Perfect Conductor

Theorem: The far field operator corresponding to a perfect conductor is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

Outline of Proof: The reciprocity principle implies that the adjoint operator F^* satisfies

$$(F^*h)(d) = \overline{(Fg)(-d)}$$

where $g(\hat{x}) = \overline{h(-\hat{x})}$. Hence F is injective if and only if F^* is injective. But $N(F^*)^\perp = \overline{F(L^2(\Omega))}$ and hence we only need to prove that F is injective.

$Fg = 0$ implies that the scattering problem with $u^i = v_g$ has vanishing far field pattern and hence using Rellich's lemma $v_g(x) = 0$ for $x \in \partial D$. Thus v_g is a Dirichlet eigenfunction unless $g = 0$.

Far Field Operator for an Inhomogeneous Medium

Recall that the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) ds(d).$$

Theorem: If F is the far field operator corresponding to an inhomogeneous medium, and $Im\, n(x) = 0$ for $x \in D$, then F is a normal operator.

Theorem: The far field operator corresponding to an inhomogeneous medium is injective with dense range if and only if there does not exist $w \in C^2(D) \cap C^1(\overline{D})$ and a Herglotz wave function v such that v, w is a solution of the interior transmission problem

$$\left.\begin{aligned} \Delta_2 v + k^2 v &= 0 \\ \Delta_2 w + k^2 n(x) w &= 0 \end{aligned}\right\} \quad \text{in } D$$

$$\left.\begin{aligned} v &= w \\ \frac{\partial v}{\partial \nu} &= \frac{\partial w}{\partial \nu}. \end{aligned}\right\} \quad \text{on } \partial D$$

Far Field Operator for an Inhomogeneous Medium

Definition: Values of k such that the interior transmission problem has a nontrivial solution are called transmission eigenvalues.

Theorem: If $Im\ n(x_0) \neq 0$ for some $x_0 \in D$ then k is not a transmission eigenvalue, i.e. the far field operator F is injective with dense range.

Proof: If there exists a nontrivial solution to the interior transmission problem then

$$
\begin{aligned}
0 &= \int_{\partial D} \left(v \frac{\partial \overline{v}}{\partial \nu} - \overline{v} \frac{\partial v}{\partial \nu} \right) ds = \int_{\partial D} \left(w \frac{\partial \overline{w}}{\partial \nu} - \overline{w} \frac{\partial w}{\partial \nu} \right) ds \\
&= \int_{D} \left(w \Delta \overline{w} - \overline{w} \Delta w \right) dx = 2ik^2 \int_{D} Im\ n|w|^2 dx.
\end{aligned}
$$

If $Im\ n(x_0) \neq 0$ then $w(x) = 0$ in a neighborhood of x_0 and by the unique continuation principle $w(x) = 0$ for $x \in D$. Then v has vanishing Cauchy data and hence $v(x) = 0$ for $x \in D \Rightarrow\Leftarrow$.

Partially Coated Perfect Conductors

If a portion of a perfectly conducting cylinder is partially coated by a dielectric, we are led to the mixed boundary value problem

$$\Delta_2 u + k^2 u = 0 \quad \text{in} \quad R^2 \setminus \overline{D}$$

$$u = u^i + u^s$$

$$u = 0 \quad \text{on } \Gamma_D$$

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \text{ on } \Gamma_I$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where λ is a positive constant and $\partial D = \Gamma_D \cup \Gamma_I$.

Theorem: The mixed boundary value problem has at most one solution.

Proof: Green's theorem and Rellich's lemma.

It is no longer appropriate to use integral equations of the second kind to obtain existence; instead integral equations of the first kind must be used.

Partially Coated Perfect Conductors

From Green's representation formula we have

$$u = S\frac{\partial u}{\partial \nu} - Du$$

where S and D are single layer and double layer potentials respectively. Applying the boundary conditions and letting ψ_I and ψ_D be the unknown boundary data for u on Γ_I and $\frac{\partial u}{\partial \nu} + ik\lambda u$ on Γ_D respectively leads to a system of integral equations of the first kind for the determination of ψ_I and ψ_D:

$$A\begin{pmatrix}\psi_D \\ \psi_I\end{pmatrix} = g$$

Theorem: In an appropriate function space, A is a Fredholm operator with index zero and A has a trivial kernel.

Corollary: A solution exists to the mixed boundary value problem.

The Linear Sampling Method in Inverse Scattering Theory

David Colton

University of Delaware

Research supported by AFOSR

The Linear Sampling Method

Consider the scattering of a time harmonic plane wave

$$U^i(x,t) = e^{ikx\cdot d - i\omega t}$$

by a sound soft obstacle $D \subset \mathbb{R}^2$.

Then, if $U^s(x,t) = u^s(x)e^{-i\omega t}$ is the scattered wave,

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

where $r = |x|$, $\hat{x} = x/|x|$, $|d| = 1$.

The inverse scattering problem is to determine D from the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ where

$$\Omega := \{x : \ |x| = 1\}.$$

The Linear Sampling Method

Now let

$$\Phi(x,z) = \frac{i}{4} H_0^{(1)}(k|x-z|)$$

which has the far field pattern

$$\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}.$$

Suppose there exists $g(\cdot, z) \in L^2(\Omega)$ such that the far field equation

$$\int_\Omega u_\infty(\hat{x}, d) g(d, z) ds(d) = \Phi_\infty(\hat{x}, z)$$

is satisfied for $z \in D$.

Then by Rellich's lemma

$$\int_\Omega u^s(x, d) g(d, z) ds(d) = -\Phi(x, z)$$

for $x \in \partial D$.

The Linear Sampling Method

Since D is sound-hard, $u^s(x,d) = -e^{ikx\cdot d}$ for $x \in \partial D$ and hence the Herglotz wave function with kernel g

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d,z)ds(d)$$

tends to infinity as $z \to \partial D \Longrightarrow$

$$\lim_{z\to\partial D} \|g(\cdot,z)\|_{L^2(\Omega)} = \infty.$$

Unfortunately, in general no solution exists to the far field equation, i.e. (assuming that k^2 is not a Dirichlet eigenvalue) the solution of the interior Dirichlet problem

$$\Delta_2 u + k^2 u = 0 \qquad \text{in } D$$
$$u = \Phi(\cdot,z) \qquad \text{on } \partial D$$

for $z \in D$ is not a Herglotz wave function!

Herglotz Wave Functions

The linear sampling method for solving the inverse scattering problem is based on the fact that although the far field equation is in general not solvable, for every $\epsilon > 0$ there does exist a "solution" with discrepancy ϵ and this approximate solution tends to infinity as $z \to \partial D$ for $z \in D$.

This fact is a corollary of the following theorem where

$$\mathbb{H}(D) := \left\{ u : \begin{array}{r} u \in H^1(D) \quad \text{and} \\ \Delta_2 u + k^2 u = 0 \text{ in } D \end{array} \right\}.$$

Theorem (Colton-Sleeman): *With respect to the $H^1(D)$ norm the set of Herglotz wave functions is dense in $\mathbb{H}(D)$.*

Herglotz Wave Functions

Outline of Proof

Step 1: Let

$$u_n(x) := J_n(kr)e^{in\theta}.$$

Then the set

$$\left\{ \frac{\partial u_n}{\partial \nu} + iu_n \right\}$$

is complete in $H^{-1/2}(\partial D)$.

Step 2: There exists a positive constant C such that if u is the unique weak solution of

$$\begin{aligned} \Delta_2 u + k^2 u = 0 \qquad & \text{in } D \\ \frac{\partial u}{\partial \nu} + iu = f \qquad & \text{on } \partial D \end{aligned}$$

for $f \in H^{-1/2}(\partial D)$ then

$$\|u\|_{H^1(D)} \leq C\|f\|_{H^{-1/2}(\partial D)}.$$

For details see
D.Colton and B.D. Sleeman, *An approximation property of importance in inverse scattering theory*, Proc. Edinburgh Math. Soc. **44** (2001), 449-454.

Herglotz Wave Functions

In the case of the inverse scattering problem with limited aperture data, the following generalization of the Colton-Sleeman theorem is important:

Theorem (Cakoni-Colton): *With respect to the $H^1(D)$ norm the set of Herglotz wave functions with kernels supported in $\Omega_0 \subset \Omega$ is dense in $\mathbb{H}(D)$.*

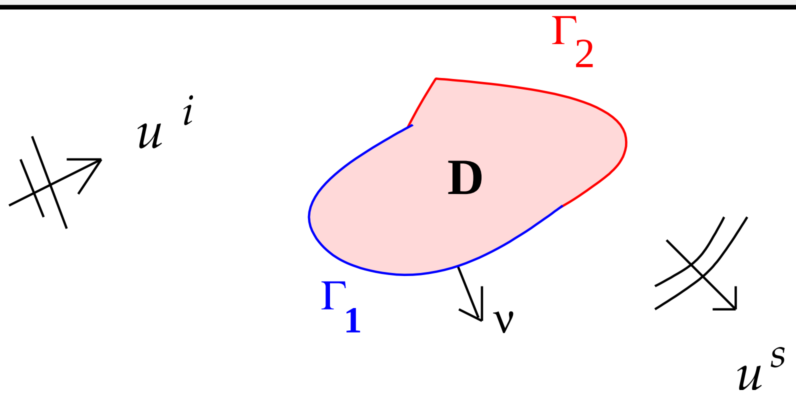
Hence, in order to reconstruct D from the far field pattern u_∞ for $d \in \omega_0$, $\hat{x} \in \Omega_1$, one must solve the far field equation

$$\int_{\Omega_0} u_\infty(\hat{x}, d) g(d, z) ds(d) = \Phi_\infty(\hat{x}, z), \qquad \hat{x} \in \Omega_1$$

and determine where $\|g(\cdot, z)\|_{L^2(\Omega_0)}$ "blows up".

This is done by using regularization methods to solve the far field equation. However, it remains to be shown that this numerical procedure recovers the Herglotz kernel g of Colton-Sleeman and Cakoni-Colton theorems! In this regard see
Tilo Arens, *Why linear sampling works*, Inverse Problems **20** (2004), 163-173.

Mixed Boundary Value Problems



$$
\begin{array}{lll}
\nabla \cdot A\nabla v + k^2 v = 0 & \text{in} & D \\
\Delta_2 u + k^2 u = 0 & \text{in} & \mathbb{R}^2 \setminus \overline{D} \\
v - u = 0 & \text{on} & \Gamma_1 \\
v - u = -i\eta(x)\dfrac{\partial u}{\partial \nu} & \text{on} & \Gamma_2 \\
\dfrac{\partial v}{\partial \nu_A} - \dfrac{\partial u}{\partial \nu} = 0 & \text{on} & \Gamma = \Gamma_1 + \Gamma_2 \\
u = u^i + u^s & & \\
\lim_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0 & &
\end{array}
$$

where

$A \in C^1(\overline{D})$, A is symmetric,
$\mathcal{R}e\left(\overline{\xi} \cdot A\xi\right) \geq \gamma|\xi|^2$ for some $\gamma > 0$, $\mathcal{I}m\left(\overline{\xi} \cdot A\xi\right) = 0$,

$\eta \in L_\infty(\Gamma_2)$ where $\eta(x) \geq \eta_0 > 0$,

$u^i(x) = e^{ikx\cdot d}$ and $\dfrac{\partial v}{\partial \nu_A} := \nu \cdot A\nabla v.$

Mixed Boundary Value Problems

Theorem: *There exists a unique solution* (v, u) *in* $H^1(D) \times H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ *to the mixed transmission problem.*

The scattered field u^s has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $r = |x|$, $\hat{x} = x/r$, k is fixed and u_∞ is the far field pattern of the scattered field u^s.

The inverse scattering problem is to determine D and η from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$.

Mixed Boundary Value Problems

A solution to the far field equation

$$\int_{\Omega} u_\infty(\hat{x}, d) g(d, z) ds(d) = \Phi_\infty(\hat{x}, z)$$

exists if and only if the solution w_z of the interior transmission problem

$$\begin{aligned}
\nabla \cdot A \nabla v_z + k^2 v_z &= 0 && \text{in} \quad D \\
\Delta_2 w_z + k^2 w_z &= 0 && \text{in} \quad D \\
v_z - w_z &= \Phi(\cdot, z) && \text{on} \quad \Gamma_1 \\
v_z - w_z &= \Phi(\cdot, z) - i\eta \frac{\partial}{\partial \nu}(w_z + \Phi(\cdot, z)) && \text{on} \quad \Gamma_2 \\
\frac{\partial v_z}{\partial \nu_A} - \frac{\partial w_z}{\partial \nu} &= \frac{\partial}{\partial \nu} \Phi(\cdot, z) && \text{on} \quad \Gamma.
\end{aligned}$$

for $z \in D$ is a Herglotz wave function.

Definition: *Values of k for which there exists a nontrivial solution to the interior transmission problem for $\Phi = 0$ are called transmission eigenvalues.*

Mixed Boundary Value Problems

Theorem: *Assume that k is not a transmission eigenvalue and that $\Re(\overline{\xi} \cdot A\xi) \geq \gamma|\xi|^2$ or $\Re(\overline{\xi} \cdot A^{-1}\xi) \geq \gamma|\xi|^2$ for some $\gamma > 1$. Then there exists a unique solution $v_z \in H^1(D)$, $w_z \in H^1(D)$ and $\dfrac{\partial w_z}{\partial \nu} \in L^2(\Gamma_2)$ to the interior transmission problem.*

Theorem (Cakoni-Colton-Monk): *The set of Herglotz wave functions v_g for all $g \in L^2(\Omega)$ is dense in*

$$\mathcal{H}(D) := \left\{ w \in \mathbb{H}(D) \quad \text{such that} \quad \frac{\partial w}{\partial \nu} \in L^2(\Gamma_2) \right\}$$

equipped with the graph norm.

Proof: Based on the ideas of Colton-Sleeman.

Determination of D and η

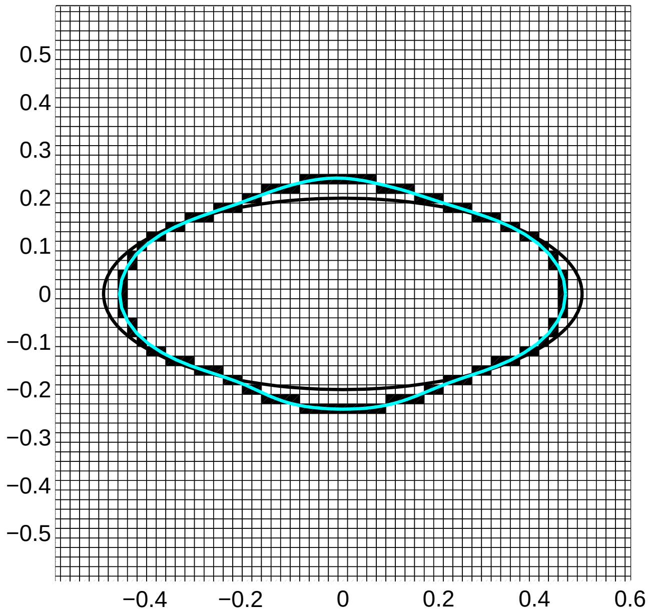
Theorem: *Assume that k is not a transmission eigenvalue and let v_z, w_z be the unique solution of the interior transmission problem. Then w_z can be approximated in $\mathcal{H}(D)$ by the Herglotz wave function v_{g_z} with kernel g_z an approximate solution of the far field equation.*

- The above theorem provides a method for approximating both D and $\|\eta\|_{L_\infty(\Gamma_2)}$. In particular, if η is a constant we have

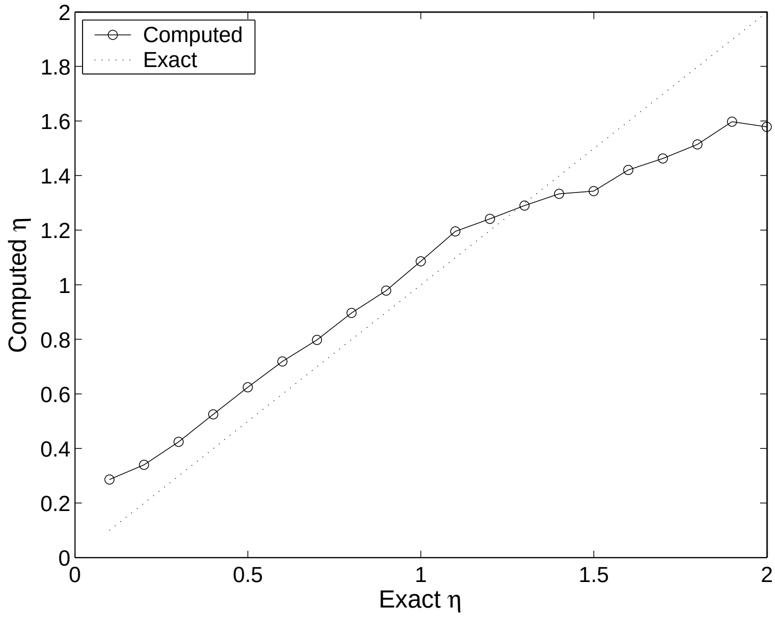
$$\eta = \frac{-\frac{1}{4} - \mathcal{I}\mathrm{m}\,(w_{z_0}(z_0))}{\|\frac{\partial}{\partial\nu}(w_{z_0} + \Phi(\cdot\,,z_0))\|^2_{L^2(\Gamma_2)}} \qquad z_0 \in D.$$

- Since Γ_2 is unknown, the above expression only provides a lower bound for η!

Determination of D and η

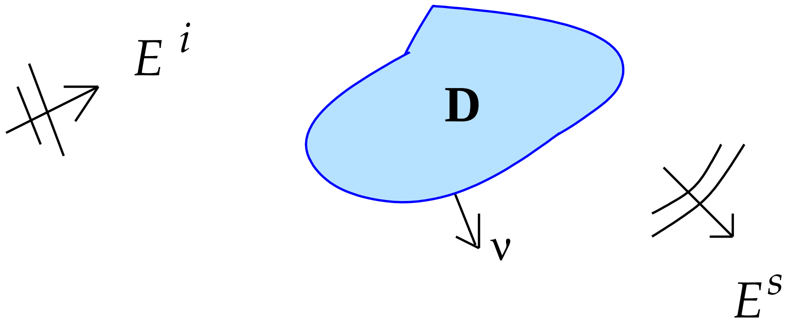


Reconstruction of D (fully coated)



Reconstruction of η (approximate D)

Maxwell's Equations



$$\text{curl}\,\text{curl}\,E - k^2 E = 0 \qquad \text{in} \qquad \mathbb{R}^3 \setminus \overline{D}$$

$$E = E^s + E^i$$

$$\nu \times E = 0 \qquad \text{on} \qquad \Gamma$$

$$\lim_{|x|\to\infty} (\text{curl}\,E^s \times x - ik|x|E^s) = 0$$

where

$$E^i(x) = \frac{i}{k}\text{curl}\,\text{curl}\,p e^{ikx\cdot d}.$$

Theorem: *There exists a unique solution E of the above scattering problem such that E is in $H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$.*

Maxwell's Equations

It can be shown that

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}, d, p) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$ The inverse scattering problem is to determine D from $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d$ on the unit sphere Ω and three linearly independent polarizations p.

We will now use the linear sampling method to determine D. To this end we define the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d)$$

and the corresponding far field equation by

$$\boxed{(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)}$$

where $E_{e,\infty}$ is the far field pattern of the electric dipole

$$E_e(x) = \frac{i}{k}\,\text{curl}\,\text{curl}\, q\, \Phi(x, z)$$

where $\Phi(x, z) = \dfrac{1}{4\pi}\dfrac{e^{ik|x-z|}}{|x-z|}$.

Maxwell's Equations

For $z \in D$ the far field equation $Fg = E_{e,\infty}$ has a unique solution if and only if the solution E of the following interior problem has a unique weak solution that is a Herglotz wave function:

$$\begin{aligned} \text{curl}\,\text{curl}\,E - k^2 E = 0 \qquad & \text{in} \qquad D \\ \nu \times (E + E_e) = 0 \qquad & \text{on} \qquad \Gamma \end{aligned}$$

There exists a unique solution in $H(\text{curl}, D)$ to the above interior problem if k is not a Maxwell eigenvalue.

Theorem (Colton-Kress): *With respect to the $H(curl, D)$ norm the set of Herglotz wave functions with kernel in $L_t^2(\Omega)$ is dense in the space of $H(curl, D)$ solutions to $curl\,curl\,E - k^2\,E = 0$.*

The above analysis now implies that D can be determined by solving the far field equation for $g(\cdot\,, z)$ using regularization methods and determining where $\|g(\cdot\,, z)\|_{L^2(\Omega)}$ "blows up".

Target Identification of Partially Coated Objects

David Colton

UNIVERSITY OF DELAWARE

Research supported by AFOSR

Mixed Boundary Value Problems

Mixed boundary value problems in electromagnetic scattering theory arise when the scattering object is a composite material such that parts of the scatterer have different electrical properties.

Such scattering objects can be:

- *Partially coated perfect conductors.*
- *Thin objects with one side a perfect conductor and the other side an imperfect conductor or dielectric.*
- *Partially coated dielectrics.*

Mixed Boundary Value Problems

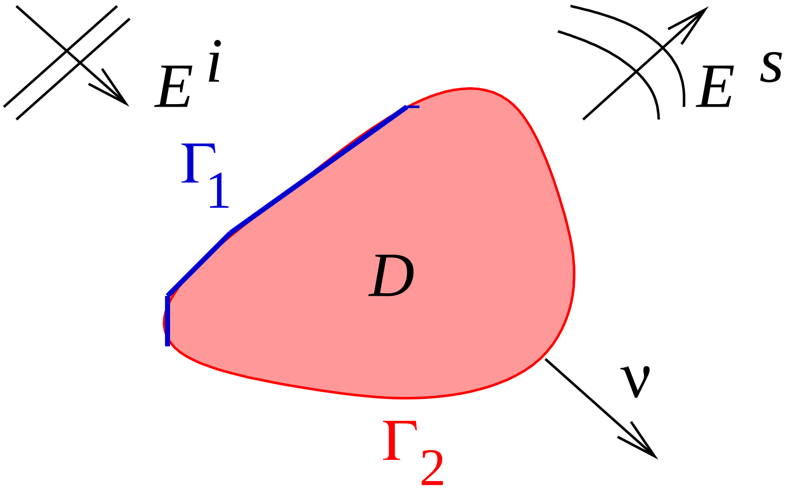
The direct scattering problem

- *The mathematical analysis of mixed boundary value problems is difficult due to the non-standard solution space.*
- *No matter how smooth the boundary data is, the change of boundary conditions causes the scattered field to be singular at the interface. This gives rise to numerical difficulties.*

The inverse scattering problem

- *Since the physical structure of the composite medium is not known a priori, the use of weak scattering approximations and/or nonlinear optimization techniques are problematic.*

Scattering by a Coated Dielectric



$$\left\{ \begin{array}{l} \nabla \times E^{ext} - ikH^{ext} = 0 \\ \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \\ \nabla \times H^{ext} + ikE^{ext} = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \nabla \times E^{int} - ikH^{int} = 0 \\ \qquad\qquad\qquad\qquad \text{in } D \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{array} \right.$$

$$\begin{aligned} \nu \times E^{ext} - \nu \times E^{int} &= 0 \quad \text{on} \quad \partial D = \Gamma_1 \cup \Gamma_2 \\ \nu \times H^{ext} - \nu \times H^{int} &= 0 \quad \text{on} \quad \Gamma_1 \\ \nu \times H^{ext} - \nu \times H^{int} &= \eta(x)(\nu \times E^{ext}) \times \nu \quad \text{on} \quad \Gamma_2 \end{aligned}$$

where the exterior field E^{ext}, H^{ext} is given by

$$E^{ext} = E^i + E^s \qquad H^{ext} = H^i + H^s,$$

E^s, H^s is the scattered field satisfying the Silver Müller radiation condition and E^i, H^i is the given incident field.

The Inverse Problem

The scattered electric field E^s has the asymptotic behaviour

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$ where $\hat{x} = x/|x|$, $\hat{x} \cdot E_\infty(\hat{x}) = 0$ and $E_\infty(\hat{x})$ is the electric far field pattern. We always assume that $k > 0$ is fixed.

If we use the incident field given by

$$E^i(x) := \frac{i}{k}\nabla \times \nabla \times p\, e^{ikx\cdot d}$$

$$H^i(x) := \nabla \times p\, e^{ikx\cdot d}$$

then

$$E_\infty(\hat{x}) := E_\infty(\hat{x}, d, p).$$

Let $\Omega := \{x : |x| = 1\}$.

The inverse scattering problem we want to solve is to determine both D and η from a knowledge of $E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega_0 \subset \Omega$, $d \in \Omega_1 \subset \Omega$ and $p \in \mathbb{R}^3$

Uniqueness Theorems

Theorem *(Cakoni-Colton)*: *D is uniquely determined by the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega_0$, $d \in \Omega_1$ and three linearly independent polarizations p_1, p_2, p_3.*

Remark: The matrix $N(x)$ is not uniquely determined by the far field pattern!

Theorem *(Cakoni-Colton-Monk)*:
Let $E^j_\infty(\hat{x}, d, p)$ be the electric far field pattern corresponding to a fixed matrix $N(x)$ but different coatings $\eta = \eta_j$, $j = 1, 2$. Assume that k is not a Maxwell eigenvalue for $D = \{x : I - N(x) \neq 0\}$. Then if $E^1_\infty(\hat{x}, d, p) = E^2_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega_0$, $d \in \Omega_1$ and three linearly independent polarizations $p \in \mathbb{R}^3$, we have that $\eta_1(x) = \eta_2(x)$ for $x \in \Gamma_2$

Electric Dipoles

The radiating solution to Maxwell's equations

$$E_e(x, z, q) := \frac{i}{k} \nabla_x \times \nabla_x \times q\,\Phi(x, z)$$
$$H_e(x, z, q) := \nabla_x \times q\,\Phi(x, z)$$

with

$$\Phi(x, z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x - z|}, \; q \in \mathbb{R}^3$$

is called the electric dipole located at z and polarized in the direction $q \in \mathbb{R}^3$.

$E_{e,\infty}(\hat{x}, z, q)$ denotes the far field pattern of the electric dipole.

Far Field Equation

We define the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_{\Omega} E_\infty(\hat{x}, d, g(d))\, ds(d).$$

Given $g \in L_t^2(\Omega)$, Fg is the far field pattern of the scattered field corresponding to the incident field being a Herglotz wave function with kernel g given by

$$E_g(x) := ik \int_{\Omega} e^{ikx\cdot d} g(d) ds(d).$$

Now consider the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)$$

Solving the Far Field Equation

Using the approximation properties of Herglotz wave functions we can prove the following theorem:

Theorem *(Cakoni-Colton-Monk)*: *For every $\epsilon > 0$ there exists g_z^ϵ such that*

$$\|Fg_z^\epsilon - E_{e,\infty}(\cdot, z, q)\|_{L^2(\Omega)} \leq \epsilon$$

and

- *For $z \in D$,* $\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} < \infty$
- *For each $\epsilon > 0$,* $\lim_{z \to \partial D} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} = \infty$
- *For $z \in \mathbb{R}^3 \setminus \overline{D}$,* $\lim_{\epsilon \to 0} \|E_{g_z^\epsilon}\|_{X(D,\Gamma_2)} = \infty.$

Here $X(D, \Gamma_2)$ is the space

$$\left\{u \in L^2(D) : \nabla \times u \in L^2(D),\ \nu \times u|_{\Gamma_2} \in L_t^2(\Gamma_2)\right\}.$$

Linear Sampling Method

The linear sampling method determines g from the far field equation $Fg = E_{e,\infty}$.

The support D can be determined by the behavior of g. In particular, $\|E_g\|_{X(D,\Gamma_2)} \to \infty$ implies $\|g\|_{L^2(\Omega)} \to \infty$.

Open Problem: In practice g is obtained by using a regularization method such as Tikhonov regularization. Does this regularized solution behave in the same way as the approximate solution g whose existence is given by the previous theorem?

This question has been answered positively in certain cases by *Arens (2004)* using the ideas of the factorization method developed by *Kirsch (1998)*.

Limited Aperture Data

In practice we have

$$\int_{\Omega_1} E_\infty(\hat{x}, d, g(d))\, ds(d) = E_{e,\infty}(\hat{x}, z, q) \quad \hat{x} \in \Omega_0.$$

Based on

Theorem *(Cakoni-Colton)*: *With respect to the $X(D, \Gamma_2)$ norm the set of Herglotz wave functions with kernel supported in a subset Ω_1 of Ω is dense in $\mathbb{H}$.*

the above discussion is applicable to the far field equation with limited aperture data.

Here

$$\mathbb{H} = \left\{ u \in X(D, \Gamma_2) : \; \nabla \times \nabla \times u - k^2 u = 0 \text{ in } D \right\}.$$

Determination of η

Assume η is a constant and let E_z, H_z, E_z^{int}, H_z^{int} be the unique solution of the interior transmission problem

$$\begin{cases} \nabla \times E_z - ikH_z = 0 \\ \qquad\qquad\qquad\qquad \text{in } D \\ \nabla \times H_z + ikE_z = 0 \end{cases}$$

$$\begin{cases} \nabla \times E_z^{int} - ikH_z^{int} = 0 \\ \qquad\qquad\qquad\qquad \text{in } D \\ \nabla \times H_z^{int} + ikN(x)E_z^{int} = 0 \end{cases}$$

$$\begin{aligned} \nu \times E_z^{int} - \nu \times E_z &= \nu \times E_e(\cdot, z, q) \quad \text{on} \quad \partial D = \Gamma_1 \cup \Gamma_2 \\ \nu \times H_z^{int} - \nu \times H_z &= \nu \times H_e(\cdot, z, q) \quad \text{on} \quad \Gamma_1 \\ \nu \times H_z^{int} - \nu \times H_z &= \begin{cases} -\eta(x)[\nu \times (E_z - E_e(\cdot, z, q)) \times \nu] \\ \quad +\nu \times H_e(\cdot, z, q) \quad \text{on} \quad \Gamma_2 \end{cases} \end{aligned}$$

where E_e, H_e is the electric dipole introduced previously.

Determination of η

Remark: The existence and uniqueness of a solution to the interior transmission problem has to date only been proved for the case when $\eta = 0$!

H.Haddar, *The interior transmission problem for anisotropic Maxwell's equations and its application to the inverse problem*, Math. Methods Applied Sciences 27 (2004), 2111-2129.

The solution E_z of the interior transmission problem (if it exists!) can be approximated by a Herglotz wave function with kernel g_z^ϵ and this g_z^ϵ is an approximate solution to the far field equation!

Determination of η

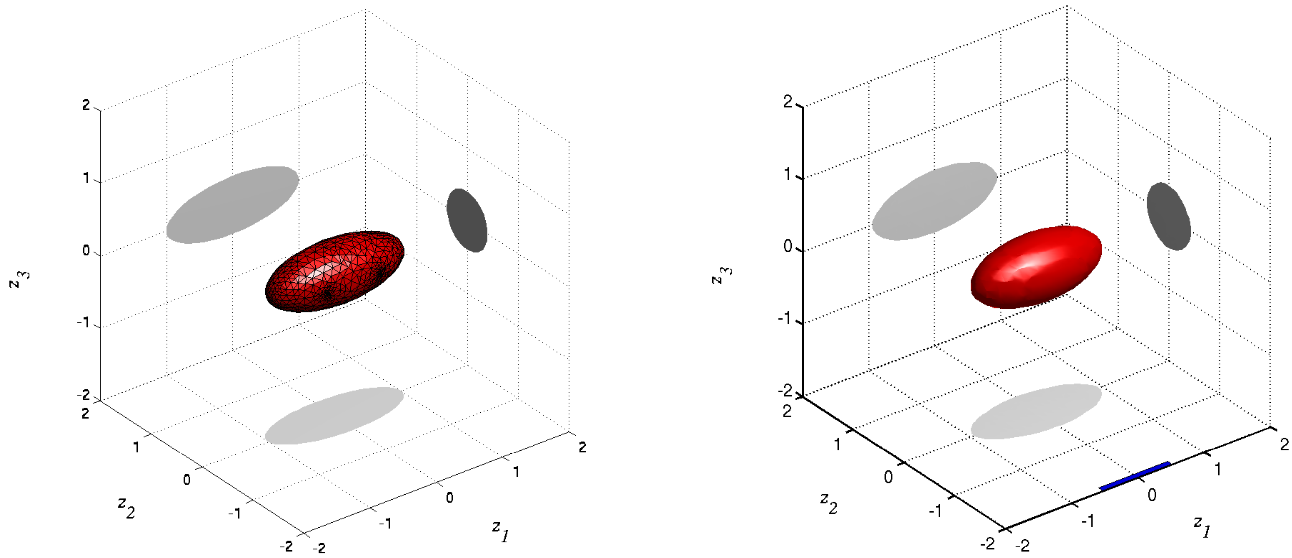
Theorem *(Cakoni-Colton-Monk)*:
Let z_0 be any arbitrary point in D and q a vector in $\mathbb{R}^3$. Then

$$\eta = \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_{z_0}(z_0)\right)}{\left\|\nu \times \left(E_{z_0}(\,\cdot\,) + E_e(\cdot\,, z_0, q)\right)\right\|^2_{L^2_t(\Gamma_2)}}.$$

Corollary: For $z_0 \in D$, $q \in \mathbb{R}^3$, we have that

$$\eta \geq \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_{z_0}(z_0)\right)}{\|E_{z_0}(\,\cdot\,) + E_e(\cdot\,, z_0, q)\|^2_{L^2(\partial D)}}.$$

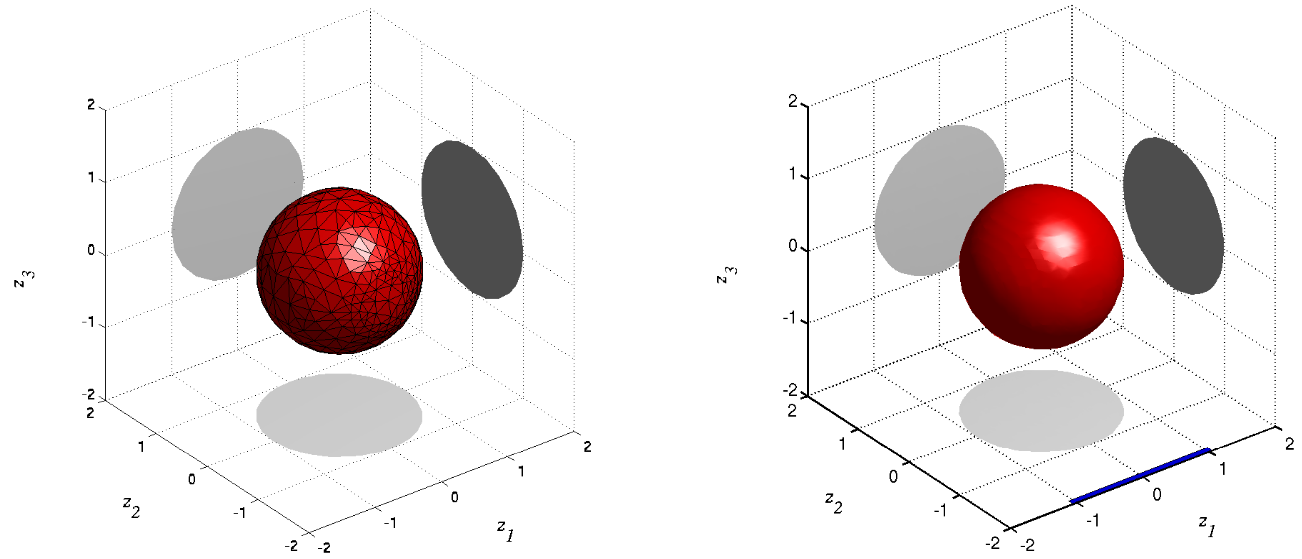
Examples of Reconstructions



Reconstruction of a fully coated ellipsoid with $\eta = 1$ and $k = 6$.

Conducting boundary condition: reconstruction of η			
Exact	Exact ∂D	LSM	LSM/bound
0.0	-0.005	-0.01	-0.004
0.1	0.09	0.16	0.07
1	0.96	0.79	0.58
2	1.15	0.94	0.66

Examples of Reconstructions



Reconstruction of a partially coated sphere. The coated portion Γ_2 is the hemisphere $x_2 > 0$. Here $\eta = 1$ and $k = 3$.

Conducting boundary condition: reconstruction of η			
Exact	Exact Γ_2	LSM	LSM/bound
0.1	0.045	0.037	0.027
1	0.94	0.52	0.43
2	2.00	0.81	0.65